Variance Estimation in Stratified Random Sampling in the Presence of Two Auxiliary Random Variables

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Abstract: ***The objective of this paper is to develop an improved population variance estimators in the presence of two auxiliary variables in stratified random sampling adapting the family of estimators proposed by Koyunchu and Kadilar (2009) for the estimation of population mean in stratified random sampling using prior information of the two auxiliary variables. In this paper, we proposed ratio-product type estimators and derived their mean square errors using first order approximation of Taylor series method. Efficiency comparisons of proposed estimators with respect to their mean square errors have been discussed and achieved improvement under certain conditions. Results are also supported by numerical analysis. Based on results, the proposed ratio- type variance estimators may be preferred over traditional ratio-type and sample estimator of population variance for the use in practical applications.***

Keywords: Variance estimator; Ratio-product type estimators, Mean square error, Auxiliary information; Efficiency; Stratified random sampling

1. Introduction

In sample surveys, it is well known that to use information of auxiliary variable(s) to estimate unknown population parameter(s) in various sampling designs. In sampling literature, many Authors have used information of auxiliary variables such as population mean, variance, kurtosis, skewness, etc to estimating population mean and variance of the study variable. Many authors who done important work in this area, were Das and Tripathi (1978), Srivastavaetal and Jhajji (1980,1983,1995), Isakietal (1983,2000), Singh and Kataria (1990), Prasad and Singh (1990,1992), Ahmedetal.(2000,2003), Gupta and Shabbir (2006). Kadilar and Cingi (2006) studied population variance of interest variable using population mean, variance, kurtosis and coefficient of variation of auxiliary variable in simple and stratified random sampling. Recently Olufadi and Kadilar (2014) estimated the population variance of interest variate in simple and two-phase sampling by using the variance of auxiliary variables and got interesting results. This paper mainly focuses on population variance estimators using prior knowledge of two auxiliary variables in stratified sampling design.

Consider a finite population P $=\{P_1, P_2, P_3, \dots, P_N\}$ of N units. Let the study and two auxiliary variables are denoted by Y, X and Z associate with each P_j (j=1,2…,N) of the population respectively. Let the population is stratified in to K strata with h^{th} stratum containing $N_h$units, where h=1,2,3,…,K such that $\sum_{h=1}^{k} N_h$=N and from the h^{th} stratum, a sample n_h is drawn by simple random sampling without replacement such that $\sum_{h=1}^{k} n_h$=n. Let (y_{hi}, x_{hi}, z_{hi}) denote the observed values of Y, X, and Z on the i^{th} unit of the h^{th} stratum where i=1,2,…,N_h. The population variance of the study variable (y) and the auxiliary variables are defined as follows.

$$(N-1){S^2}_{st,y} = \sum_{h=1}^{k}\sum_{i=1}^{N_h}(y_{hi}-\bar{Y})^2 = \sum_{h=1}^{k}\sum_{i=1}^{N_h}[(y_{hi}-\bar{Y}_h)+(\bar{Y}_h-\bar{Y})]^2$$

Where $\bar{Y}_h$the population is mean of the variate of interest in stratum h, and $y_{hi}$is the value of the i^{th} observation of interest variate in stratum h. For large sample size, assuming that $N \cong N-1$ and $N_h \cong N_h - 1$, then ${S^2}_{st,y} \cong \sum_{h=1}^{k}\omega_h {S^2}_{yh} + \sum_{h=1}^{k}\omega_h(\bar{Y}_h - \bar{Y})^2$

$\bar{y}_h = \frac{1}{n_h}\sum_{h=1}^{k} y_{hi}$ - sample mean of h^{th} stratum.

$\bar{y}_{st} = \sum_{h=1}^{k}\omega_h \bar{y}_h$ - is the sample estimator of population mean of the study variable.

$\bar{Y}_h = \frac{1}{N_h}\sum_{h=1}^{N_h} y_{hi}$ -population mean of h^{th} stratum.

${S^2}_{yh} = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}(y_{hi}-\bar{Y}_h)^2$ - population variance of h^{th} stratum.

${s^2}_{yh} = \frac{1}{n_h - 1}\sum_{i=1}^{n_h}(y_{hi}-\bar{y}_h)^2$-sample estimator of population variance in the h^{th} stratum. Similar expression are defined for the auxiliary variables x and z.

