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Likelihood Based Estimation of the Parameters of a Log-Linear Nonhomogeneous Poisson Process

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Abstract: ***Non-homogeneous Poisson process (NHPP) has widely been used over decades to model random processes that are time dependent (e.g. occurrences of serious road accidents). After the times of occurrence of a particular event have been observed, the problem of estimating the intensity function arises. In this paper, we consider maximum likelihood estimation of the parameters of a NHPP with log-linear intensity function. The maximum likelihood estimates of the unknown parameters of the intensity function are obtained numerically and the confidence intervals and regions are constructed from the respective graphs of the maximized and joint relative likelihood functions. We present simulation results which demonstrate the good performance of our confidence intervals and regions as compared to those based on large sample approaches. The large sample confidence intervals may be inaccurate in the sense that they exclude plausible parameter values and include values that are very implausible. Our approach is optimal for small sample inferences on the parameters of a log-linear NHPP.***

Keywords: Nonhomogeneous Poisson process, log-linear, relative likelihood function, estimation, intensity function

1. Introduction

Many random processes that arise naturally in daily life situations (e.g. number of patients arriving at a kidney-transplant centre) can be identified as Poisson processes [1]. The processes that are time dependent are often modelled as NHPP as illustrated by [2] and [3]. These processes have widely been used in various applied fields such as genomics, biology, imaging, meteorology, seismology, transport, communication among others [4]. A good example in meteorology was used to model the cyclone arrival times in the arctic sea using NHPP with intensity function having cyclic behaviour by [5]. The intensity function and hence the mean value function of a NHPP is usually assumed to be positive and continuous [6]. This function is parameterized in different forms of which the power law and log-linear forms are the most popular. In this paper, we consider a NHPP with log-linear intensity function which was first introduced by [7]. A NHPP $\{N(t); t > 0\}$ is said to be log-linear if its intensity function is of the form:

$$\lambda(t) = e^{\alpha+\beta t} \tag{1}$$

where α and β are the unknown parameters. The log-linear NHPP has been used mostly to model systems in transport and telecommunication [8]. The fundamental problem that has attracted a lot of attention over decades in modelling systems using log-linear NHPP is estimation of the intensity function. This estimation is believed to aid in determination of whether the rate of occurrence of events increases or decreases with time. Over the years, various researchers have derived point estimates for this model, particularly, the ML estimates derived by [8] and also interval estimates based on large sample approximation were constructed. In large sample approach of constructing interval estimates, the sampling distribution of the estimators (usually the ML estimators) is assumed to be asymptotically normal. However, for small sample sizes, interval estimates based on this approach may be inaccurate and may include values that are very implausible and may also exclude those that are plausible [9]. Interval estimation is considered as a measure of accuracy to point estimators as one can attach a degree of confidence that the true parameter value lies within a given interval. The use of RLF to obtain interval estimates is one of the best approaches to interval estimation as discussed by [9]. The RLF approach is known to yield better interval estimates than the large sample approximation approach, leading to better inference for the case of small sample sizes. Therefore, an attempt has been made in this paper to obtain the maximum likelihood estimates and to construct approximate confidence intervals and regions using the relative likelihood function approach.

The rest of the paper is organized as follows. We describe the maximum likelihood estimation for the parameters of the log-linear NHPP in section 2. In section 3, we give the simulation results. Section 4 contains the discussion of the results.

2. Parametric Estimation

2.1 Maximum Likelihood Method

Suppose that a NHPP with intensity function $\lambda(t)$ given in equation 1 is observed over a fixed interval $(0,T)$. Let n denotes the total number of events occurring during this time interval. If these events occur at the epochs $0 < t_1 < t_2 < \ldots < t_n \leq T$, then the likelihood function is given by;

$$\begin{aligned} L(\alpha,\beta) &= \left(\prod_{i=1}^{n}\lambda(t_i)\right)\exp\left(-\int_0^T \lambda(s)\,ds\right) \\ &= \exp\left(n\alpha + \beta\sum_{i=1}^{n} t_i - \frac{\exp(\alpha)}{\beta}\left(\exp(\beta T)-1\right)\right) \end{aligned} \tag{2}$$

Therefore, the log-likelihood function, obtained by taking the natural logarithm of the likelihood function, will be given by;

$$l(\alpha,\beta)=\log(L)=n\alpha+\beta\sum_{i=1}^{n}t_i-\frac{e^{\alpha}\left(e^{\beta T}-1\right)}{\beta} \tag{3}$$

The maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ are the respective values of α and β that maximize $l(\alpha,\beta)$. By partially differentiating (3) with respect to α and β we obtain the score functions which when equated to zero yields the following system of equations:

$$e^{\hat{\alpha}}=\frac{n\hat{\beta}}{e^{\hat{\beta}T}-1} \tag{4}$$

$$\sum_{i=1}^{n}t_i=\frac{nT}{1-e^{-\hat{\beta}T}}-\frac{n}{\hat{\beta}} \tag{5}$$

Equations (4) and (5) are solved simultaneously to obtain the maximum likelihood estimators for α and β. Since explicit solution for equation (5) does not exist, it may be solved numerically using the Newton Raphson method.

