On a New Certain Subclass of Meromorphically p-valent Functions Defined by a Linear Operator

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Abstract: In the present paper, we have introduced a new class of meromorphically p-valent functions \( \Sigma_p \) (\( \lambda, \mu, \eta, \alpha_1, g, s \)) defined by a linear operator \( T_{p,g,s}(\alpha_1) \). We discuss some interesting properties, like, coefficient inequality, convex set, distortion bounds, neighborhoods of a function \( f \in \Sigma_p \) and integral operator.

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1. Introduction

Let \( \Sigma_p \) denote the class of functions of the form:

\[
f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n ; \quad (a_n \geq 0; \quad p \in \mathbb{N} = \{1, 2, \ldots \}), \tag{1}
\]

which are analytic and p-valent in the punctured unit disk

\[U^* = \{z: z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{C}\setminus\{0\}.
\]

We define the Hadamard product (or Convolution) of \( f \) and \( g \) by

\[
(f \ast g)(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n b_n z^n = (g \ast f)(z), \tag{2}
\]

where \( f \) is given by (1) and \( g \) is defined as follows:

\[
g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n.
\]

For positive real values of \( \alpha_1, \ldots, \alpha_q \) and \( \beta_1, \ldots, \beta_s \) \((\beta_i \neq 0, -1, \ldots; \quad f = 1, 2, \ldots, s)\), we now define the generalized hypergeometric function

\[
qF_s(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots, \beta_s ; z) \] by

\[
qF_s(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots, \beta_s ; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n} \frac{z^n}{n!}, \tag{3}
\]

\((q \leq s + 1; \quad q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \quad z \in U)\),

where \((\theta)_v\) is the Pochhammer symbol defined

\[
(\theta)_v = \begin{cases} 1 & \text{if } v = 0, \\ \theta(\theta+1)(\theta+2)\ldots(\theta+v-1) & \text{if } v \in \mathbb{N}. \tag{4} 
\end{cases}
\]

Corresponding to the function \( h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \), defined by

\[
h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \left( z^{-p} \right)^{qF_s(\alpha_1, \ldots, \alpha_q ; \beta_1, \ldots, \beta_s ; z)} \tag{5}
\]

we consider a linear operator

\[
T_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s): \Sigma_p \rightarrow \Sigma_p
\]

which is defined by means of the following Hadamard product (or convolution):

\[
T_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z), \tag{6}
\]

We observe that, for a function \( f(z) \) of the form (1), we have

\[
T_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} (\alpha_1)_n \ldots (\alpha_q)_n \frac{a_n}{n!} z^n. \tag{7}
\]

If, for convenience, we write

\[
T_{p,g,s}(\alpha_1) = T_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s), \tag{8}
\]

then one can easily verify from the definition (6) that

\[
\begin{aligned}
&= \alpha_1 T_{p,g,s}(\alpha_1 + 1) f(z) \\
&= \alpha_1 T_{p,g,s}(\alpha_1) f(z) \tag{9}
\end{aligned}
\]

The linear operator \( T_{p,g,s}(\alpha_1) \) was investigated recently by Liu and Srivastava [8].

Some interesting Subclasses of analytic functions associated with the generalized hypergeometric function were considered recently by (for example) Dziok and Srivastava [3 and 4], Gangadharan et al. [5] and Liu [7].

Definition 1: Let \( \Sigma_p \) (\( \lambda, \mu, \eta, \alpha_1, q, s \)) be denote the new class of functions \( f \in \Sigma_p \), which satisfy the condition:

\[
\frac{\lambda}{\eta} \frac{z}{\mu} \left( T_{p,g,s}(\alpha_1) f(z) \right)'' + \frac{\alpha_1}{\mu} \left( T_{p,g,s}(\alpha_1) f(z) \right)''' < \eta, \tag{10}
\]

where \( z \in U^*; \quad 0 < \eta < \mu; \quad p \in \mathbb{N} \) and for some suitably restricted real parameters \( \lambda, \mu, \eta \).

Such type of study was carried out by several different authors for another classes, like, Nunokawa and Ahuja [9], Aouf and Hossen [1] and Cho et al. [2].