2. Adapted estimators

Koyuncu and Cem Kadilar (2009), defined the classical ratio estimator to estimate the population mean of the study variable Y in the stratified random sampling when there are two auxiliary variables as follows:

$$\bar{y}_t = \bar{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right)\left(\frac{\bar{Z}}{\bar{z}_{st}}\right) \text{ -------------------------- (1)}$$

Where $\bar{X}$ and $\bar{Z}$ are the population mean of the two auxiliary variables and $\bar{x}_{st}$,$\bar{z}_{st}$ and $\bar{y}_{st}$ are sample estimate of the population mean in stratified random sampling scheme. The regression estimator of the population mean $\bar{Y}$ also defined as:

$$\bar{Y}_{reg} = \bar{y}_{st} + \beta_1(\bar{X}-\bar{x}_{st}) + \beta_2(\bar{Z}-\bar{z}_{st}) \text{ ------ (2)}$$

Where $\beta_1 = \frac{s_{yx}}{{s^2}_x}$ and $\beta_2 = \frac{s_{yz}}{{s^2}_z}$. Adapting the estimator given in (1) and (2) to the estimator for the population variance of the study variable y and assuming the population variance of the two auxiliary variables in each stratum is known, we

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develop the following ratio -product type and regression estimators:

$$s^2{}_t = s^2{}_{st,y}\left(\frac{S^2{}_x}{s^2{}_{st,x}}\right)\left(\frac{S^2{}_z}{s^2{}_{st,z}}\right) \text{ ---------------------------- (3)}$$

$$s^2{}_{reg} = s^2{}_{st,y} + \beta_1(S^2{}_x - s^2{}_{st,x}) + \beta_2(S^2{}_z - s^2{}_{st,z}) \text{ ------(4)}$$

Where $s^2{}_{st,x} = \sum_{h=1}^{k}\omega_h s^2{}_{xh} + \sum_{h=1}^{k}\omega_h(\bar{x}_h - \bar{x}_{st})^2$, $s^2{}_{st,y} = \sum_{h=1}^{k}\omega_h s^2{}_{yh} + \sum_{h=1}^{k}\omega_h(\bar{y}_h - \bar{y}_{st})^2$, and $s^2{}_{st,z} = \sum_{h=1}^{k}\omega_h s^2{}_{zh} + \sum_{h=1}^{k}\omega_h(\bar{z}_h - \bar{z}_{st})^2$, are the sample estimator of population variance of each variables in stratified sampling scheme when neglecting population correction factor of each stratum. The mean square error of the variance estimator, given in (3) and (4), is obtained as follows:

$$MSE(s^2{}_t) \cong \frac{S^4{}_x S^4{}_z}{\nu^2{}_1\nu^2{}_2}\left[H_1 + H_2\right] \text{-------------- (5)}$$

$$MSE(s^2{}_{reg}) \cong \left[H_1 + H_3\right] \text{-------------------- (6)}$$

[see Appendix (A. 3) and (B. 1)]

3. The Proposed Estimators

In this section some variance estimators are proposed using the variance of two auxiliary variables, population kurtosis, coefficient of variation and their combination. Motivated by Cingi and Kadilar (2005a, 2006b) and Koyuncu and Kadilar (2009), the following population variance estimators are proposed in the stratified random sampling:

$$s^2{}_{pr_1} = s^2{}_{st,y}\,\frac{\left(S^2{}_x+\beta_2(x)\right)\left(S^2{}_z+\beta_2(z)\right)}{\left(s^2{}_{st,x}+\beta_2(x)\right)\left(s^2{}_{st,z}+\beta_2(z)\right)} \text{ ----------------- (7)}$$

$$s^2{}_{pr_2} = s^2{}_{st,y}\,\frac{\left(S^2{}_x C_x+\beta_2(x)\right)\left(S^2{}_z C_z+\beta_2(z)\right)}{\left(s^2{}_{st,x} C_x+\beta_2(x)\right)\left(s^2{}_{st,z} C_z+\beta_2(z)\right)} \text{ ------------ (8)}$$

The MSE of the estimators, given in (7) and (8) is found using the first degree approximation of Taylor series method as follows:

$$MSE(s^2{}_{pr_1}) \cong \frac{\left(\left(S^2{}_x+\beta_2(x)\right)\left(S^2{}_z+\beta_2(z)\right)\right)^2}{\left(\left(\nu_1+\beta_2(x)\right)\left(\nu_2+\beta_2(z)\right)\right)^2}\left\{H_1 + H_4\right\} \text{- (9)}$$

$$MSE(s^2{}_{pr_2}) \cong \frac{\left(\left(S^2{}_x C_x+\beta_2(x)\right)\left(S^2{}_z C_z+\beta_2(z)\right)\right)^2}{\left(\left(\nu_1 C_x+\beta_2(x)\right)\left(\nu_2 C_z+\beta_2(z)\right)\right)^2}\left\{H_1 + H_5\right\} \text{-- (10)}$$

(See Appendix C and D) where C_x and C_z - are population coefficient of variation of the auxiliary variables (X) and (Z) respectively. $\beta_2(x)$ and $\beta_2(z)$ are the population kurtosis of the auxiliary variables (X) and (Z) respectively. The detail derivations of all the mean square error of the estimators considered in this paper was presented in appendix at the end of the paper.

