On a New Certain Subclass of Meromorphically p-valent Functions Defined by a Linear Operator

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Abstract: ***In the present paper, we have introduced a new class of meromorphically p-valent functions $\sum_p \ (\lambda,\mu,\eta,\alpha_1,g,s)$ defined by a linear operator $T_{p,g,s(\alpha_1)}$. We discuss some interesting properties, like, coefficient inequality , convex set, distortion bounds, neighborhoods of a function $f\in\sum_p$ and integral operator.***

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1. Introduction

Let $\sum_p$ denote the class of functions of the form:

$$f(z)=\frac{1}{z^p}+\sum_{n=p}^{\infty}a_n z^n \ ; \ (a_n\geq 0;\ p\in N=\{1,2,\dots\}),(1)$$

which are analytic and p- valent in the punctured unit disk

$$\mathrm{U}^*=\{\mathrm{z}:\mathrm{z}\ \in\mathbb{C}\text{ and }0<|z|<1\}=U\backslash\{0\}.$$

We define the Hadamard product (or Convolution) of f and g by

$$(f*g)(z)=\frac{1}{z^p}+\sum_{n=p}^{\infty}a_n b_n z^n=(g*f)(z),(2)$$

where f is given by (1) and g is defined as follows:

$$g\ (z)=\frac{1}{z^p}+\sum_{n=p}^{\infty}b_n z^n.$$

For positive real values of $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_s$ $(\beta_j\neq 0,-1,\dots;\ j=1,2,\dots,s)$,we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,\dots,\alpha_q\ ;\beta_1,\dots,\beta_s;\ z)$ by

$${}_qF_s(\alpha_1,\dots,\alpha_q\ ;\ \beta_1,\dots,\beta_s;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\dots(\alpha_q)_n}{(\beta_1)_n\dots(\beta_s)_n}\frac{z^n}{n!},(3)$$

$(\mathrm{q}\leq s+1;\ q,s\in N_0=N\cup\{0\};\ z\in U)$,

where $(\theta)_v$ is the Pochhammer symbol defined

$$(\theta)_v=\begin{cases}1 & v=0\\ \theta(\theta+1)(\theta+2)\dots(\theta+v-1), & v\in N.\end{cases}(4)$$

Corresponding to the function $h_p(\alpha_1,\dots,\alpha_q;\ \beta_1,\dots,\beta_s;z)$, defined by

$$h_p(\alpha_1,\dots,\alpha_q;\ \beta_1,\dots,\beta_s;z)=z^{-p}\ {}_qF_s(\alpha_1,\dots,\alpha_q;\ \beta_1,\dots,\beta_s;z);\ (5)$$

we consider a linear operator

$T_p(\alpha_1,\dots,\alpha_q;\ \beta_1,\dots,\beta_s):\sum_p \ \to\sum_p$,

which is defined by means of the following Hadamard product (or convolution):

$$T_p(\alpha_1,\dots,\alpha_q;\ \beta_1,\dots,\beta_s)f(z)=h_p(\alpha_1,\dots,\alpha_q;\ \beta_1,\dots,\beta_s;z)*f(z).(6)$$

We observe that, for a function $f(z)$ of the form (1); we have

$$T_p(\alpha_1,\dots,\alpha_q;\ \beta_1,\dots,\beta_s)\,f(z)=\frac{1}{z^p}+\sum_{n=p}^{\infty}\frac{(\alpha_1)_n\dots(\alpha_q)_n}{(\beta_1)_n\dots(\beta_s)_n}\frac{a_n}{n!}z^n.(7)$$

If, for convenience, we write

$$T_{p,q,s}(\alpha_1)=T_p(\alpha_1,\dots,\alpha_q;\ \beta_1,\dots,\beta_s),(8)$$

then one can easily verify from the definition (6) that

$$z\left(T_{p,q,s}(\alpha_1)f(z)\right)'=\alpha_1 T_{p,q,s}(\alpha_1+1)f(z)-(\alpha_1+p)T_{p,q,s}(\alpha_1)f(z).(9)$$

The linear operator $T_{p,q,s(\alpha_1)}$ was investigated recently by Liu and Srivastava [8].

Some interesting Subclasses of analytic functions associated with the generalized hypergeometric function were considered recently by (for example) Dziok and Srivastava ([3] and [4]), Gangadharan et al. [5] and Liu [7].