2. Coefficient Inequality

First, we derive the coefficient inequality for the class \( \Sigma_p \) (\( \lambda, \mu, \eta, \alpha_1, q, s \)) contained in:
Theorem 1: Let $f \in \sum_p \ (\lambda, \mu, \eta, \alpha, q, s)$. Then $f$ is in the class 
\[ \sum_p (\lambda \mu \eta \alpha \beta)^n \frac{a_n z^n}{n!} \] if and only if 
\[ \sum_{n=p}^\infty n(\lambda(n-1)(n+p) - \eta(n-2) \prod_{k=1}^n (\alpha_k) \prod_{k=p}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ - \eta (\mu(n+1) - 1) \leq \eta \leq \mu \leq \lambda < \eta < p; \quad p \in N. \]
The result is sharp for the function 
\[ f(z) = 1 + \frac{1}{z^p} \prod_{k=1}^n (\alpha_k) \prod_{k=p}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ \leq \sum_{n=p}^\infty \lambda(n-1)(n+p) (\alpha_n) \prod_{k=1}^n (\beta_k) \frac{a_n}{n!} |z|^n \]
\[ - \eta (\mu(n+1) - 1) \leq \eta \leq \mu \leq \lambda < \eta < p; \quad p \in N. \]
\[ \text{Proof:} \] Suppose that the inequality (11) holds true and $|z| = 1$. Then, we have 
\[ |z| = 1 \implies \sum_{n=p}^\infty \lambda(n-1)(n+p) (\alpha_n) \prod_{k=1}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ - \eta (\mu(n+1) - 1) \leq \eta \leq \mu \leq \lambda < \eta < p; \quad p \in N. \]
To show the converse, suppose that 
\[ f(z) \in \sum_p \ (\lambda, \mu, \eta, \alpha, q, s). \]
Then (10), we have 
\[ f(z) = \frac{1}{z^p} \prod_{k=1}^n (\alpha_k) \prod_{k=p}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ \leq \sum_{n=p}^\infty \lambda(n-1)(n+p) (\alpha_n) \prod_{k=1}^n (\beta_k) \frac{a_n}{n!} |z|^n \]
\[ - \eta (\mu(n+1) - 1) \leq \eta \leq \mu \leq \lambda < \eta < p; \quad p \in N. \]
Sharpness of the result follows by setting 
\[ a_n = \frac{1}{n!} \eta (\mu(n+1) - 1) \]
\[ n(\alpha_1) \prod_{k=1}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ \lambda(n-1)(n+p) - \eta(n-2) \prod_{k=1}^n (\alpha_k) \prod_{k=p}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ \leq \sum_{n=p}^\infty \eta (\mu(n-1)(n+p) - \eta(n-2) \prod_{k=1}^n (\alpha_k) \prod_{k=p}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ \geq p. \]
Corollary 1: Let $f(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s)$. Then 
\[ a_n \leq \eta (\mu(p+1) - 1) \]
\[ n(\alpha_1) \prod_{k=1}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ \lambda(n-1)(n+p) - \eta(n-2) \prod_{k=1}^n (\alpha_k) \prod_{k=p}^n (\beta_k) \frac{a_n}{n!} z^n \]
\[ \geq p. \] 
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3. Convex Set

In the following theorem, we will prove the class \( \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \) is convex set.

**Theorem 2:** The class \( \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \) is a convex set.

**Proof:** Let \( f_1 \) and \( f_2 \) be the arbitrary elements of \( \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \).

Then for every \( t \) \((0 \leq t \leq 1) \), we show that \((1 - t)f_1 + tf_2 \in \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \).

Thus, we have

\[
(1 - t)f_1 + tf_2 = \frac{1}{2} + \sum_{n=p}^{\infty} \left( (1-t)a_n + tb_n \right)z^n
\]

Hence,

\[
\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2))
+ 1) \left( \frac{a_n}{\beta_n} \right)_n \left( \frac{b_n}{\beta_n} \right)_n \frac{(1-t)a_n + tb_n}{n!}
+ t \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2))
+ 1) \left( \frac{a_n}{\beta_n} \right)_n \left( \frac{b_n}{\beta_n} \right)_n \frac{b_n}{n!}
\leq (1-t)\eta(\mu(p+1) - 1) + \eta(p(\mu(p+1)-1)) = \eta(p(\mu(p+1)-1)).
\]

This completes the proof.

4. Distortion Bounds

In the following theorems, we obtain the growth and distortion bounds for the linear operator \( T_{p,q,s}(\alpha) \).

**Theorem 3:** If \( f(z) \in \Sigma_p \), then

\[
\frac{1}{z^{p+1}} \left( 2\lambda(p-1) - \eta(\mu(p-2)+1) \right) \leq \left| T_{p,q,s}(\alpha) f(z) \right| \leq \frac{1}{z^{p+1}} \left( 2\lambda(p-1) - \eta(\mu(p-2)+1) \right) \eta(\mu(p-1) - 1).
\]

The result is sharp for the function \( \alpha(z) = \frac{1}{z} \).