4. Efficiency Comparison of the Estimators

In this section, we compare the performance of the proposed estimators with other estimators considered here and some efficiency comparison condition is carry out under which the proposed estimators are more efficient than the usual sample estimator of population variance and the adapted variance estimators considered in this paper. These conditions are given as follows:

$$MSE(s^2{}_t)\text{- }MSE(s^2{}_{st,y}) < 0 \text{ if } H_1 < -\frac{H_2 S^4{}_x S^4{}_z}{S^4{}_x S^4{}_z - \nu^2{}_1\nu^2{}_2} \text{ --- (11)}$$

$$MSE(s^2{}_{reg})\text{- }MSE(s^2{}_{st,y}) < 0 \text{ if } H_3 < 0 \text{ ------------- (12)}$$

$$MSE\left(s^2{}_{pr_1}\right)\text{- }MSE(s^2{}_{st,y}) < 0 \text{ if } H_1 < -\frac{H_4\left(\left(S^2{}_x+\beta_2(x)\right)\left(S^2{}_z+\beta_2(z)\right)\right)^2}{\left(\left(S^2{}_x+\beta_2(x)\right)\left(S^2{}_z+\beta_2(z)\right)\right)^2-\left(\left(\nu_1+\beta_2(x)\right)\left(\nu_2+\beta_2(z)\right)\right)^2} \text{ ---- (13)}$$

$$MSE\left(s^2{}_{pr_2}\right)\text{- }MSE(s^2{}_{st,y}) < 0 \text{ if } H_1 < -\frac{H_5\left(\left(S^2{}_x C_x+\beta_2(x)\right)\left(S^2{}_z C_z+\beta_2(z)\right)\right)^2}{\left(\left(S^2{}_x C_x+\beta_2(x)\right)\left(S^2{}_z C_z+\beta_2(z)\right)\right)^2-\left(\left(\nu_1 C_x+\beta_2(x)\right)\left(\nu_2 C_z+\beta_2(z)\right)\right)^2} \text{ (14)}$$

Where H_i for i=2,3,4,5- is the term of each mean square error with out the common multiplier of all terms and H_1. The other method which is used to compare the performance of the proposed estimators over $s^2{}_t$ is Percent Relative Efficient (PRE). The Percent Relative Efficiencies (PREs) of the different estimators are computed with respect to the adapted estimator $s^2{}_t$ using the formula:

$$PRE(s^2{}_t, s^2{}_{pr_i}) = \frac{MSE(s^2{}_t)}{MSE(s^2{}_{pr_i})} \times 100 \text{ for i=1, 2 ---------- (15)}$$

5. Empirical Study

In this section, the performance of the suggested estimators have been analyzed with respect to the estimators considered in this paper. To achieve this, the data set of state wise area, production and productivity of major spices in India was used. In this data set, the study variable (Y) is productivity in metric tons , the first auxiliary variable (X) is area in thousand hectares , and the second auxiliary variable (Z) is production in thousand tons. From each stratum 12 states are selected. The summary of the data is given in the following tables.

Table 1: Data statistics

N_h	n_h	$\bar{X}_h$	$\bar{Y}_h$	$\bar{Z}_h$	C_{xh}	C_{yh}	C_{zh}	$\beta_2(x_h)$	$\beta_2(y_h)$	$\beta_2(z_h)$
29	12	90.2534	2.2252	150.23	1.524	0.745	1.502	6.72	2.411	12.383
29	12	90.6693	2.3486	142.95	1.515	0.781	1.722	6.23	2.476	13.564
29	12	84.9562	2.3434	138.48	1.443	0.766	1.669	5.361	2.584	14.257

Table 2: Data statistics of parameters

Parameters	Stratum I	Stratum II	Stratum III
$\theta_h(yx)$	1.643×10^{-5}	1.42×10^{-5}	1.819×10^{-6}
$\theta_h(yz)$	5.6047×10^{-6}	3.9563×10^{-6}	4.2985×10^{-6}
$\theta_h(xz)$	2.774×10^{-9}	2.782×10^{-9}	4.4645×10^{-6}

