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On Automatic Continuity in the Jordan-Banach Algebras

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Abstract: ***In this work, we are interested in continuity of Homomorphisms in the complete normalized Jordan algebras. With this intention, we extend from the results known in the case of the Banach algebras to the complete normalized Jordan algebras***

Keywords: Jordan-Banach algebra, separting space, *simple algebra, Automatic Continuity.

1. Introduction

In this work, we define for an algebra in involution * (A, *) a concept that we call * simplicity, it rests on the study of certain ideals. The interest thus is to restrict with a family of the ideals instead of considering all the left ideals. This concept of * simplicity will contribute also under investigation of the problem of the automatic continuity of the linear operators in the topological algebras. While being based on the fact that the property to be locally bounded, is more essential, for an Banach algebra, that locally convexity, we were interested in the p-normalized algebras ($0 < p \leq 1$)[4]. We propose too to generalize the results known in the case of the Banach algebras to the case of the complete p-normalized non associative algebras, in particular the non commutative Jordan algebras. We then show that if A is a *-simple p-normalized non commutative Jordan algebra, any surjective homomrphism (or with dense range) of an complete p-normalized Jordan algebra on A is continuous.

2. Preliminaries

Throughout this work, the algebras considered are supposed to be complex, non associative, unital and not necessarily commutative; and ($0 < p \leq 1$) .

An algebra over IK is a IK-vector space *A* with a bilinear map $(x, y) \to x.y$ of $A \times A$ into *A*. If this product is associative (resp commutative), we say that the algebra is associative (resp. commutative). An algebra *A* is a Jordan algebra if every $(x, y) \in A^2$ we have $xy = yx$ and $(x^2 y)x = x^2 y$. An algebra *A* is a non commutatives Jordan algebra (n.c Jordan algebra) if every $(x, y) \in A^2$ we have $x(yx) =(xy)x$ and $x^2 (yx) =(x^2 y)x$. Let *A* be an algebra, we denote by A^+ the algebra *A* equipped with its vector space structure and the product $\circ$ defined by : $x \circ y = 1/2\, (xy + yx)$ for all $x, y \in A$. The product $\circ$ is called the Jordan product. $(A^+, +, ., \circ)$ is Jordan algebra. An normed Jordan algebra *A* is a Jordan algebra equipped by a norm II . II satisfying $\|xy\| \leq \|x\| \|yx\|$ for every $(x, y) \in A^2$, if $(A, \|.\|)$ is Banach space then *A* is called a Jordan Banach algebra. Let *A* be a Jordan Banach algebra with unit *e*, if $x \in A$ such that $\|x - e\| \leq 1$, then *x* is invertible. Let A an algebra of Jordan-Banach, one calls the spectral ray of, noted r (a), the limit $\lim_{n\to+\infty} \|a^n\|^{1/n}$ if it exists.

A *-ring is a ring with a map * : A $\to$ A that is an involution. More precisely, * is required to satisfy the following properties: $(x + y)^* = x^* + y^*$, $(x y)^* = y^* x^*$, $1^* = 1$, $(x^*)^* = x$ for all x, y in A. This is also called an involutive ring, and ring with involution. Elements such that $x^* = x$ are called self-adjoint. Also, one can define *-versions of algebraic objects, such as ideal and subring, with the requirement to be *-invariant: $x \in I \Rightarrow x^* \in I$ and so on. A *-algebra A is a *-ring, with involution *, such that $(\lambda x)^* = \overline{\lambda} x^*\ \forall \lambda \in IK, x \in A$. An algebra A is called simple if it has no proper ideals. An *-algebra A is called *-simple if it has no proper *ideals.

3. Automatic Continuity

The notion of separting space characterizes the continuity of linear operator. The utility of separating space comes owing to the fact that a linear operator is bounded if, and only if, its separating space is reduced to the singleton {0} ([5]). One of the first results on the automatic continuity of homomorphism to dense image is in Rickart:

Theorem 3.1 [1] Let θ Is a homomorphism of an Banach algebra A on an strongly semi-simple Banach algebra B. if the image of θ is dense in B, then θ is automatically continuous.

Theorem 3.2 [2] Let T a homomorphism of an Jordan-Banach algebra A on an Jordan-Banach algebra B. If B is strongly semi-simple and if the image of T is dense in B, then T is automatically continuous.

Of its share, A.R. Palacios generalized the two theorems with the case of the complete normalized associative power algebras.

