International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

Some Interesting Properties of a Subclass of Meromorphic Univalent Functions Defined by Hadamard Product

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Abstract: ***In the present paper, we define a new subclass $SM_u(\tau,\gamma,\lambda,\alpha)$ of meromorphic univalent with positive coefficients defined by Hadamard product in the punctured unit disk U^*. We obtain some interesting properties, like, coefficient estimates, extreme points, distortion theorem, partial sums, integral representation.***

2014 Mathematic Subject Classification: 30C45

Keywords: Meromorphic univalent function, Hadamard product, distortion theorem, integral representation.

1. Introduction

Let M_u denote the class of functions of the form:

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n , \qquad (1)$$

which are analytic and meromorphic univalent in the punctured unit disk

$U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

Let SM_u be a subclass of M_u consisting of functions of the from:

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \ ,(a_n \geq 0). \qquad (2)$$

For the function $f \in SM_u$ given by (2) and $g \in SM_u$ defined by

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \ ,(b_n \geq 0), \qquad (3)$$

the convolution (or Hadamard product) of f and g is defined by

$$(f * g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n . \qquad (4)$$

We shall need to state the extended linear derivative operator of Ruscheweyh type for the function belong to the class SM_u

$D_*^{\lambda,1}: SM_u \to SM_u$

is defined by the following convolution:

$$D_*^{\lambda,1} f(z) = \frac{z^{-1}}{(1-z)^{\lambda+1}} * f(z)\,,(\lambda > -1\,; f \in SM_1). \ (5)$$

In terms of binomial coefficients, (5) can be written as

$$D_*^{\lambda,1} f(z) = z^{-1} + \sum_{n=1}^{\infty} \binom{\lambda+n}{n} a_n z^n \ (\lambda > -1\,; f \in SM_u). \ (6)$$

The linear operator $D^{\lambda,1}$ analogous to $D_*^{\lambda,1}$ was consider recently by Raina and Srivastava [7] on the space of analytic and p-valent functions in U($U = U^* \cup \{0\}$).

A function $f \in M_u$ is said to be in the class $M_u S$ of meromorphic univalent starlike function of order α if :

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\,, (z \in U^*, 0 \leq \alpha < 1). \quad (7)$$

A function $f \in M_u$ is said to be in the class $M_u C$ of meromorphic univalent convex function of order α if :

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,, (z \in U^*, 0 \leq \alpha < 1). \ (8)$$

Definition 1: A function $f \in SM_u$ is said to be in the class of $SM_u(\tau,\gamma,\lambda,\alpha)$ if it satisfies the following condition:

$$\left| \frac{\frac{z\tau}{2}\frac{(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))} + \frac{\tau}{2}}{\alpha(1+3\gamma) + \frac{2z\alpha\gamma(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))}} \right| < 1, (9)$$

for $0 < \gamma < 1, 0 < \tau < 1, 0 < \alpha < 1$.

Atshan and Kulkarni [3], Atshan and Buti [1], Atshan and Joudah [2], Dziok et al. [4], Khairnar and More [5] and Najafzadeh and Ebadian [6] studied meromorphic univalent and Multivalent functions for different classes.

2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for a function f to be in the class $SM_u(\tau,\gamma,\lambda,\alpha)$.

Theorem 1: Let $f \in SM_u$ .Then $f \in SM_u(\tau,\gamma,\lambda,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \binom{\lambda+n}{n} \left(\frac{\tau}{2}(n+1) + \gamma + \alpha(1+\gamma(3+2n)\right) a_n b_n \leq \alpha(1+\gamma), (10)$$

where $0 < \gamma < 1, 0 < \tau < 1, 0 < \alpha < 1$.