S_{yx}	50.5266	72.5386	60.66
S_{yz}	24	2.532	0.927
S_{xz}	25758.621	23551.724	20482.7586

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Table 3: Values of parameters

ν_0 =3.378592 ν_1 = 18960.84 ν_2 = 64358.93

S^2_x =4400 S^2_z =14945.833

$\beta_2 = 5.24\times 10^{-6}$

$\beta^*_1 =$ 9.61×10^{-6} $\beta^*_2 = 4.12 \times 10^{-4}$

$\beta_1 = 3.11\times 10^{-5}$

C_z =1.631 C_x =1.5 $\beta_2(x)$= 6.5522 $\beta_2(z)$ =14.4724

Table 4: Summary of μ_{rsth}

μ_{rsth}	Stratum I	Stratum II	Stratum III
μ_{300h}	2.890	4.771	4.506
μ_{210h}	-103.789	-127.5024	-121.621
μ_{201h}	-135.134	-197.1155	-194.107
μ_{120h}	-10344.828	-12896.552	-8379.31
μ_{102h}	55517.241	52034.483	41034.483
μ_{030h}	4827586.207	4620689.655	2965517.24
μ_{021h}	5413793.103	4620689.655	3551724.138
μ_{012h}	12206896.552	11068965.517	9413793.103
μ_{003h}	46551724.138	44827586.207	37586206.897

Table 5: PRE of the different estimators with respect to s^2_t

Estimators	$s^2_{st,y}$	s^2_t	s^2_{reg}	$s^2_{pr_1}$	$s^2_{pr_2}$
PRE	4.52	100	4.3458	107.163	113.5684

Table 6: Estimators with their MSE values

Estimators	$s^2_{st,y}$	s^2_t	s^2_{reg}	$s^2_{pr_1}$	$s^2_{pr_2}$
MSE values	0.420353	0.019	0.4372	0.01772	0.01673

6. Conclusion

Table 5 reveals that the suggested estimators $s^2_{pr_i}$, for i= 1,2 has the highest PRE among other estimators considered in this paper. So that the suggested estimators in stratified random sampling provides a sufficient improvement in variance estimation compared to the s^2_t. It is also observed from Table 5 that the sample and regression estimators are less efficient than s^2_t.Table 6shows that the proposed estimators of S^2_y is more efficient than the traditional estimator of population variance of interest variable in stratified random sampling according to the data set of a population considered in this paper. Theoretically, it has been established that, in general, the regression type estimator is more efficient than the ratio-type estimators. However, in this paper the regression estimator of S^2_y is not efficient than the sample estimator and the proposed ratio-type estimators of population variance of interest variable. From the above results and discussion it is observed that incorporating prior information's obtained from the two auxiliary variables improves population variance of interest variable in stratified random sampling scheme. As a recommendation based on results, the proposed ratio-type variance estimators may be preferred over traditional ratio- type and sample estimator of population variance for the use in practical applications. This paper can be improved by adding higher order Taylor series terms. In forthcoming studies, we recommended to develop improved variance estimators by adapting the estimators of Rajesh Singi and Mukesh Kumar (2012).

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Appendixes

Appendix A

The MSE of the ratio type variance estimator in the stratified random sampling in the presence of two auxiliary variables can be obtained using the first degree approximation in the Taylor series method defined by

$$MSE(s^2{}_t) \cong \sum_{h=1}^{k} d_h \Sigma_h d'_h \quad \text{--} \quad (A.1)$$

Where

$d_h = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 \end{bmatrix}$ *such that*

$d_1 = \frac{\partial}{\partial a} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$ $d_2 = \frac{\partial}{\partial b} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$,

$d_3 = \frac{\partial}{\partial c} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$

$d_4 = \frac{\partial}{\partial d} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$,

$d_5 = \frac{\partial}{\partial e} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$

$d_6 = \frac{\partial}{\partial f} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$,

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$d_7 = \frac{\partial}{\partial g} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$

$d_8 = \frac{\partial}{\partial h} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$,