Definition 1: Let $\sum_p \ (\lambda,\mu,\eta,\alpha_1,q,s)$be denote the new class of functions $f\in\sum_p$, which satisfy the condition:

$$\left|\frac{\lambda(p+2)z^2\left(T_{p,q,s}(\alpha_1)f(z)\right)''+\lambda z^3\left(T_{p,q,s}(\alpha_1)f(z)\right)'''}{\mu z^2\left(T_{p,q,s}(\alpha_1)f(z)\right)''-(\mu-1)z\left(T_{p,q,s}(\alpha_1)f(z)\right)'}\right|<\eta,(10)$$

where $z\in U^*;\ 0\leq\eta<p;\ p\in N$ and for some suitably restricted real parameters λ, and μ.

Such type of study was carried out by several different authors for another classes, like, Nunokawa and Ahuja [9], Aouf and Hossen [1] and Cho et al. [2].

2. Coefficient Inequality

First, we derive the coefficient inequality for the class $\sum_p \ (\lambda,\mu,\eta,\alpha_1,q,s)$ contained in:

Theorem 1: Let $f \in \sum_p$ .Then f is in the class $\sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$ if and only if

$$\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2)+1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} \leq \eta p(\mu(p+1)-1), (11)$$

where $0 < \eta < p;\ \frac{1}{2} < \mu < \lambda < p;\ and\ p \in N.$

The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{n!\ \eta p(\mu(p+1)-1)}{n(\lambda(n-1)(n+p) - \eta(\mu(n-2)+1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}} z^n, n \geq p. (12)$$

Proof: Suppose that the inequality (11) holds true and $|z| = 1$. Then, we have

$$|\lambda(p+2)z^2(T_{p,q,s}(\alpha_1)f(z))'' + \lambda z^3(T_{p,q,s}(\alpha_1)f(z))'''| - \eta|(\mu-1)z(T_{p,q,s}(\alpha_1)f(z))' + \mu z^2(T_{p,q,s}(\alpha_1)f(z))''|$$

$$= \left|\sum_{n=p}^{\infty} \lambda n(n-1)(n+p) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} z^n\right| - \eta \left| p(\mu(p+2)-1)z^{-p} - \sum_{n=p}^{\infty} n(\mu(2-n)-1) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} z^n \right|$$

$$\leq \sum_{n=p}^{\infty} \lambda n(n-1)(n+p) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} |z|^n - \eta p(\mu(p+2)-1)|z|^{-p} + \sum_{n=p}^{\infty} \eta n(\mu(2-n)-1) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} |z|^n$$

$$= \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2)+1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} - \eta p(\mu(p+1)-1) \leq 0,$$

by hypothesis. Thus by maximum modulus principle, $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$.

To show the converse, suppose that $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$. Then (10), we have

$$\left|\frac{\lambda(p+2)z^2(T_{p,q,s}(\alpha_1)f(z))'' + \lambda z^3(T_{p,q,s}(\alpha_1)f(z))'''}{\mu z^2(T_{p,q,s}(\alpha_1)f(z))'' - (\mu-1)z(T_{p,q,s}(\alpha_1)f(z))'}\right| =$$

$$\left|\frac{\sum_{n=p}^{\infty} \lambda n(n-1)(n+p) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} z^n}{p(\mu(p+2)-1)z^{-p} - \sum_{n=p}^{\infty} n(\mu(2-n)-1) \frac{(\alpha_1)_n \dots (\alpha_g)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} z^n}\right| < \eta.$$

Since Re $(z) \leq |z|$ for all z $(z \in U)$, we have

$$Re\left\{\frac{\sum_{n=p}^{\infty} \lambda n(n-1)(n+p) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} z^n}{p(\mu(p+2)-1)z^{-p} - \sum_{n=p}^{\infty} n(\mu(2-n)-1) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} z^n}\right\} < \eta. (13)$$

We choose the value of z on the real axis so that $z(T_{p,q,s}(\alpha_1)f(z))'$ is real.

Upon clearing the denominator of (13) and letting $z \to 1^-$, through real values so we can write (13) as

$$\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2)+1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} \leq \eta p(\mu(p+1)-1).$$

Sharpness of the result follows by setting

$$f(z) = \frac{1}{z^p} + \frac{n!\ \eta p(\mu(p+1)-1)}{n(\lambda(n-1)(n+p) - \eta(\mu(n-2)+1)) \frac{(\alpha_1)_n \dots (\alpha_g)_n}{(\beta_1)_n \dots (\beta_s)_n}} z^n, (n \geq p).$$

Corollary 1: Let $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$.Then

$$a_n \leq \frac{\eta p(\mu(p+1)-1)n!}{n(\lambda(n-1)(n+p) - \eta(\mu(n-2)+1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}},$$

$(n \geq p)$.

3. Convex Set

In the following theorem, we will prove the class $\sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$ is convex set.

Theorem 2: The class $\sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$ is a convex set.

