Some Interesting Properties of a Subclass of Meromorphic Univalent Functions Defined by Hadamard Product

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Abstract: ***In the present paper, we define a new subclass $SM_u(\tau,\gamma,\lambda,\alpha)$ of meromorphic univalent with positive coefficients defined by Hadamard product in the punctured unit disk U^*. We obtain some interesting properties, like, coefficient estimates, extreme points, distortion theorem, partial sums, integral representation.***

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1. Introduction

Let M_u denote the class of functions of the form:

$$f(z)=z^{-1}+\sum_{n=1}^{\infty}a_nz^n, \qquad (1)$$

which are analytic and meromorphic univalent in the punctured unit disk
$U^*=\{z\in\mathbb{C}:0<|z|<1\}$.
Let SM_u be a subclass of M_u consisting of functions of the from:

$$f(z)=z^{-1}+\sum_{n=1}^{\infty}a_nz^n\ ,(a_n\geq 0). \qquad (2)$$

For the function $f\in SM_u$ given by (2) and $g\in SM_u$ defined by

$$g(z)=z^{-1}+\sum_{n=1}^{\infty}b_nz^n\ ,(b_n\geq 0), \qquad (3)$$

the convolution (or Hadamard product) of f and g is defined by

$$(f*g)(z)=z^{-1}+\sum_{n=1}^{\infty}a_nb_nz^n. \qquad (4)$$

We shall need to state the extended linear derivative operator of Ruscheweyh type for the function belong to the class SM_u
$D_*^{\lambda,1}:SM_u\to SM_u$
is defined by the following convolution:

$$D_*^{\lambda,1}f(z)=\frac{z^{-1}}{(1-z)^{\lambda+1}}*f(z)\ ,(\lambda>-1\ ;f\in SM_1). \qquad (5)$$

In terms of binomial coefficients, (5) can be written as

$$D_*^{\lambda,1}f(z)=z^{-1}+\sum_{n=1}^{\infty}\binom{\lambda+n}{n}a_nz^n\ (\lambda>-1\ ;f\in SM_u). \qquad (6)$$

The linear operator $D^{\lambda,1}$ analogous to $D_*^{\lambda,1}$ was consider recently by Raina and Srivastava [7] on the space of analytic and p-valent functions in U($U=U^*\cup\{0\}$).

A function $f\in M_u$ is said to be in the class M_uS of meromorphic univalent starlike function of order α if :

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\ ,(z\in U^*,0\leq\alpha<1). \qquad (7)$$

A function $f\in M_u$ is said to be in the class M_uC of meromorphic univalent convex function of order α if :

$$-Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\ ,(z\in U^*,0\leq\alpha<1). \qquad (8)$$

Definition 1: A function $f\in SM_u$ is said to be in the class of $SM_u(\tau,\gamma,\lambda,\alpha)$ if it satisfies the following condition:

$$\left|\frac{\dfrac{\dfrac{z\tau}{2}(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))}+\dfrac{\tau}{2}}{\alpha(1+3\gamma)+\dfrac{2z\alpha\gamma(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))}}\right|<1, \qquad (9)$$

for $0<\gamma<1, 0<\tau<1, 0<\alpha<1$.
Atshan and Kulkarni [3], Atshan and Buti [1], Atshan and Joudah [2], Dziok et al. [4], Khairnar and More [5] and Najafzadeh and Ebadian [6] studied meromorphic univalent and Multivalent functions for different classes.

2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for a function f to be in the class $SM_u(\tau,\gamma,\lambda,\alpha)$.

Theorem 1: Let $f\in SM_u$ .Then $f\in SM_u(\tau,\gamma,\lambda,\alpha)$ if and only if

$$\sum_{n=1}^{\infty}\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\gamma+\alpha(1+\gamma(3+2n)\right)a_nb_n\leq\alpha(1+\gamma), \qquad (10)$$

where $0<\gamma<1, 0<\tau<1, 0<\alpha<1$.

The result is sharp for the function

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)\right)b_n} z^n, n \in N.$$

