

On a New Certain Subclass of Meromorphically p -valent Functions Defined by a Linear Operator

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Abstract: In the present paper, we have introduced a new class of meromorphically p -valent functions $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$ defined by a linear operator $T_{p,q,s}(\alpha_1)$. We discuss some interesting properties, like, coefficient inequality, convex set, distortion bounds, neighborhoods of a function $f \in \Sigma_p$ and integral operator.

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1. Introduction

Let Σ_p denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n; \quad (a_n \geq 0; p \in N = \{1, 2, \dots\}), \quad (1)$$

which are analytic and p -valent in the punctured unit disk

$$U^* = \{z: z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}.$$

We define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n b_n z^n = (g * f)(z), \quad (2)$$

where f is given by (1) and g is defined as follows:

$$g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n.$$

For positive real values of $\alpha_1, \dots, \alpha_q$ and β_1, \dots, β_s ($\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, s$), we now define the generalized hypergeometric function

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \text{ by}$$

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{z^n}{n!}, \quad (3)$$

$$(q \leq s + 1; q, s \in N_0 = N \cup \{0\}; z \in U),$$

where $(\theta)_v$ is the Pochhammer symbol defined

$$(\theta)_v = \begin{cases} 1 & v = 0 \\ \theta(\theta + 1)(\theta + 2) \dots (\theta + v - 1) & v \in N. \end{cases} \quad (4)$$

Corresponding to the function $h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$, defined by

$$h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z^{-p} {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z); \quad (5)$$

we consider a linear operator

$$T_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s): \Sigma_p \rightarrow \Sigma_p,$$

which is defined by means of the following Hadamard product (or convolution):

$$T_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) = h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z). \quad (6)$$

We observe that, for a function $f(z)$ of the form (1); we have

$$T_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} z^n. \quad (7)$$

If, for convenience, we write

$$T_{p,q,s}(\alpha_1) = T_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s), \quad (8)$$

then one can easily verify from the definition (6) that

$$\begin{aligned} z(T_{p,q,s}(\alpha_1)f(z))' &= \alpha_1 T_{p,q,s}(\alpha_1 + 1)f(z) \\ &- (\alpha_1 + p)T_{p,q,s}(\alpha_1)f(z). \end{aligned} \quad (9)$$

The linear operator $T_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [8].

Some interesting Subclasses of analytic functions associated with the generalized hypergeometric function were considered recently by (for example) Dziok and Srivastava ([3] and [4]), Gangadharan et al. [5] and Liu [7].

Definition 1: Let $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$ be denote the new class of functions

$f \in \Sigma_p$, which satisfy the condition:

$$\left| \frac{\lambda(p+2)z^2(T_{p,q,s}(\alpha_1)f(z))'' + \lambda z^3(T_{p,q,s}(\alpha_1)f(z))'''}{\mu z^2(T_{p,q,s}(\alpha_1)f(z))'' - (\mu-1)z(T_{p,q,s}(\alpha_1)f(z))'} \right| < \eta, \quad (10)$$

where $z \in U^*$; $0 \leq \eta < p$; $p \in N$ and for some suitably restricted real parameters λ , and μ .

Such type of study was carried out by several different authors for another classes, like, Nunokawa and Ahuja [9], Aouf and Hossen [1] and Cho et al. [2].

2. Coefficient Inequality

First, we derive the coefficient inequality for the class $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$ contained in:

Theorem 1: Let $f \in \Sigma_p$. Then f is in the class $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$ if and only if

$$\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} \leq \eta p(\mu(p+1) - 1), \quad (11)$$

where $0 < \eta < p$; $\frac{1}{2} < \mu < \lambda < p$; and $p \in \mathbb{N}$.

