1. learning the methods for scientific learning,
2. developing abilities for inductive and deductive reasoning,
3. developing abilities to perform elementary modeling and use of the existing or creating new models to learn the properties of the objects,
4. developing skills to classify the learned objects, systematizing knowledge, discovering the cause – consequence and structured relationships between the objects on the level of the already attained knowledge.
5. developing skills to choose tools and methods to obtain the set objective, according to the existing conditions,
6. developing skills to discover the connection between the learned material with the practical work in life, and
7. learning the basic qualities, characteristic of the scientific thinking.

The special developing functions of the tasks include:
1. forming and development of knowledge and skills to deductively affirm and negate mathematical statements,
2. developing skills to plan the solving of a given problem, to exclude the unimportant information in the problem and to add information that is missing, to choose methods, tools and operations to solve the problem and to check the accuracy and the sense of the solution,
3. forming a clear image about the logical structure of the mathematics course, and the fact that the abstract character of mathematics as a science is the basic reason for its many uses in the remaining sciences, in techniques and life in general,
4. developing skills to define the mathematical terms,
5. developing and perfecting skills for making fast and accurate calculations, with or without the help of technical aids, and
6. perfecting the skills to use the language of the mathematical symbolism.

2. Steps in Solving Mathematical Problems

One of the objectives of the mathematical education is the realization of the functions of the mathematical problems. This can be accomplished by solving specific mathematical problems, and in order to do this, the students have to adequately learn the methodology of mathematical problem solving. Taking into consideration the previously mentioned, it is necessary to devote special attention to the methodology of mathematical problem solving starting from the initial education. This methodology, according to many authors, recognizes the following four steps in mathematical problem solving:

a) step one – *understanding the problem*, making a preliminary analysis of the problem, i.e. crystallizing the condition and the conclusion, analysis of all statements presented in the condition and the conclusion, determining which of the objects in the problem are given facts and which are unknown, i.e. what needs to be calculated in the problem, proven or constructed. In other words, in this step the students assess the information in the task, which can be:
   - *basic information*, which characterizes the type of problem and directs the students towards the common ideas and methods for solving problems of the specific type. The separating of this information aims for the students to see what is essential in the problem, which is an assumption for generalization of the knowledge. Hence, if the teachers do not overcome the specific in the given problem and focus on its essential elements, then we are safe to say that the educational function of the problem solving process has been neglected.
   - *specific information*, which is characteristic of the specific problem, however not of the problem type. This information determines the individuality of the problem and instructs the students to concretize the common ideas and methods, determined by the basic information, and to devise a plan to solve the specific problem, which is why separating this information directs the students in each problem, apart from the general, to look for specific things, which develops habits for careful observation of the objects and essential study of the relations between the objects and the data related to them, and
   - *non-essential information*, which does not influence the solution to the problem, but can influence the pedagogical value of the problem. For example, through the content of the so-called word problems, the educational component can be strengthened, followed by correlation to other school subjects, etc. This step is managed by the teacher. The teachers use different approaches, such as reading the entire text of the problem or separate parts, illustrating the problem with a drawing, scheme, sketch or a model, etc. Regardless of which approach is chosen, it is important that the students understand the problem well. Namely, the teachers who skip to step two immediately, without making sure that the problem is understood, make a rough methodological mistake.

b) step two – *developing an idea and devising a plan for solving the problem*, which requires the highest intellectual involvement and starts parallel to understanding the problem. While carrying out this step, learning, i.e. the way to the solution should be revealed gradually. However, in the interest of time, the teacher, or a student, often present a readymade solution, an action which annuls the importance of the development of an idea and devising a plan to solve the problem, which is methodologically wrong. Namely, the greatest amount of time should be devoted to this step, since the functions of the problems are best realized by performing these steps. The following two cases are possible when realizing this step:
   - the type of the problem is familiar, the structure of the solution and the solving method are known, thus, in this case the plan for solving is to concretize the known structure and method and use them. When solving problems of this type, the student carries out the following activities: *identifies the type of problem, chooses the adequate algorithm and uses it in the specific problem, formulates the answer and discusses it.*
   - the structure of the problem is unknown, and the algorithm for solving is unknown as well. There is no pattern which would produce a solution to these problems. Hence, we could say that they have the

greatest educational and developing value. However, these problems are rare in initial education, since they are introduced later - when the students have attained a greater quantum of mathematical knowledge.

c) step three – *practical realization of the plan for solving the problem*, where the students face the least amount of difficulties. However, this step is given the least amount of time in practice, since it requires the solution to be correct, complete, and as short as possible, which is why sometimes, more attention should be given to training the students to form solutions to the problems. It would be beneficial to address this issue in initial education, however the forming of the solution can fully be mastered in the period from grade V in primary school to year I in high school, when the students learn the majority of the mathematical symbolism and terminology, when they are trained to form solutions and proofs of theorems, i.e. when they develop skills to organize the judgments in a logical order. This step should be given more attention, since it establishes the knowledge and skills of the students, and they also learn how to adequately express themselves mathematically in oral and written form.

d) Step four – *additional work after the problem is solved* – which is an essential step. Therefore, after the problem is solved or after several similar problems are solved, it is desirable to pay attention to some of the following questions:
- Is the result correct and why? (Check should be made, if there is a possibility.)
- Which other problems can be solved with the method used for solving this problem?
- Which other methods can be used to solve this problem and which method would be the most efficient in this case?
- What is interesting and important about the solution?
- Can this problem be generalized?
- What else can be found in the problem, apart from what is required?
- Which other data can lead to the elements requested by the task?