$d_9 = \frac{\partial}{\partial i} h(a,b,c,d,e,f,g,h,i)|_{s^2{}_{yh},\bar{Y}_h,\bar{Y},s^2{}_{xh},\bar{X}_h,\bar{X},s^2{}_{zh},\bar{Z}_h,\bar{Z}}$ *and*

$$\Sigma_h = \begin{bmatrix} \sigma^2{}_1 & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} & \sigma_{17} & \sigma_{18} & \sigma_{19} \\ \sigma_{21} & \sigma^2{}_2 & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} & \sigma_{27} & \sigma_{28} & \sigma_{29} \\ \sigma_{31} & \sigma_{32} & \sigma^2{}_3 & \sigma_{34} & \sigma_{35} & \sigma_{36} & \sigma_{37} & \sigma_{38} & \sigma_{39} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma^2{}_4 & \sigma_{45} & \sigma_{46} & \sigma_{47} & \sigma_{48} & \sigma_{49} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma^2{}_5 & \sigma_{56} & \sigma_{57} & \sigma_{58} & \sigma_{59} \\ \sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma^2{}_6 & \sigma_{67} & \sigma_{68} & \sigma_{69} \\ \sigma_{71} & \sigma_{72} & \sigma_{73} & \sigma_{74} & \sigma_{75} & \sigma_{76} & \sigma^2{}_7 & \sigma_{78} & \sigma_{79} \\ \sigma_{81} & \sigma_{82} & \sigma_{83} & \sigma_{84} & \sigma_{85} & \sigma_{86} & \sigma_{87} & \sigma^2{}_8 & \sigma_{89} \\ \sigma_{91} & \sigma_{92} & \sigma_{93} & \sigma_{94} & \sigma_{95} & \sigma_{96} & \sigma_{97} & \sigma_{98} & \sigma^2{}_9 \end{bmatrix} \text{-------------------------}[A.2]$$

Here $h(a,b,c,d,e,f,g,h,i) = h\left(s^2{}_{yh}, \bar{y}_h, \bar{y}_{st}, s^2{}_{xh}, \bar{x}_h, \bar{x}_{st}, s^2{}_{zh}, \bar{z}_h, \bar{z}_{st}\right)$ and Σ_h is the variance-covariance matrixes of $h(a,b,c,d,e,f,g,h,i)$. Note that $\bar{X}_{st} = \sum_{h=1}^{k} \omega_h \bar{X}_h = \bar{X}$,
$\bar{Y}_{st} = \sum_{h=1}^{k} \omega_h \bar{Y}_h = \bar{Y}$ and $\bar{Z}_{st} = \sum_{h=1}^{k} \omega_h \bar{Z}_h = \bar{Z}$. According to equation (A.2), we obtain d_h for the estimator, $s^2{}_t$ as follows,

Let $v_0 = \sum_{h=1}^{k} \omega_h S^2{}_{yh} + \sum_{h=1}^{k} \omega_h (\bar{Y}_h - \bar{Y})^2$

$v_1 == \sum_{h=1}^{k} \omega_h S^2{}_{xh} + \sum_{h=1}^{k} \omega_h (\bar{X}_h - \bar{X})^2$

$v_2 == \sum_{h=1}^{k} \omega_h S^2{}_{zh} + \sum_{h=1}^{k} \omega_h (\bar{Z}_h - \bar{Z})^2$, then we have

$$d_h = \frac{s^2{}_X s^2{}_z}{v_1 v_2} \begin{bmatrix} \omega_h \; 2\omega_h(\bar{Y}_h - \bar{Y}) \; -2\omega_h(\bar{Y}_h - \bar{Y}) - \frac{v_0 \omega_h}{v_1} \; - \frac{2v_0\omega_h(\bar{X}_h - \bar{X})}{v_1} \; \frac{2v_0\omega_h(\bar{X}_h - \bar{X})}{v_1} \; - \frac{v_0\omega_h}{v_2} \\ - \frac{2v_0\omega_h(\bar{Z}_h - \bar{Z})}{v_2} \; \frac{2v_0\omega_h(Z_h - \bar{Z})}{v_2} \end{bmatrix}$$

We obtain the MSE of $s^2{}_t$ using (A.1), as

$$MSE(s^2{}_t) \cong \frac{s^4{}_x s^4{}_z}{v^2{}_1 v^2{}_2} \left[H_1 + H_2\right] \text{--} \text{(A.3)}$$

Where $H_1 = \sum_{h=1}^{k} \omega^2{}_h V\left(s^2{}_{yh}\right) + 4\sum_{h=1}^{k} \omega^2{}_h\left(\bar{Y}_h - \bar{Y}\right)\left[COV\left(\bar{y}_h, s^2{}_{yh}\right) - COV\left(\bar{y}_{st}, s^2{}_{yh}\right)\right] + 4 \sum_{h=1}^{k} \omega^2{}_h\left(\bar{Y}_h - \bar{Y}\right)^2\left[V\left(\bar{y}_h\right) - 2COV\left(\bar{y}_h, \bar{y}_{st}\right) + V\left(\bar{y}_{st}\right)\right]$