The result is sharp for the function

$$\begin{aligned} f(z) &= \frac{1}{z^p} \\ &+ \frac{n! \eta p(\mu(p+1) - 1)}{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}} z^n, \quad n \geq p. \end{aligned} \quad (12)$$

Proof: Suppose that the inequality (11) holds true and $|z| = 1$. Then, we have

$$\begin{aligned} &|\lambda(p+2)z^2(T_{p,q,s}(\alpha_1)f(z))'' + \lambda z^3(T_{p,q,s}(\alpha_1)f(z))''' \\ &- \eta|(\mu-1)z(T_{p,q,s}(\alpha_1)f(z))' \\ &+ \mu z^2(T_{p,q,s}(\alpha_1)f(z))''| \end{aligned}$$

$$= \left| \sum_{n=p}^{\infty} \lambda n(n-1)(n+p) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} z^n \right|$$

$$\begin{aligned} &- \eta \left| p(\mu(p+2) - 1)z^{-p} \right. \\ &- \sum_{n=p}^{\infty} n(\mu(2-n)) \\ &- \left. 1) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} z^n \right| \end{aligned}$$

$$\left| \frac{\sum_{n=p}^{\infty} \lambda n(n-1)(n+p) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} z^n}{p(\mu(p+2) - 1)z^{-p} - \sum_{n=p}^{\infty} n(\mu(2-n) - 1) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} z^n} \right| < \eta.$$

Since $\text{Re}(z) \leq |z|$ for all $z (z \in U)$, we have

$$\text{Re} \left\{ \frac{\sum_{n=p}^{\infty} \lambda n(n-1)(n+p) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} z^n}{p(\mu(p+2) - 1)z^{-p} - \sum_{n=p}^{\infty} n(\mu(2-n) - 1) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} z^n} \right\} < \eta. \quad (13)$$

We choose the value of z on the real axis so that $z(T_{p,q,s}(\alpha_1)f(z))'$ is real.

Upon clearing the denominator of (13) and letting $z \rightarrow 1^-$, through real values so we can write (13) as

$$\begin{aligned} &\sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) \\ &+ 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} \\ &\leq \eta p(\mu(p+1) - 1). \end{aligned}$$

Sharpness of the result follows by setting

$$\begin{aligned} &\leq \sum_{n=p}^{\infty} \lambda n(n-1)(n+p) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} |z|^n \\ &- \eta p(\mu(p+2) - 1)|z|^{-p} \\ &+ \sum_{n=p}^{\infty} \eta n(\mu(2-n)) \\ &- 1) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} |z|^n \end{aligned}$$

$$= \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n a_n}{(\beta_1)_n \dots (\beta_s)_n n!} - \eta p(\mu(p+1) - 1) \leq 0,$$

by hypothesis. Thus by maximum modulus principle, $f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$.

To show the converse, suppose that

$f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$. Then (10), we have

$$\left| \frac{\lambda(p+2)z^2(T_{p,q,s}(\alpha_1)f(z))'' + \lambda z^3(T_{p,q,s}(\alpha_1)f(z))'''}{\mu z^2(T_{p,q,s}(\alpha_1)f(z))'' - (\mu-1)z(T_{p,q,s}(\alpha_1)f(z))'} \right| =$$

$$\begin{aligned} f(z) &= \frac{1}{z^p} \\ &+ \frac{n! \eta p(\mu(p+1) - 1)}{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}} z^n, \quad (n \geq p). \end{aligned}$$

Corollary 1: Let $f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$. Then

$$\begin{aligned} &a_n \\ &\leq \frac{\eta p(\mu(p+1) - 1)n!}{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}} \end{aligned}$$

$(n \geq p).$

3. Convex Set

In the following theorem, we will prove the class $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$ is convex set.

Theorem 2: The class $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$ is a convex set.

Proof: Let f_1 and f_2 be the arbitrary elements of

$\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$.

Then for every $t (0 \leq t \leq 1)$, we show that $(1-t)f_1 + tf_2 \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$.

Thus, we have

$$(1-t)f_1 + tf_2 = \frac{1}{z^p} + \sum_{n=p}^{\infty} [(1-t)a_n + tb_n]z^n.$$

Hence,

$$\begin{aligned} & \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{[(1-t)a_n + tb_n]}{n!} \\ &= (1-t) \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{a_n}{n!} \\ &+ t \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{b_n}{n!} \\ &\leq (1-t)\eta(\mu(p+1) - 1) + t\eta p(\mu(p+1) - 1) \\ &= \eta p(\mu(p+1) - 1). \end{aligned}$$

This completes the proof.

4. 4. Distortion Bounds

In the following theorems, we obtain the growth and distortion bounds for the linear operator $T_{p,q,s}(\alpha_1)$.