Proof: Suppose that the inequality (10) holds true and $|z| = 1$. Then from (9), we have

$$\left|\frac{z\tau}{2}\left(D_*^{\lambda,1}(f*g)(z)\right)' + \frac{\tau}{2}\left(D_*^{\lambda,1}(f*g)(z)\right)\right| - \left|\alpha(1+\gamma)\left(D_*^{\lambda,1}(f*g)(z)\right) + \gamma(D_*^{\lambda,1}(f*g)(z)) + z\alpha\gamma\left(D_*^{\lambda,1}(f*g)(z)\right)'\right|$$

$$= \left|\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n\right| - \left|\alpha(1+3\gamma)(z^{-1} + \sum_{n=1}^{\infty}\binom{\lambda+n}{n}a_nb_nz^n) + 2z\alpha\gamma\,(-z^{-2} + \sum_{n=1}^{\infty}n\binom{\lambda+n}{n}a_nb_nz^{n-1})\right|$$

$$= \left|\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n\right| - \left|\alpha(1+3\gamma)z^{-1} + \alpha(1+3\gamma)\sum_{n=1}^{\infty}\binom{\lambda+n}{n}a_nb_nz^n - 2\alpha\gamma z^{-1} + \sum_{n=1}^{\infty}2\alpha\gamma n\binom{\lambda+n}{n}a_nb_nz^n\right|$$

$$= \left|\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n\right| - \left|\alpha(1+\gamma)z^{-1} + \sum_{n=1}^{\infty}\alpha(1+\gamma(3+2n))\binom{\lambda+n}{n}a_nb_nz^n\right| \quad (11)$$

$$\le \sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_n - \alpha(1+\gamma) + \sum_{n=1}^{\infty}\alpha(1+\gamma(3+2n))\binom{\lambda+n}{n}a_n b_n$$

$$= \sum_{n=1}^{\infty}\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))\right)a_n b_n - \alpha(1+\gamma) \le 0,$$

by hypothesis.

Hence, by maximum modulus principle, $f \in SM_u(\tau,\gamma,\lambda,\alpha)$

Conversely, assume that defined by (2) is un the class $SM_u(\tau,\gamma,\lambda,\alpha)$.

Then from (9), we have

$$\left|\frac{\frac{z\tau}{2}(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))} + \frac{\tau}{2} \over \alpha(1+3\gamma) + \frac{2z\alpha\gamma(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n}{(\alpha+\alpha\gamma)z^{-1} + \sum_{n=1}^{\infty}\alpha\left(1+\gamma(3+2n)\right)\binom{\lambda+n}{n}a_nb_nz^n}\right| < 1.$$

Since Re (z) ≤ | z | for all z (z ∈ U*), we get

$$\text{Re}\left\{\frac{\sum_{n=1}^{\infty}\frac{\tau}{2}(n+1)\binom{\lambda+n}{n}a_nb_nz^n}{(\alpha+\alpha\gamma)z^{-1} + \sum_{n=1}^{\infty}\alpha\left(1+\gamma(3+2n)\right)\binom{\lambda+n}{n}a_nb_nz^n}\right\} < 1. \quad (12)$$

We choose the value of z on the real axis so that $\frac{z(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))}$ *is real.*

Let z → 1⁻ through real values, so we can write (12) as

$$\sum_{n=1}^{\infty}\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))\right)a_nb_n \le \alpha(1+\gamma).$$

Finally, sharpness follows if we take

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{\left(\frac{\tau}{2}(n+1)+\alpha\left(1+\gamma(3+2n)\right)\right)b_n\binom{\lambda+n}{n}}z^n, n = 1,2,\dots. \quad (13)$$

The proof is complete.

Corollary 1: Let $f \in SM_u(\tau,\gamma,\lambda,\alpha)$. Then

$$a_n \le \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\alpha\left(1+\gamma(3+2n)\right)\right)b_n}, n = 2,3,\dots. \quad (14)$$

3. Distortion Bounds

Next, we obtain the growth and distortion bounds for the class $SM_u(\tau,\gamma,\lambda,\alpha)$.

Theorem 2: If $f \in SM_u(\tau,\gamma,\lambda,\alpha)$ and $b_n \ge b_1 (n \ge 1)$, then

$$\frac{1}{r} - \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}r \le |f(z)| \le \frac{1}{r} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}r, (|z| = r < 1),$$

and

$$\frac{1}{r^2} - \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}r \le |f'(z)| \le \frac{1}{r^2} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}r, (|z| = r < 1).$$

The result is sharp for the function:

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}z. (15)$$

Proof: Since

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n ,$$

then

$$|f(z)| = \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n\right|$$

$$\le \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n = \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n$$

$$\le \frac{1}{r} + r\sum_{n=1}^{\infty} a_n \ (16)$$

Since for n ≥ 1,

$$(1+\lambda)(\tau+\alpha(1+5\gamma))b_1 \le \binom{\lambda+n}{n}(\frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)))b_n.$$

By Theorem (1), we have

$$(1+\lambda)(\tau+\alpha(1+5\gamma))b_1 \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} \binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n))\right) a_n b_n.$$

$$\le \alpha(1+\gamma).$$

That is

$$\sum_{n=1}^{\infty} a_n \le \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1}.$$

Using the above inequality in (16), we have

$$|f(z)| \le \frac{1}{r} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1} r,$$

and

$$|f(z)| \ge \frac{1}{r} - \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1} r.$$