$H_2 = -4\sum_{h=1}^{k} v_0\omega^2{}_h(\bar{Y}_h - \bar{Y})\left[\frac{1}{v_1}\left(cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})\right) + \frac{1}{v_2}\left(cov(\bar{y}_h, s^2{}_{zh}) - cov(\bar{y}_{st}, s^2{}_{zh})\right)\right] - 2\sum_{h=1}^{k} \frac{v_0\omega^2{}_h}{v_1} cov\left(s^2{}_{xh}, s^2{}_{yh}\right) - 4\sum_{h=1}^{k} \frac{v_0\omega^2{}_h(\bar{X}_h - \bar{X})}{v_1}\left[cov\left(\bar{x}_h, s^2{}_{yh}\right) - cov\left(\bar{x}_{st}, s^2{}_{yh}\right) - \frac{v_0}{v_1}\left(cov(\bar{x}_h, s^2{}_{xh}) - cov(\bar{x}_{st}, s^2{}_{xh})\right) - \frac{v_0}{v_2}\left(cov(\bar{x}_h, s^2{}_{zh}) - cov(\bar{x}_{st}, s^2{}_{zh})\right)\right] - 2\sum_{h=1}^{k} \frac{v_0\omega^2{}_h}{v_2} cov\left(s^2{}_{zh}, s^2{}_{yh}\right) - 8\sum_{h=1}^{k} \frac{1}{v_1} v_0\omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{X}_h - \bar{X})[cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h) + cov(\bar{y}_{st}, \bar{x}_{st})] - 8\sum_{h=1}^{k} \frac{v_0\omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})}{v_2}[cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h) + cov(\bar{y}_{st}, \bar{z}_{st})] + \sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h}{v^2{}_1} v(s^2{}_{xh}) + 2\sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h}{v_1 v_2} cov(s^2{}_{xh}, s^2{}_{zh}) + 4\sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h(\bar{X}_h - \bar{X})^2}{v^2{}_1}[v(\bar{x}_h) - 2cov(\bar{x}_h, \bar{x}_{st}) + v(\bar{x}_{st})] + \sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h}{v^2{}_2} v(s^2{}_{zh}) - 8\sum_{h=1}^{k} \frac{v_0\,\omega^2{}_h(\bar{X}_h - \bar{X})(\bar{Z}_h - \bar{Z})}{v_1 v_2}[cov(\bar{x}_h, \bar{z}_{st}) - cov(\bar{x}_h, \bar{z}_h) + cov(\bar{x}_{st}, \bar{z}_h) - cov(\bar{x}_{st}, \bar{z}_{st})] + 4\sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h(\bar{Z}_h - \bar{Z})^2}{v^2{}_2}[v(\bar{z}_h) - 2cov(\bar{z}_h, \bar{z}_{st}) + v(\bar{z}_{st})] - 4\sum_{h=1}^{k} \frac{v_0\omega^2{}_h(\bar{Z}_h - \bar{Z})}{v_2}\left[cov\left(\bar{z}_h, s^2{}_{yh}\right) - cov\left(\bar{z}_{st}, s^2{}_{yh}\right) - \frac{v_0}{v_1}\left(cov(\bar{z}_h, s^2{}_{xh}) - cov(\bar{z}_{st}, s^2{}_{xh})\right) - \frac{v_0}{v_2}\left(cov(\bar{z}_h, s^2{}_{zh}) - cov(\bar{z}_{st}, s^2{}_{zh})\right)\right]$

Appendix B

The MSE of the regression estimator for the population variance in the stratified random sampling in the presence of two auxiliary variables can be obtained using the first degree approximation in the Taylor series method as follows:

$d_h = [\,\omega_h \; 2\omega_h(\bar{Y}_h - \bar{Y}) \; -2\omega_h(\bar{Y}_h - \bar{Y}) \; -\beta_1\omega_h \; -2\beta_1\omega_h(\bar{X}_h - \bar{X}) \; 2\beta_1\omega_h(\bar{X}_h - \bar{X}) \; -\beta_2\omega_h \; -2\beta_2\omega_h(\bar{Z}_h - \bar{Z}) \; 2\beta_2\omega_h(\bar{Z}_h - \bar{Z})\,]$ *and* Σ_h *,using (A.1) and (A.2) ,*

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$MSE(s^2{}_{reg}) \cong [H_1 + H_3]$ -- (B.1)