In other words, the creative work after the reformulation of the problem should be realized in this step, i.e. in this step, the students should move from the area of current to the area of future development. This is why, while carrying out this step, the teacher should approach the students very carefully, and it is desirable to carry out this step with each student individually or in smaller groups.

Before we analyze some examples, which were used with students of the adequate grades in practice, we should note that the students have to learn the given didactic scheme for solving problems by repeating it several times. Thereby, having into consideration that the most important step in the process of problem solving is developing an idea and devising a plan for solving, which is related to understanding the problem, we would like to mention once again that these steps should be given most of the attention, and the students should not be offered readymade solutions in the interest of time. Also, the complete learning of these two steps from the methodology of problem solving is especially important in order to eliminate the errors which the students make when solving the problems.

Example 1 (second grade). There are 8 sparrows in the nearby park, and there are 5 sparrows in the nest in the tree in Maria's yard. How many more sparrows are there in the park?

During the first step – understanding the problem, it is important that the students understand that this is a *word problem (basic information)*, and that there are more objects of the same type in one place (*specific information*), as well as that there is a park and a nest in Maria's yard (*non-essential information*).

During the second step – developing an idea and devising a plan for solving the problem, the students should understand that the problem requires an answer about how many more sparrows there are in the park as opposed to the nest, which means that they need to understand that this problem is solved with subtraction, hence, in this case the plan of solving is specifying the known structure and method and their application.

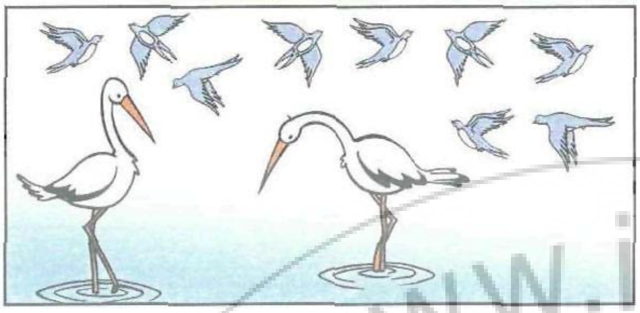
During the third step - practical realization of the plan for solving the problem, we actually perform the two previous steps, and the work with students of this age has shown that this is best done with the following scheme:

What facts are given in the problem?	There are 8 sparrows in the park. There are five sparrows in the nest.
What facts are not given in the problem?	How many more sparrows are there in the park?
How to find the facts which are not given?	By using the operation subtraction.
How to record the calculations?	8-5
How to calculate?	8-5=3
How to write the answer to the question?	There are 3 more sparrows in the park than in the nest.

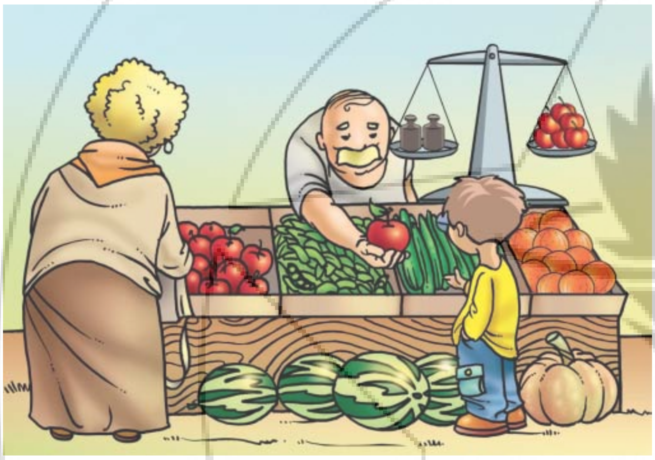
During the fourth step - additional work after the problem is solved, the students first need to check, i.e. calculate that the 5 sparrows in the nest and the 3 more in the park add up to 8, 5+3=8, which is correct. Further on, under the supervision of the teacher, the students need to see that with the information given in the problem, they can calculate the total number of sparrows 8+5=13 sparrows. ■

It is desirable to repeat the procedure for solving this type of problems in the further work, i.e. for example, to solve the following problems.

Example 2 (second grade). Two storks were hunting frogs in the nearby pond. Nine swallows have landed to collect mud for their nests. How many more swallows were there than storks?