The result is sharp for the function

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))b_n} z^n, n \in N.$$

Proof: Suppose that the inequality (10) holds true and $|z| = 1$. Then from (9), we have

$$\left|\frac{z\tau}{2}\left(D_*^{\lambda,1}(f*g)(z)\right)' + \frac{\tau}{2}\left(D_*^{\lambda,1}(f*g)(z)\right)\right| - \left|\alpha(1+\gamma)\left(D_*^{\lambda,1}(f*g)(z)\right) + \gamma(D_*^{\lambda,1}(f*g)(z)) + z\alpha\gamma\left(D_*^{\lambda,1}(f*g)(z)\right)'\right|$$

$$= \left|\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n\right| - \left|\alpha(1+3\gamma)(z^{-1}+\sum_{n=1}^{\infty}\binom{\lambda+n}{n}a_nb_nz^n) + 2z\alpha\gamma\,(-z^{-2} + \sum_{n=1}^{\infty}n\binom{\lambda+n}{n}a_nb_nz^{n-1})\right|$$

$$= \left|\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n\right| - \left|\alpha(1+3\gamma)z^{-1} + \alpha(1+3\gamma)\sum_{n=1}^{\infty}\binom{\lambda+n}{n}a_nb_nz^n - 2\alpha\gamma z^{-1} + \sum_{n=1}^{\infty}2\alpha\gamma n\binom{\lambda+n}{n}a_nb_nz^n\right|$$

$$= \left|\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n\right| - \left|\alpha(1+\gamma)z^{-1} + \sum_{n=1}^{\infty}\alpha(1+\gamma(3+2n))\binom{\lambda+n}{n}a_nb_nz^n\right| \quad (11)$$

$$\le \sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_n - \alpha(1+\gamma) + \sum_{n=1}^{\infty}\alpha(1+\gamma(3+2n))\binom{\lambda+n}{n}a_nb_n$$

$$= \sum_{n=1}^{\infty}\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))\right)a_nb_n - \alpha(1+\gamma) \le 0,$$

by hypothesis.

Hence, by maximum modulus principle, $f \in SM_u(\tau,\gamma,\lambda,\alpha)$

Conversely, assume that defined by (2) is un the class $SM_u(\tau,\gamma,\lambda,\alpha)$.

Then from (9), we have

$$\left|\frac{\frac{z\tau}{2}(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))} + \frac{\tau}{2} \Big/ \left(\alpha(1+3\gamma) + \frac{2z\alpha\gamma(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))}\right)\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n}{(\alpha+\alpha\gamma)z^{-1}+\sum_{n=1}^{\infty}\alpha(1+\gamma(3+2n))\binom{\lambda+n}{n}a_nb_nz^n}\right| < 1.$$

Since Re (z) ≤ │z│ for all z (z ∈ U*), we get

$$\operatorname{Re}\left\{\frac{\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n}{(\alpha+\alpha\gamma)z^{-1}+\sum_{n=1}^{\infty}\alpha(1+\gamma(3+2n))\binom{\lambda+n}{n}a_nb_nz^n}\right\} < 1. \quad (12)$$

We choose the value of z on the real axis so that $\frac{z(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))}$ *is real.*

Let z → 1^- through real values, so we can write (12) as

$$\sum_{n=1}^{\infty}\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))\right)a_nb_n \le \alpha(1+\gamma).$$

Finally, sharpness follows if we take

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{\left(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))\right)b_n\binom{\lambda+n}{n}}z^n, n = 1,2,\ldots. \quad (13)$$

The proof is complete.

Corollary 1: Let $f \in SM_u(\tau,\gamma,\lambda,\alpha)$. Then

$$a_n \le \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))b_n}, n = 2,3,\ldots. \quad (14)$$

3. Distortion Bounds

Next, we obtain the growth and distortion bounds for the class $SM_u(\tau,\gamma,\lambda,\alpha)$.

Theorem 2: If $f \in SM_u(\tau,\gamma,\lambda,\alpha)$ and $b_n \ge b_1 (n \ge 1)$, then

$$\frac{1}{r} - \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}r \le |f(z)| \le \frac{1}{r} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}r, (|z| = r < 1),$$

and

$$\frac{1}{r^2} - \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}r \le |f'(z)| \le \frac{1}{r^2} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}r, (|z| = r < 1).$$

The result is sharp for the function:

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1} z. (15)$$

Proof: Since

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n ,$$

then

$$|f(z)| = \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right|$$

$$\leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n = \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n$$

$$\leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n \quad (16)$$

Since for $n \geq 1$,

$$(1+\lambda)(\tau+\alpha(1+5\gamma))b_1 \leq \binom{\lambda+n}{n} (\frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)))b_n.$$

By Theorem (1), we have

$$(1+\lambda)\big(\tau+\alpha(1+5\gamma)\big)b_1 \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \binom{\lambda+n}{n} \left(\frac{\tau}{2}(n+1) + \alpha\big(1+\gamma(3+2n)\big)\right) a_n\, b_n.$$

$\leq \alpha(1+\gamma)$.