Where $H_3 =$

$-2\beta_1 \sum_{h=1}^{k} \omega^2{}_h cov(s^2{}_{xh}, s^2{}_{yh}) + 4\beta_1 \sum_{h=1}^{k} \omega^2{}_h (\bar{X}_h - \bar{X}) [cov(\bar{x}_{st}, s^2{}_{yh}) - cov(\bar{x}_h, s^2{}_{yh}) + \beta_1 (cov(\bar{x}_h, s^2{}_{xh}) - cov(\bar{x}_{st}, s^2{}_{xh})) + \beta_2 (cov(\bar{x}_h, s^2{}_{zh}) - cov(\bar{x}_{st}, s^2{}_{zh}))] - 2\beta_2 \sum_{h=1}^{k} \omega^2{}_h cov(s^2{}_{yh}, s^2{}_{zh}) + 4\beta_2 \sum_{h=1}^{k} \omega^2{}_h (\bar{Z}_h - \bar{Z}) [cov(\bar{z}_{st}, s^2{}_{yh}) - cov(\bar{z}_h, s^2{}_{yh}) + \beta_1 (cov(\bar{z}_h, s^2{}_{xh}) - cov(\bar{z}_{st}, s^2{}_{xh})) + \beta_2 (cov(\bar{z}_h, s^2{}_{zh}) - cov(\bar{z}_{st}, s^2{}_{zh}))] + 4 \sum_{h=1}^{k} \omega^2{}_h (\bar{Y}_h - \bar{Y}) [\beta_1 (cov(\bar{y}_{st}, s^2{}_{xh}) - cov(\bar{y}_h, s^2{}_{xh})) - \beta_2 (cov(\bar{y}_h, s^2{}_{zh}) + cov(\bar{y}_{st}, s^2{}_{zh}))] + 2\beta_1\beta_2 \sum_{h=1}^{k} \omega^2{}_h cov(s^2{}_{xh}, s^2{}_{zh}) + 8\beta_2 \sum_{h=1}^{k} \omega^2{}_h (\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})[cov(\bar{y}_h, \bar{z}_{st}) - cov(\bar{y}_h, \bar{z}_h) + cov(\bar{y}_{st}, \bar{z}_h) - cov(\bar{y}_{st}, \bar{z}_{st})] + 8\beta_1 \sum_{h=1}^{k} \omega^2{}_h (\bar{Y}_h - \bar{Y})(\bar{X}_h - \bar{X})[cov(\bar{y}_h, \bar{x}_{st}) - cov(\bar{y}_h, \bar{x}_h) + cov(\bar{y}_{st}, \bar{x}_h) - cov(\bar{y}_{st}, \bar{x}_{st})] + 8 \beta_1\beta_2 \sum_{h=1}^{k} \omega^2{}_h (\bar{X}_h - \bar{X})(\bar{Z}_h - \bar{Z}) [cov(\bar{x}_h, \bar{z}_h) - cov(\bar{x}_h, \bar{z}_{st}) - cov(\bar{x}_{st}, \bar{z}_h) + cov(\bar{x}_{st}, \bar{z}_{st})] + \beta^2{}_1 \sum_{h=1}^{k} \omega^2{}_h\, v(s^2{}_{xh}) + 4\beta^2{}_1 \sum_{h=1}^{k} \omega^2{}_h (\bar{X}_h - \bar{X})^2 [v(\bar{x}_h) - 2cov(\bar{x}_h, \bar{x}_{st}) + v(\bar{x}_{st})] + \beta^2{}_2 \sum_{h=1}^{k} \omega^2{}_h\, v(s^2{}_{zh}) + 4\beta^2{}_2 \sum_{h=1}^{k} \omega^2{}_h (\bar{Z}_h - \bar{Z})^2 [v(\bar{z}_h) - 2cov(\bar{z}_h, \bar{z}_{st}) + v(\bar{z}_{st})]$

Appendix C

The MSE of the proposed estimator $s^2{}_{pr_1}$ and $s^2{}_{pr_2}$ for the population variance in the stratified random sampling in the presence of two auxiliary variables can be obtained using the first degree approximation in the Taylor series method as follows:

$$d_h = \frac{(s^2{}_x + \beta_2(x))(s^2{}_z + \beta_2(z))}{(v_1 + \beta_2(x))(v_2 + \beta_2(z))} \left[\omega_h \; 2\omega_h(\bar{Y}_h - \bar{Y}) \; - 2\omega_h(\bar{Y}_h - \bar{Y}) - \frac{v_0\omega_h}{(v_1+\beta_2(x))} - \frac{2v_0\omega_h(\bar{X}_h - \bar{X})}{(v_1+\beta_2(x))} \; \frac{2v_0\omega_h(\bar{X}_h - \bar{X})}{(v_1+\beta_2(x))} - \frac{v_0\omega_h}{(v_2+\beta_2(z))} - \frac{2v_0\omega_h(\bar{Z}_h - \bar{Z})}{(v_2+\beta_2(z))} \; \frac{2v_0\omega_h(\bar{Z}_h - \bar{Z})}{(v_2+\beta_2(z))} \right]$$