The result is sharp for the function:

$$f(z) = \frac{1}{z} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1} z,$$

Similarly, we have

$$|f'(z)| \ge \frac{1}{r^2} - \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1},$$

and

$$|f'(z)| \le \frac{1}{r^2} + \frac{\alpha(1+\gamma)}{(1+\lambda)(\tau+\alpha(1+5\gamma))b_1},$$

4. Partial Sums

Theorem 3: Let $f \in SM_u$ be given by (2) and the partial sums $S_1(z)$ and $S_k(z)$ be defined by $S_1(z) = z^{-1}$ and

$$S_k(z) = z^{-1} + \sum_{n=1}^{k-1} a_n z^n , (k > 1).$$

Also, suppose that

$$\sum_{n=1}^{\infty} d_n a_n \le 1, \left(d_n = \frac{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1) + \alpha(1+\gamma(3+2n)))}{\alpha(1+\gamma)} \right). (17)$$

Then, we have

$$Re\left\{\frac{f(z)}{s_k(z)}\right\} > 1 - \frac{1}{d_k}, (18)$$

and

$$Re\left\{\frac{s_k(z)}{f(z)}\right\} > \frac{d_k}{1+d_k} \ (z \in U, k > 1). (19)$$

Each of the bounds in (18) and (19) is the best possible for n∈ ℕ.

Proof: For the coefficients d_n given by (17), it is not difficult to verify that

$$d_{n+1} > d_n > 1, n = 1,2, \dots .$$

Therefore, by using the hypothesis (17), we have

$$\sum_{n=1}^{k-1} a_n + d_k \sum_{n=k}^{\infty} a_n \le \sum_{n=1}^{\infty} d_n a_n \le 1. (20)$$

By setting

$$g_1(z) = d_k\left(\frac{f(z)}{s_k(z)} - \left(1 - \frac{1}{d_k}\right)\right) = 1 + \frac{d_k \sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n z^{n+1}} \ (21)$$

and applying (20),we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{d_k \sum_{n=k}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n - d_k \sum_{n=k}^{\infty} a_n} \le 1, (22)$$

which readily yields the assertion (18), if we take

$$f(z) = z^{-1} - \frac{z^k}{d_k}, (23)$$

then

$$\frac{f(z)}{s_k(z)} = 1 - \frac{z^k}{d_k} \to 1 - \frac{1}{d_k} \ (z \to 1^-),$$

which shows that the bound (18) is the best possible for each n∈ N.

Similarly, if we take

$$g_2(z) = (1+d_k)\left(\frac{s_k(z)}{f(z)} - \frac{d_k}{1+d_k}\right) = 1 + \frac{(1+d_k)\sum_{n=k}^{\infty} a_n z^{n+1}}{1 + \sum_{n=1}^{k-1} a_n z^{n+1}},$$

and make use of (20), we obtain

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1+d_k)\sum_{n=k}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n + (1-d_k)\sum_{n=k}^{\infty} a_n} \le 1, (24)$$

which leads us to the assertion (19). The bounds in (18) and (19) is sharp with the function given by (21).

Theorem 4: If f(z) of the form (2) satisfy the condition (10). Then

$$Re\left\{\frac{f'(z)}{s'_k(z)}\right\} > 1 - \frac{k+1}{d_{k+1}}$$

$$Re\left\{\frac{f(z)}{s'_k(z)}\right\} > \frac{d_{m+1}}{k+1+d_{k+1}},$$

where

$$d_n \geq \begin{cases} n \ for \ n = 2,3,\dots,m \\ \dfrac{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))b_n}{\alpha(1+\gamma)} \ for \ n \\ \quad = m+1, m+2, m+3, \dots . \end{cases}$$

The bounds are sharp, with the extremal function f(z) of the form (15)

Proof: The proof is analogous to that Theorem 3, and we omit details.

5. Integral Representation

Theorem 5: Let $f \in SM_u(\tau,\gamma,\lambda,\alpha)$. Then

$$D_*^{\lambda,1}(f*g)(z) = exp\int_0^z \frac{\vartheta(z)\alpha(1+3\gamma)-\frac{\tau}{2}}{\frac{\tau}{2}-\vartheta(z)2\alpha\gamma}dt,$$

where $|\vartheta(z)|<1, z \in U$.