2.2 Likelihood Confidence Region for α and β

One of the main objectives of this paper was to construct approximate confidence region for the parameters of the intensity function given in (1). This is achieved by using the joint relative likelihood function of α and β. The joint relative likelihood function (RLF) of α and β denoted by $R(\alpha,\beta)$, is defined as the ratio of the likelihood function $L(\alpha,\beta)$ to its maximum value $L(\hat{\alpha},\hat{\beta})$ [9]. This is given by;

$$R(\alpha,\beta)=\frac{L(\alpha,\beta)}{L(\hat{\alpha},\hat{\beta})}=\frac{e^{n\alpha+\beta\sum_{i=1}^{n}t_i-\frac{e^{\alpha}\left(e^{\beta T}-1\right)}{\beta}}}{e^{n\hat{\alpha}+\hat{\beta}\sum_{i=1}^{n}t_i-\frac{e^{\hat{\alpha}}\left(e^{\hat{\beta}T}-1\right)}{\hat{\beta}}}} \tag{6}$$

The $100p\%$ likelihood region is the set of all parameter values (α,β) such that $R(\alpha,\beta)\geq p$ and the closed curve $R(\alpha,\beta)=p$ which forms the boundary of this region is the $100p\%$ likelihood contour. The $100p\%$ likelihood region for α and β is an approximate $100(1-p)\%$ confidence region.

2.3 Likelihood Confidence Intervals

The significance of interval estimates is to confirm the accuracy of the point estimates. We approximate separate confidence intervals of α and β were obtained using their maximized relative likelihood functions $R_{\max}(\alpha)$ and $R_{\max}(\beta)$ respectively. The maximized likelihood function of α $R_{\max}(\alpha)$ is obtained by maximizing $R(\alpha,\beta)$ over β with α fixed. That is

$$R_{\max}(\alpha)=R(\alpha,\hat{\beta})=\frac{e^{n\hat{\alpha}+\beta\sum_{i=1}^{n}t_i-\frac{e^{\hat{\alpha}}\left(e^{\beta T}-1\right)}{\beta}}}{e^{n\hat{\alpha}+\beta\sum_{i=1}^{n}t_i-\frac{e^{\hat{\alpha}}\left(e^{\hat{\beta}T}-1\right)}{\hat{\beta}}}} \tag{7}$$

The $100p\%$ maximized likelihood interval estimate for β will be the set of all values for which $R_{\max}(\beta)\geq p$. The endpoints of this interval estimate are obtained as the solution of the equation, $r_{\max}(\beta)-\log(p)=0$ which can be solved numerically. The likelihood interval contains β values such that, for some α, the pair (α,β) are contained in the $100p\%$ likelihood region.

Similarly, $R_{\max}(\alpha)=R(\alpha,\hat{\beta})$

$$=\frac{e^{n\alpha+\hat{\beta}\sum_{i=1}^{n}t_i-\frac{e^{\alpha}\left(e^{\hat{\beta}T}-1\right)}{\hat{\beta}}}}{e^{n\hat{\alpha}+\hat{\beta}\sum_{i=1}^{n}t_i-\frac{e^{\hat{\alpha}}\left(e^{\hat{\beta}T}-1\right)}{\hat{\beta}}}} \tag{8}$$

and the $100p\%$ maximized likelihood interval for α will be the set of all values for which $R_{\max}(\alpha)\geq p$. This interval contains α values such that, for some β, the pair (α,β) are contained in the $100p\%$ likelihood region. To obtain the desired approximate confidence interval, the $100p\%$ likelihood interval is computed and the value of p that gives the desired coverage probability is selected. For instance, the 14.7% likelihood interval corresponds approximately to 95% confidence interval.

3. Simulation Results

Simulated data sets from log-linear NHPP with parameter set $(\alpha,\beta)=(0.5,0.1)$ for different small sample sizes were utilised to illustrate both point and interval estimation procedures described in section 2. The data were simulated using the thinning algorithm. The ML estimates for (α,β) for varying sample sizes are given in table 1 and the likelihood intervals for the two parameters are given in table 2.