**Proof:** Let \( f(z) \in \Sigma_p \). Then by Theorem 1, we get

\[
p(2\lambda(p-1) - \eta(\mu(p-2)) + 1) \left( \frac{a_p}{\beta_p} \right)_p \leq \sum_{n=p}^{\infty} a_n \leq \eta(p(\mu(p-1)-1)).
\]

5. \( \delta \)-Neighborhood of a function \( f \in \Sigma_p \)

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruscheweyh [10], we begin by introducing here the \( \delta \)-Neighborhood of a function \( f \in \Sigma_p \) of the form (1) by means of the definition below:

\[
N_\delta(f) = \left\{ g \in \sum_p : \left| g(z) - 1 \right| = \frac{1}{z^{p+1}} \left( 2\lambda(p-1) - \eta(\mu(p-2)+1) \right) \eta(\mu(p-1)-1) \right\}
\]

The result is sharp for the function \( f(z) \) is given by (15).

**Proof:** The proof is similar to that of Theorem 3.
Particularly for the identity function \( f(z) = \frac{1}{z^p} \), we have

\[
N_p(\delta) = \left\{ \delta \in \mathbb{C} : \sum_{n=0}^{\infty} |a_n - b_n| \leq \delta \right\}.
\]

**Theorem 5**: If \( g(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \) and

\[
y = 1 - \frac{\delta(\lambda(p-1)2\mu - \eta(\mu(p-2) - 1))}{p(\lambda(p-1)2\mu - \eta(\mu(p-2) + 1))} \frac{(\alpha_1)_p \cdots (\alpha_q)_p (\beta_1)_p \cdots (\beta_s)_p}{(\beta_1)_p \cdots (\beta_s)_p}.
\]

**Proof**: Let \( f(z) \in N_p(g) \). Then, we find from (20) that

\[
\sum_{n=p}^{\infty} |a_n - b_n| \leq \delta,
\]

which implies the coefficient inequality

\[
\sum_{n=p}^{\infty} |a_n - b_n| \leq \delta \frac{1}{p}(n \geq p).
\]

Since \( g(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \), then by using Theorem (1), we get

\[
\sum_{n=p}^{\infty} b_n \leq \frac{\eta(\mu(p+1) - 1)p!}{(\lambda(p-1)2\mu - \eta(\mu(p-2) + 1))}\frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p},
\]

so that

\[
\frac{f(z)}{g(z)} - 1 < \frac{\sum_{n=p}^{\infty} |a_n - b_n|}{1 - \sum_{n=p}^{\infty} b_n} \leq \frac{\delta}{p} \frac{\delta(\lambda(p-1)2\mu - \eta(\mu(p-2) - 1))}{p(\lambda(p-1)2\mu - \eta(\mu(p-2) + 1))} \frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p} = 1 - y.
\]

Hence, by Definition 2, \( f(z) \in \sum_{p,y} (\lambda, \mu, \eta, \alpha, q, s) \) for \( y \) given by (22). This complete the proof.

### 6. Radii of starlikeness and convexity:

In the following Theorems, we discuss the radii starlikeness and convexity.

**Theorem 6**: If \( f(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \), then \( f(z) \) is multivalent meromorphic starlike of order \( \phi(0 \leq \phi < \pi) \) in the disk \(|z| < r_1\), where

\[
r_1 = \inf \left\{ (p+\theta)n(\lambda(n-1)(n+p) - \eta(n(n-2)+1)) \frac{(\alpha_1)_n \cdots (\alpha_q)_n (\beta_1)_n \cdots (\beta_s)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right\}^{1/p+q}
\]

\[
, n \geq p.
\]

The result is sharp for the function \( f(z) \) is given by (12).

**Proof**: It is sufficient to show that

\[
|zf'(z) + p| \leq p - \theta \text{ for } |z| < r_1, \text{ (23)}
\]

But

\[
\frac{|zf'(z) + pf(z)|}{f(z)} \leq \sum_{n=p}^{\infty} |a_n| z^{n+p} |1 + \sum_{n=p}^{\infty} a_n z^n|^{n+p}.
\]

Thus, (23) will be satisfied if

\[
\sum_{n=p}^{\infty} |a_n| z^{n+p} |1 + \sum_{n=p}^{\infty} a_n z^n|^{n+p} \leq p - \theta \text{, or } \sum_{n=p}^{\infty} |a_n| z^{n+p} |1 + \sum_{n=p}^{\infty} a_n z^n|^{n+p} \leq 1 \text{, (24)}.
\]

Since \( f(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \), we have
\[ \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n \leq 1. \]

Hence, (24) will be true if
\[ \frac{(n + 2p - \theta)|z|^{n+p}}{p - \theta} \leq \frac{n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n}{\eta p(\mu(p+1) - 1)n!}, \]
or equivalently
\[ |z| \leq \left( \frac{(p - \theta)n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n}{n(n + 2p - \theta) - \eta(\mu(p+1) - 1)n!} \right)^{\frac{1}{p + n + p}}, \quad n \geq p, \]
which follows the result.