Proof: By putting

$$\frac{z(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))} = Q(z)$$

in (9), we have

$$\left|\frac{\frac{\tau}{2}Q(z)+\frac{\tau}{2}}{\alpha(1+3\gamma)+2\alpha\gamma Q(z)}\right| < 1,$$

or equivalently

$$\frac{\frac{\tau}{2}Q(z)+\frac{\tau}{2}}{\alpha(1+3\gamma)+2\alpha\gamma Q(z)} = \vartheta(z), (|\vartheta(z)|<1, z \in U).$$

So

$$\frac{(D_*^{\lambda,1}(f*g)(z))'}{(D_*^{\lambda,1}(f*g)(z))} = \frac{\vartheta(z)\alpha(1+3\gamma)-\frac{\tau}{2}}{z(\frac{\tau}{2}-\vartheta(z)2\alpha\gamma)},$$

after integration , we have

$$log(D_*^{\lambda,1}(f*g)(z)) = \int_0^z \frac{\vartheta(z)(\alpha(1+3\gamma))-\frac{\tau}{2}}{\frac{\tau}{2}-\vartheta(z)2\alpha\gamma}dt.$$

Therefore

$$D_*^{\lambda,1}(f*g)(z) = exp\int_0^z \frac{\vartheta(z)\alpha(1+3\gamma)-\frac{\tau}{2}}{\frac{\tau}{2}-\vartheta(z)2\alpha\gamma}dt,$$

and this gives the required result.

6. Extreme Points

Theorem 6: Let $f_0(z) = z^{-1}$ and

$$f_n(z) = z^{-1} + \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}z^n, (n \geq 1). (25)$$

Then $f \in SM_u(\tau,\gamma,\lambda,\alpha)$, if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty}\mu_n f_n(z), (\mu_n \geq 0, \sum_{n=0}^{\infty}\mu_n = 1). (26)$$

Proof: Suppose that

$$f(z) = \sum_{n=0}^{\infty}\mu_n f_n(z), \text{where } \mu_n \geq 0, \sum_{n=0}^{\infty}\mu_n = 1.$$

Then

$$f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty}\mu_n f_n(z)$$

$$f(z) = z^{-1} + \sum_{n=1}^{\infty}\frac{\alpha(1+\gamma)\mu_n}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))b_n}z^n$$

$$= z^{-1} + \sum_{n=1}^{\infty}\ell_n z^n,$$

where

$$\ell_n = \frac{\alpha(1+\gamma)\mu_n}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}.$$

Therefore

$$\sum_{n=1}^{\infty}\mu_n \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}\left(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n))\right)b_n}. \frac{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}{\alpha(1+\gamma)}$$

$$= \sum_{n=1}^{\infty}\mu_n = 1-\mu_0 \leq 1.$$

So by Theorem (1), $f \in SM_u(\tau,\gamma,\lambda,\alpha)$.

Conversely, we suppose $f \in SM_u(\tau,\gamma,\lambda,\alpha)$. By (4), we have

$$a_n \leq \frac{\alpha(1+\gamma)}{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}, n \geq 1.$$

We set,

$$\mu_n = \frac{\binom{\lambda+n}{n}(\frac{\tau}{2}(n+1)+\alpha(1+\gamma(3+2n)))b_n}{\alpha(1+\gamma)}a_n, n \geq 1,$$

and

$$\mu_0 = 1 - \sum_{n=1}^{\infty}\mu_n.$$

Then, we have

$$f(z) = \sum_{n=0}^{\infty}\mu_n f_n(z)$$

$$f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty}\mu_n f_n(z).$$

Hence the results follows.

References

[1] W. G. Atshan and R. H. Buti, Fractional calculus of a class of univalent functions with negative coefficients defined by Hadamard product with Rafid operator, European J. Pure Appl. Math., 4(2)(2011),162-173.

[2] W. G. Atshan and A. S. Joudah, Subclass of meromorphic univalent functions defined by Hadamard product with multiplier transformation, Int. Math. Forum, 6(46)(2011),2279-2292.

[3] W. G. Atshan and S. R. Kulkarni, On application of differential subordination for certain subclass of

meromorphically p-valent functions with positive coefficients defined by linear operator, J. Ineq. Pure Appl. Math., 10(2)(2009), Article 53, 11 pages.

[4] J. Dziok, G. Murugusundaramoorthy and J. Sokol, On certain class of meromorphic functions with positive coefficients, Acta Math. Scientia, 4(3213)(2012),1376-1390.

[5] S. M. Khairnar and M. More, On a class of meromorphic multivalent functions with negative coefficients defined by Ruschewyh derivative Int. Math. Forum, 3(22)(2008), 1087-1097.

[6] S. Najafzadeh and A. Ebadian, Convex family of meromorphically multivalent functions on connected sets, Math. Com. Mod., 57(2013), 301-305.

[7] R. K. Raina and H. M. Srivastava, Incluion and neighborhoods properties of some analytic and multivalnt functions, J. Inequal. Pure Appl. Math., 7(1)(2006), Article 5,1-6.

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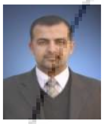

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