Table 1: The ML estimates for different values of n.

n	$\hat{\alpha}$	$\hat{\beta}$
21	0.5467	0.0638
23	0.5440	0.0825
24	0.5945	0.0810
28	0.4611	0.1360
30	0.4895	0.1435

Table 2: The 95% approximate confidence intervals for different values of n

		RLF approach
n	α	β
21	0.0866, 0.9492	0.0000, 0.1345
23	0.1059, 0.9325	0.0035, 0.1491
24	0.1663, 0.9700	0.0049, 0.1463
28	0.0667, 0.8100	0.0700, 0.1940
30	0.1094, 0.8240	0.0776, 0.1993

Based on the same five datasets, the interval estimates for β obtained using the large sample approximation procedure are given in table3.

Table 3: The 95% approximate confidence intervals for different values of n.

N	β
21	0.0136, 0.2289
23	0.0316, 0.2394
24	0.0302, 0.2344
28	0.0790, 0.2821
30	0.0852, 0.2853

The interval estimates for β (obtained using both approaches) corresponding to the sample size n=23 are plotted on the likelihood function to illustrate the plausibility of an interval estimate for the case of small sample sizes. This plot is presented in figure 1.

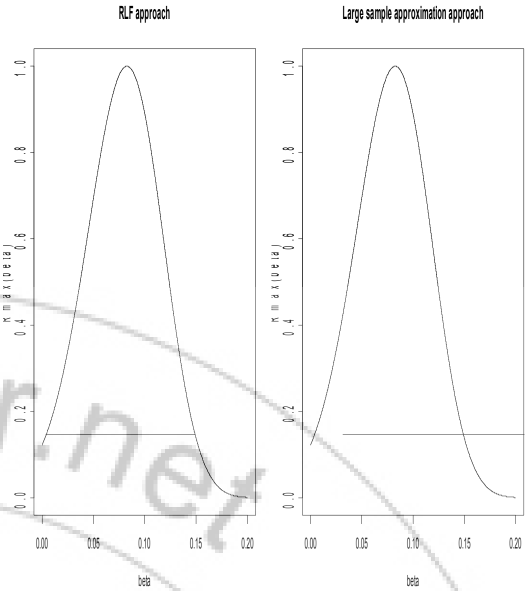

Figure 1: Plots of maximized relative likelihood function of **β** for n=23

A separate data set was simulated for different large sample sizes and parameter set $(\alpha,\beta)=(0.5,0.1)$. Using this data set, the maximum likelihood estimates for the two parameters and the 95% approximate confidence intervals were computed and are given in tables 4 and 5 respectively.

Table 4: The ML estimates for different values of n

n	$\hat{\alpha}$	$\hat{\beta}$
56	0.4502	0.1027
69	0.4705	0.0861
77	0.5156	0.1025
82	0.5453	0.1056
93	0.4530	0.1032

Table 5: The 95% approximate confidence intervals for different values of n

n	α	β
56	0.1756, 0.6995	0.0757, 0.1273
69	0.2251, 0.6983	0.0660, 0.1045
77	0.2835, 0.7311	0.0821, 0.1207
82	0.3211, 0.7545	0.0860, 0.1231
93	0.2428, 0.6496	0.0874, 0.1177

Based on the large sample data, the 95% approximate confidence intervals for β were also computed using the large sample approximation approach and the results are presented in table 6.

Table 6: The 95% approximate confidence intervals for different values of n

n	β
56	0.0749, 0.1663
69	0.0630, 0.1338
77	0.0759, 0.1514
82	0.0785, 0.1530
93	0.0769, 0.1435

As in the case of small samples, the interval estimates for β (obtained using both approaches) corresponding to the sample size $n = 69$ are plotted on the likelihood function to illustrate the plausibility of an interval estimate for the case of large sample sizes. This plot is presented in figure 2.

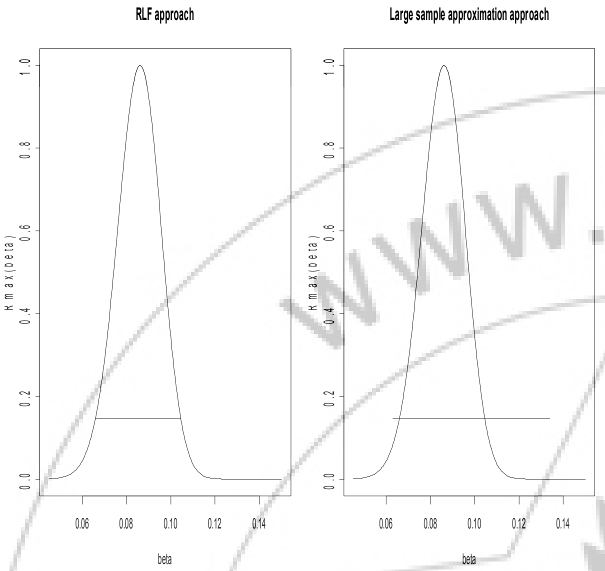

Figure 2: Plots of maximized relative likelihood function of $\boldsymbol{\beta}$ for n=69

Using an additional data set that was simulated for $n = 26$ and parameter set $(\alpha, \beta) = (0.5, 0.1)$, the log linear intensity function parameters were estimated as $\left(\hat{\alpha} = 0.5302, \hat{\beta} = 0.1091\right)$ and the 95% approximate confidence intervals were $(0.1199, 0.8911)$ and $(0.0372, 0.1704)$ for α and β, respectively. Further, the joint estimation for the two parameters was done using the joint relative likelihood function in which, the 75%, 50% and 10% contour likelihood regions for appropriate values of α and β were plotted as shown in figure 3.