Theorem 3: If $f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$, then

$$\begin{aligned} & \frac{1}{r^p} - \frac{\eta(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1))} r^p \leq |T_{p,q,s}(\alpha_1)f(z)| \\ & \leq \frac{1}{r^p} + \frac{\eta(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1))} r^p, (|z| = r < 1). \end{aligned}$$

The result is sharp for the function

$$\begin{aligned} & f(z) = \frac{1}{z^p} + \frac{\eta p(\mu(p+1) - 1)p!}{p(\lambda(p-1)2p - \eta(\mu(p-2) + 1))} z^p \cdot \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} \end{aligned} \tag{15}$$

Proof: Let $f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$. Then by Theorem 1, we get

$$\begin{aligned} & p(2\lambda p(p-1) - \eta(\mu(p-2) + 1)) \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} \cdot \frac{1}{p!} \sum_{n=p}^{\infty} a_n \end{aligned}$$

$$\leq \sum_{n=p}^{\infty} n(\lambda(n-1)(n+p))$$

$$\begin{aligned} & - \eta(\mu(n-2) + 1) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{a_n}{n!} \\ & \leq \eta p(\mu(p+1) - 1), \end{aligned}$$

$$\begin{aligned} & \text{or} \\ & \sum_{n=p}^{\infty} a_n \leq \frac{\eta p(\mu(p+1) - 1)p!}{p(2\lambda p(p-1) - \eta(\mu(p-2) + 1))} \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} \end{aligned} \tag{16}$$

$$\begin{aligned} & |T_{p,q,s}(\alpha_1)f(z)| \leq \frac{1}{|z|^p} + \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{a_n}{n!} |z|^n \\ & \leq \frac{1}{|z|^p} + \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{|z|^p}{p!} \sum_{n=p}^{\infty} a_n \\ & = \frac{1}{r^p} + \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} \cdot \frac{r^p}{p!} \sum_{n=p}^{\infty} a_n \\ & \leq \frac{1}{r^p} + \frac{\eta(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1))} r^p. \end{aligned} \tag{17}$$

Similarly,

$$\begin{aligned} & |T_{p,q,s}(\alpha_1)f(z)| \geq \frac{1}{|z|^p} - \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{a_n}{n!} |z|^n \\ & \geq \frac{1}{|z|^p} - \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \cdot \frac{|z|^p}{p!} \sum_{n=p}^{\infty} a_n \\ & = \frac{1}{r^p} - \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} \cdot \frac{r^p}{p!} \sum_{n=p}^{\infty} a_n \\ & \geq \frac{1}{r^p} - \frac{\eta(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1))} r^p. \end{aligned} \tag{18}$$

From (17) and (18), we get (14) and the proof is complete.

Theorem 4: If $f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$, then

$$\begin{aligned} & \frac{-p}{r^{p+1}} - \frac{\eta p(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1))} r^{p-1} \leq |(T_{p,q,s}(\alpha_1)f(z))'| \leq \\ & \frac{-p}{r^{p+1}} + \frac{\eta p(\mu(p+1) - 1)}{(2\lambda p(p-1) - \eta(\mu(p-2) + 1))} r^{p-1}, (|z| = r < 1). \end{aligned} \tag{19}$$

The result is sharp for the function $f(z)$ is given by (15).

Proof: The proof is similar to that of Theorem 3.

5. δ -Neighborhood of a function $f \in \Sigma_p$:

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruscheweyh [10], we begin by introducing here the δ -Neighborhood of a function $f \in \Sigma_p$ of the form (1) by means of the definition below:

$$\begin{aligned} & N_{\delta}(f) = \left\{ g \in \Sigma_p : g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n \text{ and } \sum_{n=p}^{\infty} n|a_n - b_n| \leq \delta, 0 \leq \delta < 1 \right\}. \end{aligned} \tag{20}$$

Particularly for the identity function $(z) = \frac{1}{z^p}$, we have

$$N_\delta(e) = \left\{ g \in \sum_p : g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n \text{ and } \sum_{n=p}^{\infty} n|b_n| \leq \delta \right\}. \quad (21)$$

Definition 2: A function $f(z) \in \sum_p$ is said to be in the class $\sum_{p,y}(\lambda, \mu, \eta, \alpha_1, q, s)$, if there exists function $g(z) \in \sum_p(\lambda, \mu, \eta, \alpha_1, q, s)$, such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - y, (z \in y, 0 \leq y < 1).$$