Example 3 (second grade). Mr. Marko sold a 10 kg watermelon to Mrs. Maria, and 2 kg of apples to Boban. How many more kilograms of fruit did Mr. Marko sell to Mrs. Maria?



Further on, after the teacher has concluded that the students have successfully learned the methodology of solving problems, the simplified table for solving problems given in the example below, can be used.

Example 4 (second grade). Maya bought two types of cookies for the birthday party: circle-shaped and heart-shaped. She placed the cookies on two plates: she placed 3 circle-shaped and 9 heart-shaped cookies on the first plate, and 3 circle-shaped and 5 heart-shaped cookies on the second plate. Which plate contained more cookies and how many more?

In order to solve this problem it is necessary to fully perform the first two steps, but we can use the following table during the third step:

What are the facts?	There were 3 circle-shaped and 9 heart-shaped cookies on the first plate There were 3 circle-shaped and 5 heart-shaped cookies on the second plate
What is the unknown?	How many cookies were on each plate? Which plate had more cookies and how many more?
Calculation:	3+9=12, 3+5=8 and 12-8=4
Answer:	The first plate contains 4 more cookies.

Furthermore, the fourth step should be fully realized regarding these and other, similar examples, since in this case, other requirements can be made.

Further on, several examples need to be solved with the use of the simplified table. It is desirable to gradually avoid illustrations, by substituting them with adequate simplified representations at first, all in order to reduce the influence of the visual component. Namely, according to our observation of two different groups of students, the constant use of illustrations, in one of the groups, proved to be an obstacle in the development of the thinking properties, and especially of the elasticity and depth of thinking.

Regarding the learning of the methodology of solving word problems, which are predominant in third, fourth and fifth grade, there are no implications regarding the previously stated. This is why, here we will only mention that it is necessary to repeat all four steps in detail, from time to time, while the main focus should be placed on the use of the simplified table, in order to bring the methodology of solving problems to the level of an algorithm procedure. Further on, it is especially important that the students learn nonstandard algorithms for problem solving in their initial education, to develop the thinking properties and find more ways to solve one problem. The latter, for example, can be achieved by solving problems like this.

Example 5 (fifth grade). Several passengers got on the bus at the first bus station. Half of them got off the bus at the second station and two other got on the bus. Half of the passengers got off the bus at the third station, while three new got on. At the fourth station, half of the passengers got off the bus again, and four more got on and at this time the number of passengers was seven. How many passengers did get on the bus at the first station?

A small number of the students which were assigned with this problem, solved it with the method of algebra analysis, i.e. offered the following solution, which we only edited:
We will mark the number of passengers who got on the bus at the first station with x. *Then, after the second station we will have* $x:2+2$ *passengers, and after the third* $(x:2+2):2+3$ *passengers. According to this, after the fourth station we will have*

$$[(x:2+2):2+3]:2+4$$

passengers, which gives the equation

$$[(x:2+2):2+3]:2+4=7,$$

with the solution $x=8$.

None of the students offered a solution which directly used the analytical method which is acceptable even for students from third and fourth grade. The solution is as follows:

At the fourth station, 4 passengers got on, and since afterwards there were 7 remaining, we conclude that before these passengers got on the bus, there were 3 passengers and this is half of the number of passengers who traveled between the third and fourth station, which means that the number of passengers is 6.

According to this, we have $7 \to 3 \to 6$.

Half of the passengers got off the bus at the third station, and three more got on it. Since there were 6 passengers, we get $6 \to 3 \to 6$. *This means that the number of passengers before the third station was 6.*

Half of the passengers got off the bus at the second station, and two more got on, hence, $6 \to 4 \to 8$. *Finally, the number of passengers who traveled between the first and the second station is 8, which means that 8 passengers got on at the first station.*

We can write the previously mentioned with the help of the following scheme

$$7 \to 3 \to 6 \to 3 \to 6 \to 4 \to 8.$$

Nevertheless, after the previous algorithm was elaborated with the students, almost half of them used it to solve problems of the mentioned type without difficulties. Naturally, when learning the mentioned algorithm, special attention was given to the steps for solving mathematical problems.

3. Conclusion

In this paper we addressed the methodology of solving problems and we analyzed how it is learned by the students from initial education. We analyzed the functions of the problems and the steps for solving mathematical problems, for which we gave two separate examples. From the analyses alone, as well as the activities with the students of this age, we may conclude that:

- The complete understanding of the methodology of solving word problems decreases the number of errors made by the students while solving word and other problems,
- The illustrative component is especially important in the initial education, however the presence of illustrations needs to be decreased gradually, both in the formulations of the problems and in the solving process. At first they will be substituted with adequate simplified representations, which will directly help to develop the thinking properties, and especially the elasticity and depth of thinking,
- The learning of the methodology of solving problems has to include problems with different didactic value (algorithm, semi-algorithm and heuristic; problems for learning, exercising, etc.). The teachers need to stimulate discovering different ways to solve one given problem, a procedure which develops the thinking properties of the students and directly helps the students to move from the area of current to the area of future development.

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