That is

$$\sum_{n=1}^{\infty} a_n \leq \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}.$$

Using the above inequality in (16), we have

$$|f(z)| \leq \frac{1}{r} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1} r,$$

and

$$|f(z)| \geq \frac{1}{r} - \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1} r.$$

The result is sharp for the function:

$$f(z) = \frac{1}{z} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1} z,$$

Similarly, we have

$$|f'(z)| \geq \frac{1}{r^2} - \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1},$$

and

$$|f'(z)| \leq \frac{1}{r^2} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1},$$

4. Partial Sums

Theorem 3: Let $f \in SM_u$ be given by (2) and the partial sums $S_1(z)$ and $S_k(z)$ be defined by $S_1(z) = z^{-1}$ and

$$S_k(z) = z^{-1} + \sum_{n=1}^{k-1} a_n z^n \ ,(k>1).$$

Also, suppose that

$$\sum_{n=1}^{\infty} d_n a_n \leq 1, \left(d_n = \frac{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)))}{\alpha(1+\gamma)} \right). (17)$$

Then, we have

$$Re\left\{\frac{f(z)}{s_k(z)}\right\} > 1 - \frac{1}{d_k}, (18)$$

and

$$Re\left\{\frac{s_k(z)}{f(z)}\right\} > \frac{d_k}{1+d_k} \ (z \in U, k>1). (19)$$

Each of the bounds in (18) and (19) is the best possible for n∈ ℕ.

Proof: For the coefficients d_n given by (17), it is not difficult to verify that

$$d_{n+1} > d_n > 1, n = 1,2, \dots .$$

Therefore, by using the hypothesis (17), we have

$$\sum_{n=1}^{k-1} a_n + d_k \sum_{n=k}^{\infty} a_n \leq \sum_{n=1}^{\infty} d_n a_n \leq 1. (20)$$

By setting

$$g_1(z) = d_k \left(\frac{f(z)}{s_k(z)} - \left(1 - \frac{1}{d_k}\right) \right) = 1 + \frac{d_k \sum_{n=k}^{\infty} a_n\, z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n\, z^{n+1}} \ (21)$$

and applying (20),we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_k \sum_{n=k}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n - d_k \sum_{n=k}^{\infty} a_n} \leq 1, (22)$$

which readily yields the assertion (18), if we take

$$f(z) = z^{-1} - \frac{z^k}{d_k} \ ,(23)$$

then

$$\frac{f(z)}{s_k(z)} = 1 - \frac{z^k}{d_k} \to 1 - \frac{1}{d_k} \ (z \to 1^-),$$

which shows that the bound (18) is the best possible for each n∈ N.

Similarly, if we take

$$g_2(z) = (1+d_k)\left(\frac{s_k(z)}{f(z)} - \frac{d_k}{1+d_k} \right) = 1 + \frac{(1+d_k)\sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n z^{n+1}},$$

and make use of (20), we obtain

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1+d_k)\sum_{n=k}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n + (1-d_k)\sum_{n=k}^{\infty} a_n} \leq 1 \ ,(24)$$

which leads us to the assertion (19). The bounds in (18) and (19) is sharp with the function given by (21).

Theorem 4: If f(z) of the form (2) satisfy the condition (10). Then

$$Re\left\{\frac{f'(z)}{s'_k(z)}\right\} > 1 - \frac{k+1}{d_{k+1}}$$

$$Re\left\{\frac{f(z)}{s'_k(z)}\right\} > \frac{d_{m+1}}{k+1+d_{k+1}},$$

where

$$d_n \geq \begin{cases} n \ for\ n = 2,3,\dots,m \\ \dfrac{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))b_n}{\alpha(1+\gamma)} \ for\ n \\ = m+1, m+2, m+3, \dots. \end{cases}$$

The bounds are sharp, with the extremal function f(z) of the form (15)

Proof: The proof is analogous to that Theorem 3, and we omit details.