Theorem 5: If $g(z) \in \sum_p(\lambda, \mu, \eta, \alpha_1, q, s)$ and

$$y = 1 - \frac{\delta(\lambda(p-1)2p - \eta(\mu(p-2) - 1)) \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p}}{p(\lambda(p-1)2p - \eta(\mu(p-2) + 1)) \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} - \eta(\mu(p+1) - 1)p!}. \quad (22)$$

Proof: Let $f(z) \in N_\delta(g)$. Then, we find from (20) that

$$\sum_{n=p}^{\infty} n|a_n - b_n| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=p}^{\infty} |a_n - b_n| \leq \frac{\delta}{p}, (n \geq p).$$

Since $g(z) \in \sum_p(\lambda, \mu, \eta, \alpha_1, q, s)$, then by using Theorem (1), we get

$$\sum_{n=p}^{\infty} b_n \leq \frac{\eta(\mu(p+1) - 1)p!}{(\lambda(p-1)2p - \eta(\mu(p-2) + 1)) \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p}},$$

so that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{n=p}^{\infty} |a_n - b_n|}{1 - \sum_{n=p}^{\infty} b_n} \leq \frac{\delta}{p} \frac{\delta(\lambda(p-1)2p - \eta(\mu(p-2) - 1)) \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p}}{(\lambda(p-1)2p - \eta(\mu(p-2) + 1)) \frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p} - \eta(\mu(p+1) - 1)p!} = 1 - y.$$

Hence, by Definition 2, $f(z) \in \sum_{p,y}(\lambda, \mu, \eta, \alpha_1, q, s)$ for y given by (22).

This complete the proof.

6. Radii of starlikeness and convexity:

In the following Theorems, we discuss the radii starlikeness and convexity.

Theorem 6: If $f(z) \in \sum_p(\lambda, \mu, \eta, \alpha_1, q, s)$, then $f(z)$ is multivalent meromorphic starlike of order $\theta (0 \leq \theta < p)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_n \left\{ \frac{(p - \theta)n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}^{\frac{1}{n+p}}}{(n + 2p - \theta)\eta p(\mu(p+1) - 1)n!} \right\}$$

, $n \geq p$.

The result is sharp for the function $f(z)$ is given by (12).

Proof: It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq p - \theta \text{ for } |z| < r_1. \quad (23)$$

But

$$\left| \frac{zf'(z) + pf(z)}{f(z)} \right| = \left| \frac{\sum_{n=p}^{\infty} (n+p)a_n z^{n+p}}{1 + \sum_{n=p}^{\infty} a_n z^{n+p}} \right| \leq \frac{\sum_{n=p}^{\infty} (n+p)a_n |z|^{n+p}}{1 - \sum_{n=p}^{\infty} a_n |z|^{n+p}}.$$

or if

$$\frac{\sum_{n=p}^{\infty} (n+p)a_n |z|^{n+p}}{1 - \sum_{n=p}^{\infty} a_n |z|^{n+p}} \leq p - \theta,$$

$$\sum_{n=p}^{\infty} \frac{(n + 2p - \theta)a_n}{p - \theta} |z|^{n+p} \leq 1. \quad (24)$$

Since $f(z) \in \sum_p(\lambda, \mu, \eta, \alpha_1, q, s)$, we have

Thus, (23) will be satisfied if

$$\sum_{n=p}^{\infty} \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p(\mu(p+1) - 1)n!} a_n \leq 1.$$

Hence, (24) will be true if

$$\frac{(n+2p-\theta)}{p-\theta} |z|^{n+p} \leq \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p(\mu(p+1) - 1)n!},$$

or equivalently

$$|z| \leq \left\{ \frac{(p-\theta)n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{(n+2p-\theta) - \eta(\mu(p+1) - 1)n!} \right\}^{\frac{1}{n+p}}, n \geq p$$

which follows the result.

Theorem 7: If $f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$, then $f(z)$ is multivalent meromorphic convex of order θ ($0 \leq \theta < p$) in the disk $|z| < r_2$, where

$$r_2 = \inf_n \left\{ \frac{p(p-\theta)n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{n(n+2p-\theta) - \eta(\mu(p+1) - 1)n!} \right\}^{\frac{1}{n+p}}, n \geq p.$$

The result is sharp for the function $f(z)$ is given by (12).