Proof: Let f_1 and f_2 be the arbitrary elements of $\sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$.

Then for every t $(0 \leq t \leq 1)$, we show that $(1-t)f_1 + tf_2 \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$.

Thus, we have

$$(1-t)f_1 + tf_2 = \frac{1}{z^p} + \sum_{n=p}^{\infty} [(1-t)a_n + tb_n]z^n.$$

Hence,

$$\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1))\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{[(1-t)a_n + tb_n]}{n!}$$

$$= (1-t)\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1))\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{a_n}{n!}$$

$$+t\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1))\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{b_n}{n!}$$

$$\leq (1-t)\eta(\mu(p+1) - 1) + t\eta p(\mu(p+1) - 1)$$
$$= \eta p(\mu(p+1) - 1).$$

This completes the proof.

4. 4.Distortion Bounds

In the following theorems, we obtain the growth and distortion bounds for the linear operator $T_{p,q,s}(\alpha_1)$.

Theorem 3: If $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$, then

$$\frac{1}{r^p} - \frac{\eta(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1)} r^p \leq |T_{p,q,s}(\alpha_1)f(z)| \leq \frac{1}{r^p} + \frac{\eta(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1)} r^p, (|z| = r < 1).$$

The result is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{\eta p(\mu(p+1) - 1)p!}{p(\lambda(p-1)2p - \eta(\mu(p-2) + 1))\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p}} z^p. (15)$$

Proof: Let $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$.Then by Theorem 1, we get

$$p(2\lambda p(p-1) - \eta(\mu(p-2) + 1))\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} \cdot \frac{1}{p!}\sum_{n=p}^{\infty} a_n$$

$$\leq \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1))\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{a_n}{n!} \leq \eta p(\mu(p+1) - 1),$$

or

$$\sum_{n=p}^{\infty} a_n \leq \frac{\eta p(\mu(p+1) - 1)p!}{p(2\lambda p(p-1) - \eta(\mu(p-2) + 1))\frac{(\alpha_1)_p \dots (\alpha_g)_p}{(\beta_1)_p \dots (\beta_s)_g}} \quad (16)$$

$$|T_{p,q,s}(\alpha_1)f(z)| \leq \frac{1}{|z|^p} + \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{a_n}{n!}|z|^n$$

$$\leq \frac{1}{|z|^p} + \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{|z|^p}{p!}\sum_{n=p}^{\infty} a_n$$

$$= \frac{1}{r^p} + \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} \cdot \frac{r^p}{p!}\sum_{n=p}^{\infty} a_n$$

$$\leq \frac{1}{r^p} + \frac{\eta(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1)} r^p. (17)$$

Similarly,

$$|T_{p,q,s}(\alpha_1)f(z)| \geq \frac{1}{|z|^p} - \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{a_n}{n!}|z|^n$$

$$\geq \frac{1}{|z|^p} - \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{|z|^p}{p!}\sum_{n=p}^{\infty} a_n$$

$$= \frac{1}{r^p} - \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} \cdot \frac{r^p}{p!}\sum_{n=p}^{\infty} a_n$$

$$\geq \frac{1}{r^p} - \frac{\eta(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1)} r^p. (18)$$

From (17) and (18), we get (14) and the proof is complete.

Theorem 4: If $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$, then

$$\frac{-p}{r^{p+1}} - \frac{\eta p(\mu(p+1)-1)}{(2\lambda p(p-1) - \eta(\mu(p-2)+1)} r^{p-1} \leq |(T_{p,q,s}(\alpha_1)f(z))'| \leq \frac{-p}{r^{p+1}} + \frac{\eta p(\mu(p+1)-1)}{(2\lambda p(p-1) - \eta(\mu(p-2)+1)} r^{p-1}, (|z| = r < 1). (19)$$

The result is sharp for the function $f(z)$ is given by (15).

Proof: The proof is similar to that of Theorem 3.

5. δ-Neighborhood of a function $f \in \sum_p$:

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruscheweyh [10], we begin by introducing here the δ-Neighborhood of a function $f \in \sum_p$ of the form (1) by means of the definition below:

$$N_\delta(f) = \left\{ g \in \sum_p : g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n \text{ and } \sum_{n=p}^{\infty} n|a_n - b_n| \leq \delta, 0 \leq \delta < 1 \right\}. (20)$$

Particularly for the identity function $(z) = \frac{1}{z^p}$, we have

$$N_\delta(e) = \left\{ g \in \sum_p : g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n \text{ and } \sum_{n=p}^{\infty} n|b_n| \le \delta \right\}. (21)$$