5. Integral Representation

Theorem 5: Let $f \in SM_u(\tau,\gamma,\lambda,\alpha)$. Then

$$D_*^{\lambda,1}(f*g)(z) = exp\int_0^z \frac{\vartheta(z)\alpha(1+3\gamma)-\frac{\tau}{2}}{\frac{\tau}{2}-\vartheta(z)2\alpha\gamma}dt,$$

where $|\vartheta(z)|<1, z\in U$.

Proof: By putting

$$\frac{z(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))} = Q(z)$$

in (9), we have

$$\left|\frac{\frac{\tau}{2}Q(z)+\frac{\tau}{2}}{\alpha(1+3\gamma)+2\alpha\gamma Q(z)}\right| < 1,$$

or equivalently

$$\frac{\frac{\tau}{2}Q(z)+\frac{\tau}{2}}{\alpha(1+3\gamma)+2\alpha\gamma Q(z)} = \vartheta(z), (|\vartheta(z)|<1, z\in U).$$

So

$$\frac{(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))} = \frac{\vartheta(z)\alpha(1+3\gamma)-\frac{\tau}{2}}{z(\frac{\tau}{2}-\vartheta(z)2\alpha\gamma)},$$

after integration , we have

$$log(D_*^{\lambda,1}(f*g)(z)) = \int_0^z \frac{\vartheta(z)(\alpha(1+3\gamma))-\frac{\tau}{2}}{\frac{\tau}{2}-\vartheta(z)2\alpha\gamma}dt.$$

Therefore

$$D_*^{\lambda,1}(f*g)(z) = exp\int_0^z \frac{\vartheta(z)\alpha(1+3\gamma)-\frac{\tau}{2}}{\frac{\tau}{2}-\vartheta(z)2\alpha\gamma}dt,$$

and this gives the required result.

6. Extreme Points

Theorem 6: Let $f_0(z)=z^{-1}$ and

$$f_n(z) = z^{-1} + \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))b_n}z^n, (n \geq 1). (25)$$

Then $f\in SM_u(\tau,\gamma,\lambda,\alpha)$, if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty}\mu_n f_n(z), (\mu_n \geq 0, \sum_{n=0}^{\infty}\mu_n = 1). (26)$$

Proof: Suppose that

$$f(z) = \sum_{n=0}^{\infty}\mu_n f_n(z), \text{where } \mu_n \geq 0, \sum_{n=0}^{\infty}\mu_n = 1.$$

Then

$$f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty}\mu_n f_n(z)$$

$$f(z) = z^{-1} + \sum_{n=1}^{\infty}\frac{\alpha(1+\gamma)\mu_n}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))b_n}z^n = z^{-1} + \sum_{n=1}^{\infty}\ell_n z^n,$$

where

$$\ell_n = \frac{\alpha(1+\gamma)\mu_n}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}.$$

Therefore

$$\sum_{n=1}^{\infty}\mu_n \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))\right)b_n}. \frac{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}{\alpha(1+\gamma)} = \sum_{n=1}^{\infty}\mu_n = 1-\mu_0 \leq 1.$$

So by Theorem (1), $f\in SM_u(\tau,\gamma,\lambda,\alpha)$.
Conversely, we suppose $f\in SM_u(\tau,\gamma,\lambda,\alpha)$. By (4), we have

$$a_n \leq \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}, n\geq 1.$$

We set,

$$\mu_n = \frac{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}{\alpha(1+\gamma)}a_n, n\geq 1,$$

and

$$\mu_0 = 1-\sum_{n=1}^{\infty}\mu_n.$$

Then, we have

$$f(z) = \sum_{n=0}^{\infty}\mu_n f_n(z)$$

$$f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty}\mu_n f_n(z).$$

Hence the results follows.

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