Thus, (25) will be satisfied if

Proof: It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} + 1 + p \right| \leq p - \theta \text{ for } |z| < r_2. \quad (25)$$

or if

$$\frac{\sum_{n=p}^{\infty} n(n+p)a_n |z|^{n+p}}{p - \sum_{n=p}^{\infty} n a_n |z|^{n+p}} \leq p - \theta,$$

But

$$\left| \frac{zf''(z)}{f'(z)} + 1 + p \right| = \left| \frac{zf''(z) + (1+p)f'(z)}{f'(z)} \right| \leq \frac{\sum_{n=p}^{\infty} n(n+p)a_n |z|^{n+p}}{p - \sum_{n=p}^{\infty} n a_n |z|^{n+p}}.$$

$$\sum_{n=p}^{\infty} \frac{(n+2p-\theta)a_n}{p(p-\theta)} |z|^{n+p} \leq 1. \quad (26)$$

Since $f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$, we have

$$\sum_{n=p}^{\infty} \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p(\mu(p+1) - 1)n!} a_n \leq 1.$$

Hence, (26) will be true if

$$\frac{n(n+2p-\theta)}{p(p-\theta)} |z|^{n+p} \leq \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p(\mu(p+1) - 1)n!},$$

or equivalently

$$|z| \leq \left\{ \frac{p(p-\theta)n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{n(n+2p-\theta)\eta(\mu(p+1) - 1)n!} \right\}^{\frac{1}{n+p}}, n \geq p,$$

which follows the result.

Theorem 8: Let the function $f(z)$ be given by (1) in the class $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$. Then, the integral operator

$$\omega(z) = \varepsilon \int_0^1 u^\varepsilon f(uz) du, \quad (0 < u \leq 1, 0 < \varepsilon < \infty), \quad (27)$$

is in the class $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$, where

$$\tau = \frac{\varepsilon(\lambda(p-1)2p - \eta)(\mu(p+1) - 1) + (\varepsilon + p + 1)(\lambda(p-1)2p - \eta)(\mu(p+1) - 1)}{(\varepsilon + p + 1)(p+1)(\lambda(p-1)2p - \eta(\mu(p-2) + 1))\eta\varepsilon(p-2)(\mu(p+1) - 1)}.$$

The result is sharp for the function $f(z)$ given by (15).

Proof: Let

$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n$ is in the class $\Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$.

Then

$$\begin{aligned} \omega(z) &= \varepsilon \int_0^1 u^\varepsilon f(uz) du \\ &= \varepsilon \int_0^1 \left(\frac{u^{\varepsilon-1}}{z^p} - \sum_{n=p}^{\infty} u^{n+\varepsilon} a_n z^n \right) d\varepsilon \\ &= \frac{1}{z^p} + \sum_{n=p}^{\infty} \frac{\varepsilon}{\varepsilon + n + 1} a_n z^n. \end{aligned}$$

It is enough to show that

$$\sum_{n=p}^{\infty} \frac{\varepsilon n (\lambda(n-1)(n+p) - \eta(\tau(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} a_n}{(\varepsilon + n + 1) \eta p (\tau(p+1) - 1) n!} \leq 1. \quad (28)$$

Since $f(z) \in \Sigma_p(\lambda, \mu, \eta, \alpha_1, q, s)$, then by Theorem 1, we get or equivalently

$$\tau \leq \frac{\varepsilon(\lambda(n-1)(n+p) - \eta)(\mu(p+1) - 1) + (\varepsilon + n + 1)(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1))}{(\varepsilon + n + 1)(p+1)(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) + \eta\varepsilon(n-2)(\mu(p+1) - 1)} = \omega(n).$$

A simple computation will show that $\omega(n)$ is increasing function of n .

This means that $\omega(n) \geq \omega(p)$. Using this, we obtain the result.

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$$\sum_{n=p}^{\infty} \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} a_n}{\eta p (\mu(p+1) - 1) n!} \leq 1.$$

Note that (28) is satisfied if

$$\begin{aligned} &\frac{\varepsilon n (\lambda(n-1)(n+p) - \eta(\tau(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{(\varepsilon + n + 1) \eta p (\tau(p+1) - 1) n!} \\ &\leq \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n}}{\eta p (\mu(p+1) - 1) n!} \end{aligned}$$

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