**Theorem 7:** If \( f(z) \in \sum_p (\lambda, \mu, \eta, a_1, q, s) \), then \( f(z) \) is multivalent meromorphic convex of order \( \theta \) \((0 \leq \theta < p)\) in the disk \( |z| < r_2 \), where
\[ r_2 = \inf_n \left\{ \frac{p(\theta)n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n}{n(n + 2p - \theta) - \eta(\mu(p+1) - 1)n!} \right\}^{\frac{1}{p + n + p}}, \quad n \geq p. \]

The result is sharp for the function \( f(z) \) given by (12).

**Proof:** It is sufficient to show that
\[ \left| \frac{zf''(z)}{f(z)} + 1 + p \right| \leq p - \theta \text{ for } |z| < r_2. \quad (25) \]
But
\[ \left| \frac{zf''(z)}{f(z)} + 1 + p \right| = \left| \frac{zf''(z) + (1 + p)f'(z)}{f(z)} \right| \leq \frac{\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n}{\sum_{n=p}^{\infty} p(n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n)|z|^{n+p}}. \]
Since \( f(z) \in \sum_p (\lambda, \mu, \eta, a_1, q, s) \), we have
\[ \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n \leq 1. \]
Hence, (26) will be true if
\[ \frac{n(n + 2p - \theta)}{p(\theta)n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n} \leq \frac{n(n + 2p - \theta) - \eta(\mu(p+1) - 1)n!}{\eta p(\mu(p+1) - 1)n!}, \]
or equivalently
\[ |z| \leq \left( \frac{p(\theta)n(\lambda(n-1)(n+p) - \eta(n(n-2) + 1))(a_n \ldots (a_q))_n}{n(n + 2p - \theta) - \eta(\mu(p+1) - 1)n!} \right)^{\frac{1}{p + n + p}}, \quad n \geq p, \]
which follows the result.

**Theorem 8:** Let the function \( f(z) \) be given by (1) in the class \( \sum_p (\lambda, \mu, \eta, a_1, q, s) \). Then, the integral operator
\[ \omega(z) = \epsilon \int_0^1 u^\tau f(uz)du, \quad (0 < u \leq 1, 0 < \epsilon < \infty), \quad (27) \]
is in the class \( \sum_p (\lambda, \mu, \eta, a_1, q, s) \), where
\[ \tau = \frac{\epsilon(\lambda + \mu)(p+1) + (\epsilon + p + 1)(\lambda(p+1)(p+1) - \eta\mu(p+1) - 1)(\lambda(p+1)(p+1) - \eta\mu(p+1) - 1))}{(\epsilon + p + 1)(\lambda(p+1)(p+1) - \eta\mu(p+1) - 1)(\lambda(p+1)(p+1) - \eta\mu(p+1) - 1))}. \]
The result is sharp for the function \( f(z) \) given by (15).
Proof: Let 
\[ f(z) = \frac{z}{z^p} + \sum_{n=p}^{\infty} a_n z^n \]
is in the class \( \sum (\lambda, \mu, \eta, \alpha, \eta, \mu, q, s) \).
Then
\[ \omega(z) = \varepsilon \int_0^1 u^n f(uz) du = \frac{1}{z^p} + \sum_{n=p}^{\infty} \frac{\varepsilon}{\varepsilon + n + 1} a_n z^n. \]
It is enough to show that
\[ \sum_{n=p}^{\infty} \frac{\varepsilon n \lambda(n-1)(n+p) - \eta \mu(n-2) + 1}{\varepsilon + n + 1} \frac{(a_1)_n \cdots (a_q)_n}{(\beta_1)_n \cdots (\beta_q)_n} \leq 1. \ (28) \]

Since \( f(z) \in \sum (\lambda, \mu, \eta, \mu, q, s) \), then by Theorem 1, we get
or equivalently
\[ \omega(n) \geq \omega(p). \]
Using this, we obtain the result.

A simple computation will show that \( \omega(n) \) is increasing function of \( n \).
This means that \( \omega(n) \geq \omega(p) \). Using this, we obtain the result.

References