Theorem 3.3 [3] Any homomorphism with dense image of an complete normalized associative power algebras A on an strongly semi-simple complete normalized associative power algebras is automatically continuous.

Question: is there the automatic continuity of a homomorphism with dense range of an n.c Jordan-Banach algebra in an semi-simple n.c Jordan-Banach algebra?

Definition: 3.1
Let T a linear application of a complete p-normalized space X in a complete p-normalized space Y. Then, the separating space σ (T) of Y is the subset of Y defined by: σ(T) = {y ϵ Y/ ∃$(x_n)_n$ C X : $x_n \xrightarrow{\|\ \|_p} 0$ et $T(x_n) \xrightarrow{\|\ \|_p} y$ }.

Proposition 3.1
Let X and Y two complete p-normalized space, then the separating space σ (T) of any a linear application T : X → Y is a closed subspace of Y.

Proof
Evidently σ(T) is a subspace vector of Y. Let $(y_k)_k$ a sequence in σ(T) converging to y of Y. we prove that y ∈ σ(T). for every k ∈ IN*, y_k ∈ σ(T). Then there is a sequence $(y_{k,n})_n$ ⊂ X such that: $x_{n,k} \xrightarrow{\|\ \|_p} 0$ and $T(x_{n,k}) \xrightarrow{\|\ \|_p} y_k$. Let us choose two sequence $(z_k)_k$ $(y_k)_k$ in X such that, for every k ∈ IN*, $\| z_k\|_p < 1/k$ and $\| T(z_k)- y \|_p <$ 1/k. then, we are; $z_k \xrightarrow{\|\ \|_p} 0$ and $T(z_k) - y_k \xrightarrow{\|\ \|_p} 0$. On other hand, for every k ∈ IN*, we are : $T(z_k) – y = (T(z_k) - y_k) + (y_k – y)$. then, $z_k \xrightarrow{\|\ \|_p} 0$ et $T(z_k) - y \xrightarrow{\|\ \|_p} 0$. Consequently, y ∈ σ(T).

Proposition 3.2
Let T a linear application of a complete p-normalized unital algebra A in a complete p-normalized unital algebra B. then, if T is surjective, the separating space σ (T) is proper ideal of Y.

Proof
Let b ∈ B and y ∈ σ(T).
y ∈ σ(T), there then there exists a sequence $(a_n)_n$ ⊂ A such that: $a_n \xrightarrow{\|\ \|_p} 0$ et $T(a_n) \xrightarrow{\|\ \|_p} y$. Suppose that T is surjective, then there exists a ∈ A such that T (a) = B; Then, $a_n a \xrightarrow{\|\ \|_p} 0$ et $T(a_n a) = T(a_n)(T(a) = T(a_n)b \xrightarrow{\|\ \|_p}$ yb, Consequently yb ∈ σ(T). Let us show that σ (T) is a proper ideal of B. As A and B are unital, T $(e_A) = e_B$. For all A, a ∈ A, Sp(T(a)) ⊂ Sp(a). Then $r_B(T(a)) \leq r_A(a)$ a ∈ A. Let c an element of center of B, then $r_B(T(a)) \leq r_B(c - T(a)) + r_B(T(a)) \leq$ II c – T(a)II_p + IIA II_p Suppose that e_B ∈ σ(T). Then, there exists a $(a_n)_n$ ⊂ A such that: $a_n \xrightarrow{\|\ \|_p} 0$ et $T(a_n) \xrightarrow{\|\ \|_p} e_B$. As e_B is an element of center of B, $r_B(T(e_A)) = r_B(e_B) \leq$ II $e_B – T(a_n)II_p$ + IIa_n $II_p \xrightarrow{\|\ \|_p} 0$
What contradicts the fact that $r_B(e_B) = 1$.

Proposition 3.4 Let A an *simple non associative algebra which is not simple. Then, there exists a subalgebra simple unit I of A such as A = I ⊕ I*.