Theorem 5: If $g(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$ and

$$y = 1 - \frac{\delta(\lambda(p-1)2p - \eta(\mu(p-2)-1))\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p}}{p(\lambda(p-1)2p - \eta(\mu(p-2)+1))\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} - \eta(\mu(p+1)-1)p!}. (22)$$

Proof: Let $f(z) \in N_\delta(g)$. Then, we find from (20) that

$$\sum_{n=p}^{\infty} n|a_n - b_n| \le \delta,$$

which implies the coefficient inequality

$$\sum_{n=p}^{\infty} |a_n - b_n| \le \frac{\delta}{p}, (n \ge p).$$

Since $g(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$, then by using Theorem (1), we get

$$\sum_{n=p}^{\infty} b_n \le \frac{\eta(\mu(p+1)-1)p!}{(\lambda(p-1)2p - \eta(\mu(p-2)+1))\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p}},$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{n=p}^{\infty} |a_n - b_n|}{1 - \sum_{n=p}^{\infty} b_n} \le \frac{\delta}{p} \frac{\delta(\lambda(p-1)2p - \eta(\mu(p-2)-1))\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p}}{(\lambda(p-1)2p - \eta(\mu(p-2)+1))\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} - \eta(\mu(p+1)-1)p!} = 1 - y.$$

Hence, by Definition 2, $f(z) \in \sum_{p,y} (\lambda, \mu, \eta, \alpha_1, q, s)$ for y given by (22). This complete the proof.

Definition 2: A function $f(z) \in \sum_p$ is said to be in the class $\sum_{p,y} (\lambda, \mu, \eta, \alpha_1, q, s)$, if there exists function $g(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$, such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - y, (z \in y, 0 \le y < 1).$$

6. Radii of starlikeness and convexity:

In the following Theorems, we discuss the radii starlikeness and convexity.

Theorem 6: If $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$, then $f(z)$ is multivalent meromorphic starlike of order $\theta(0 \le \theta < p)$ in the disk $|z| < r_1$, where

$$r_1 = inf_n \left\{ \frac{(p-\theta)n(\lambda(n-1)(n+p) - \eta(\mu(n-2)+1))\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{(n+2p-\theta)\eta p(\mu(p+1)-1)n!} \right\}^{\frac{1}{n+p}}, n \ge p.$$

The result is sharp for the function f(z) is given by (12).

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} + p\right| \le p - \theta \text{ for } |z| < r_1. (23)$$

But

$$\left|\frac{zf'(z)+pf(z)}{f(z)}\right| = \left|\frac{\sum_{n=p}^{\infty}(n+p)a_n z^{n+p}}{1+\sum_{n=p}^{\infty} a_n z^{n+p}}\right| \le \frac{\sum_{n=p}^{\infty}(n+p)a_n|z|^{n+p}}{1-\sum_{n=p}^{\infty} a_n|z|^{n+p}}.$$

Thus, (23) will be satisfied if

$$\frac{\sum_{n=p}^{\infty}(n+p)a_n|z|^{n+p}}{1-\sum_{n=p}^{\infty} a_n|z|^{n+p}} \le p - \theta,$$

or if

$$\sum_{n=p}^{\infty} \frac{(n+2p-\theta)a_n}{p-\theta}|z|^{n+p} \le 1. (24)$$

Since $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, \mathcal{g}, s)$, we have

$$\sum_{n=p}^{\infty} \frac{n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p(\mu(p+1)-1)n!} a_n \le 1.$$

Hence, (24) will be true if

$$\frac{(n+2p-\theta)}{p-\theta}|z|^{n+p} \le \frac{n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p(\mu(p+1)-1)n!},$$

or equivalently

$$|z| \le \left\{\frac{(p-\theta)n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{(n+2p-\theta)-\eta(\mu(p+1)-1)n!}\right\}^{\frac{1}{n+p}}, n \ge p$$

which follows the result.

Theorem 7: If $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$, then f(z) is multivalent meromorphic convex of order $\theta(0 \le \theta < p)$ in the disk $|z| < r_2$, where

$$r_2 = inf_n \left\{\frac{p(p-\theta)n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{n(n+2p-\theta)-\eta(\mu(p+1)-1)n!}\right\}^{\frac{1}{n+p}}, n \ge p.$$

The result is sharp for the function $f(z)$ is given by (12).