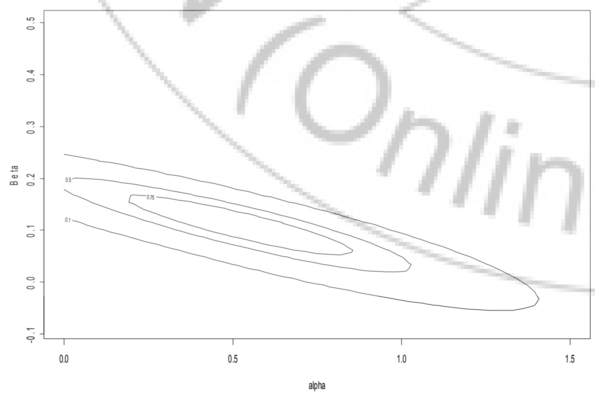

Figure 3: The contour likelihood regions for α and β

One thousand data sets were then simulated for the determination of coverage probability for the 14.7% likelihood intervals (approximate 95% confidence interval) and the 5% likelihood region (95% approximate confidence region). The coverage probability for the approximate interval for α was estimated to be 0.944 while the coverage probability for the confidence interval for β was estimated to be 0.945. For the joint estimation, the coverage probability was estimated to be 0.947.

4. Discussion

In this paper, we have considered interval estimation for the parameters of a log-linear NHPP based on the likelihood approach. We have used the relative likelihood function to construct the likelihood intervals of specified levels and used them to approximate the desired confidence intervals for the parameters of the intensity function. Our simulation results in tables 2, 3,5 and 6 demonstrate that the relative likelihood function approach yields narrower interval estimates than the large sample approximation approach for both small and large samples. For instance, when the sample size is $n = 21$ the width of the interval estimate obtained using the relative likelihood function is 0.1345 while the width under the large sample approximation approach is 0.2153. It is evident in figures 1 and 2 that interval estimates obtained, for both small and large sample sizes, using the large sample approximation approach may be imprecise and may include values that are very implausible and exclude those that are plausible. Employment of the RLF approach to interval estimation is thus convenient in that it is applicable to both small and large samples. The technique applied in this paper can also be applied to other NHPP with other forms of intensity functions such as the weibull process and the linear rate.

References

[1] Slavik, J. (2004) Estimation of Parameters in Nonhomogeneous Poisson Process. Journal of Information, Control and Management Systems, vol. 2, (2004), No. 2

[2] Kuhl, M. E., Deo, S. C. and Wilson, J. R. (2008). Smooth Flexible Models of Non-homogeneous Poisson Process Using One or More Process Realizations. Proceedings of the 2008 Winter Simulation Conference. S. J. Mason, R. R. Hill, L. Monch, O. Rose, T. Jefferson, J. W. Fowler eds.

[3] Leemis, L. M. (1991). Nonparametric Estimation of the Cummulative Intensity Function for a Non-homogeneous Poisson Process. Management Science.27 (7): 886-900

[4] Belitser, E., Serra, P. and Zanten, H. V. (2012).Estimating the Period of a Cyclic Nonhomogeneous Posson Process. Scadinavian Journal of Statistics.40 issue 2 204-218

[5] Lee, S., Wilson, J.R. and Crawford, M.M. (1991). Modelling and Simulation of a non-homogeneous Poisson Process Having Cyclic Behaviour, communications in statistics-simulation and Computation, 20:2&3, 777-809

[6] Leemis, L. M. (1990). Nonparametric Estimation of the Cumulative Intensity Function for a Non-homogeneous Poisson Process. Management Science 37

[7] Cox, D. R. and Lewis, P.A.W. (1966).The Statistical Analysis of Series of Events, Chapman and Hall, London.
[8] Khvatskin, L. V., Frenkel, I. B., and Gertsbakh, I. B., (2003). Parameter Estimation and Hypothesis Testing of Nonhomogeneous Poisson Process. Transport and Telecommunication.Vol.4, N 2, 2003
[9] Kalbfleisch, L. G. (1985). Probability and Statistical Inference.2nd edition, Springer-Verag, New York.

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