Proof
Let I a proper ideal of A. I ∩ I* is one *ideal, therefore I ∩ I* = {0} or I ∩ I* = A. If I ∩ I* = A. If I ∩ I* = A then I = A, which is absurd. From where I ∩ I* = {0}. There is also I + I* is one *ideal, then I + I* = {0} or I + I* = A. If I + I* = {0}, then I = {0}, which contradicts the fact that I am *proper*. Therefore, A = I ⊕ I*. Let J a ideal of A such as J ⊆ I. According to what precedes, A = J ⊕ J*. Let i ∈ I, then there exists j, j' ∈ J such that i = j + j'*. However i - j = j'* ∈ I ∩ I* = {0}, from where i = j, therefore I = J. Consequently, I am a minimal ideal of A. Let J an ideal of I, then J is an ideal of A. Indeed, let a ∈ A and j ∈ J , then it exists i,i'* such that a = i + i'*. From where aj = (i + i'*)j = ij + i'*j. However, i'*j ∈ I*I ⊆ I ∩ I* = {0}, consequently aj = ij ∈ J. As I am a minimal ideal, then J = {0} or I = J. Thus, I am simple subalgebra. On other hand, I am unital and if 1 indicates the unit of A, then there exists e, e' ∈ I such that 1 = e + e'*. Let x ∈ I, we are: x = x1 = xe + x e'*, but x - xe = x e'* ∈ I ∩ I* = {0}, from where x = xe. In the same way, we checked that x = xe. Consequently, I am unital of unit e.

Lemma 3.1 Let T a homomorphism of an complete p-normalized n.c Jordan algebra of A in an complete p-normalized n.c Jordan algebra B, for any element a in A, r(T(a)) ≤ r(a).

Proof
Let A and C under maximum commutative associative subalgebra of A containing a (the existence of is proven by the lemma of Zorn). Let D under maximum commutative associative algebra containing T (C) in B. Still Let us note per T the definite restriction of T of C in D. it is clear that $Sp_D((T(a)) \subseteq Sp_C(a)$. it thus results from it that:
r(T(a)) Sp_D {|λ|/ λ ∈ Sp_D((T(a))} ≤ Sup{|λ|/ λ ∈ Sp_C(a)} = r(a)

Proposition 3.5 Let T a homomorphism of an complete p-normalized n.c Jordan algebra A on an complete p-normalized n.c Jordan algebra B then, if B is simple and if T is surjective (or with dense range), T is continuous.

Proof
Let σ (T) the separating space of T in B which is simple, therefore σ (T) = {0} or σ (T) = B. the last case is impossible, indeed: Let e the unit of B. we have: 1 = r(e) = r(e – T(a) + T(a)). However T (a) and e – T (a) commutate, therefore: r(e – T(a) + T(a)) ≤ r(e – T(a))+ r(T(a)). From where, for any element a of A, , 1 ≤$\|e – T(a)\|_p$+ $\|a\|_p$. If e ∈ σ(T), then there exists a sequence $(a_n)_n$ n≥1 which converges to 0 in A such that the sequence $(T(a_n))_n$ n≥1 converges to e in B. but 1 ≤$\|e – T(a)\|_p$+ $\|a\|_p$ →0, which impossible. From where, σ (T) = {0}. Consequently, T is continuous.

Theorem 3.1
Let T a homomorphism of an complete p-normalized n.c Jordan algebra A on an complete p-normalized n.c Jordan *algebra B then, if B is *simple and if T is surjective (or with dense range), T is continuous.

Proof

Let B is an algebra *simple, there exists simple unital subalgebra I of B such that : B = I $\oplus$ + I* (Proposal 1.1); following algebraic isomorphism: B $\approx$ B/I*, one deduces that I am a maximum ideal of B. From where I (resp; I*) is closed in B. Consequently, I (resp; I*) is a complete p-normalized subalgebra. Let us consider: Pr1: B $\rightarrow$ I (resp. Pr2: B $\rightarrow$ I*) the canonical projection of B on I (resp. of B on I*). Since Pr1 (resp. Pr2) is a continuous epimorphism, then according to the proposal (1.2) Pr1 T (resp. Pr2 T) is continuous. Consequently, T = $(Pr_1 + Pr_2) \circ T = Pr_1 \circ T + Pr_2 \circ T$ is continuous.

Corollary 3.1 Let (A, $\|.\|_p$) an complete p-normalized *-simple n.c Jordan algebra. Then, we have:

1) All the complete complete p-normed on A are equivalent.
2) The involution * is automatically continuous.

Proof

1) It is enough to apply the previous theorem to the Identity of A.
2) That is to say q the linear application of A in IR^+ defined by $q(x) = \| x^* \|_p$ ($x \in A$).

We checks easily that Q is a complete p-normed on A. And according i), Q is equivalent to II x*IIp

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