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+1+p\right| \le p-\theta \text{ for } |z| < r_2. (25)$$

But

$$\left|\frac{zf''(z)}{f'(z)}+1+p\right| = \left|\frac{zf''(z)+(1+p)f'(z)}{f'(z)}\right| \le \frac{\sum_{n=p}^{\infty} n(n+p)a_n|z|^{n+p}}{p-\sum_{n=p}^{\infty} na_n|z|^{n+p}}.$$

Thus, (25) will be satisfied if

$$\frac{\sum_{n=p}^{\infty} n(n+p)a_n|z|^{n+p}}{p-\sum_{n=p}^{\infty} n\, a_n|z|^{n+p}} \le p-\theta ,$$

or if

$$\sum_{n=p}^{\infty} \frac{(n+2p-\theta)a_n}{p(p-\theta)}|z|^{n+p} \le 1. (26)$$

Since $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$, we have

$$\sum_{n=p}^{\infty} \frac{n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p(\mu(p+1)-1)n!} a_n \le 1.$$

Hence, (26) will be true if

$$\frac{n(n+2p-\theta)}{p(p-\theta)}|z|^{n+p} \le \frac{n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p(\mu(p+1)-1)n!},$$

or equivalently

$$|z| \le \left\{\frac{p(p-\theta)n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{n(n+2p-\theta)\eta(\mu(p+1)-1)n!}\right\}^{\frac{1}{n+p}}, n \ge p,$$

which follows the result.

7. Integral Operator

Theorem 8: Let the function $f(z)$ be given by (1) in the class $\sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$. Then, the integral operator

$$\omega(z) = \varepsilon \int_0^1 u^{\varepsilon} f(uz)du, (0 < u \le 1, 0 < \varepsilon < \infty), (27)$$

is in the class $\sum_p (\lambda, \mu, \eta, \alpha_1, q, s)$, where

$$\tau = \frac{\varepsilon\big(\lambda(p-1)2p-\eta\big)(\mu(p+1)-1)+(\varepsilon+p+1)(\lambda(p-1)2p-\eta)(\mu(p+1)-1)\big)}{(\varepsilon+p+1)(p+1)\big(\lambda(p-1)2p-\eta(\mu(p-2)+1)\big)\eta\varepsilon(p-2)(\mu(p+1)-1)}.$$

The result is sharp for the function $f(z)$ given by (15).

Proof: Let
$f(z)=\frac{1}{z^p}+\sum_{n=p}^{\infty} a_n z^n$ is in the class $\sum_p\ (\lambda,\mu,\eta,\alpha_1,q,s)$.
Then

$$\omega(z)=\varepsilon\int_0^1 u^{\varepsilon} f(uz)du$$
$$=\varepsilon\int_0^1 \left(\frac{u^{\varepsilon-1}}{z^n}-\sum_{n=p}^{\infty} u^{n+\varepsilon} a_n z^n\right)d\varepsilon$$
$$=\frac{1}{z^p}+\sum_{n=p}^{\infty}\frac{\varepsilon}{\varepsilon+n+1} a_n z^n .$$

It is enough to show that

$$\sum_{n=p}^{\infty}\frac{\varepsilon n\big(\lambda(n-1)(n+p)-\eta(\tau(n-2)+1)\big)\frac{(\alpha_1)_n\dots(\alpha_q)_n}{(\beta_1)_n\dots(\beta_s)_n}}{(\varepsilon+n+1)\eta p(\tau(p+1)-1)n!}a_n \le 1. \quad (28)$$

Since $f(z)\in\sum_p(\lambda,\mu,\eta,\alpha_1,q,s)$, then by Theorem 1, we get

$$\sum_{n=p}^{\infty}\frac{n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n\dots(\alpha_q)_n}{(\beta_1)_n\dots(\beta_s)_n}}{\eta p(\mu(p+1)-1)n!}a_n \le 1.$$

Note that (28) is satisfied if

$$\frac{\varepsilon n\big(\lambda(n-1)(n+p)-\eta(\tau(n-2)+1)\big)\frac{(\alpha_1)_n\dots(\alpha_q)_n}{(\beta_1)_n\dots(\beta_s)_n}}{(\varepsilon+n+1)\eta p(\tau(p+1)-1)n!} \le \frac{n\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)\frac{(\alpha_1)_n\dots(\alpha_q)_n}{(\beta_1)_n\dots(\beta_s)_n}}{\eta p(\mu(p+1)-1)n!}$$

or equivalently

$$\tau\le\frac{\varepsilon\big(\lambda(n-1)(n+p)-\eta\big)(\mu(p+1)-1)+(\varepsilon+n+1)(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1))}{(\varepsilon+n+1)(p+1)\big(\lambda(n-1)(n+p)-\eta(\mu(n-2)+1)\big)+\eta\varepsilon(n-2)(\mu(p+1)-1)}=\omega(n).$$

A simple computation will show that ω(n) is increasing function of n.
This means that $\omega(n)\ge\omega(p)$ .Using this, we obtain the result.

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