Using (A.1) and (A.2), we have

$MSE(s^2{}_{pr_1}) \cong \frac{((s^2{}_x + \beta_2(x))(s^2{}_z + \beta_2(z)))^2}{((v_1 + \beta_2(x))(v_2 + \beta_2(z)))^2} \{H_1 + H_4\}$ -- (C.1)

Where

$H_4 = \Big\{ -4 \sum_{h=1}^{k} \frac{v_0\omega^2{}_h(\bar{Y}_h - \bar{Y})}{(v_1+\beta_2(x))} (cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})) - 4 \sum_{h=1}^{k} \frac{v_0\omega^2{}_h(\bar{Y}_h - \bar{Y})}{(v_2+\beta_2(z))} (cov(\bar{y}_h, s^2{}_{zh}) - cov(\bar{y}_{st}, s^2{}_{zh})) - 2 \sum_{h=1}^{k} \frac{v_0\omega^2{}_h}{(v_1+\beta_2(x))} cov(s^2{}_{xh}, s^2{}_{yh}) - 4 \sum_{h=1}^{k} \frac{v_0\omega^2{}_h(\bar{X}_h - \bar{X})}{(v_1+\beta_2(x))} \Big[cov(\bar{x}_h, s^2{}_{yh}) - cov(\bar{x}_{st}, s^2{}_{yh}) - \frac{v_0}{(v_1+\beta_2(x))} (cov(\bar{x}_h, s^2{}_{xh}) - cov(\bar{x}_{st}, s^2{}_{xh})) - \frac{v_0}{(v_2+\beta_2(z))} (cov(\bar{x}_h, s^2{}_{zh}) - cov(\bar{x}_{st}, s^2{}_{zh})) \Big] - 2 \sum_{h=1}^{k} \frac{v_0\omega^2{}_h}{(v_2+\beta_2(z))} cov(s^2{}_{zh}, s^2{}_{yh}) - 8 \sum_{h=1}^{k} \frac{v_0\omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{X}_h - \bar{X})}{(v_1+\beta_2(x))} [cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h) + cov(\bar{y}_{st}, \bar{x}_{st})] - 8 \sum_{h=1}^{k} \frac{v_0\omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})}{(v_2+\beta_2(z))} [cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h) + cov(\bar{y}_{st}, \bar{z}_{st})] + \sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h}{(v_1+\beta_2(x))^2} v(s^2{}_{xh}) + 2 \sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h}{(v_1+\beta_2(x))(v_2+\beta_2(z))} cov(s^2{}_{xh}, s^2{}_{zh}) + 4 \sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h(\bar{X}_h - \bar{X})^2}{(v_1+\beta_2(x))^2} [v(\bar{x}_h) - 2cov(\bar{x}_h, \bar{x}_{st}) + v(\bar{x}_{st})] + \sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h}{(v_2+\beta_2(z))^2} v(s^2{}_{zh}) - 8 \sum_{h=1}^{k} \frac{v_0\,\omega^2{}_h(\bar{X}_h - \bar{X})(\bar{Z}_h - \bar{Z})}{(v_1+\beta_2(x))(v_2+\beta_2(z))} [cov(\bar{x}_h, \bar{z}_{st}) - cov(\bar{x}_h, \bar{z}_h) + cov(\bar{x}_{st}, \bar{z}_h) - cov(\bar{x}_{st}, \bar{z}_{st})] + 4 \sum_{h=1}^{k} \frac{v^2{}_0\omega^2{}_h(\bar{Z}_h - \bar{Z})^2}{(v_2+\beta_2(z))^2} [v(\bar{z}_h) - 2cov(\bar{z}_h, \bar{z}_{st}) + v(\bar{z}_{st})] - 4 \sum_{h=1}^{k} \frac{v_0\,\omega^2{}_h(\bar{Z}_h - \bar{Z})}{(v_2+\beta_2(z))} \Big[cov(\bar{z}_h, s^2{}_{yh}) - cov(\bar{z}_{st}, s^2{}_{yh}) - \frac{v_0}{(v_1+\beta_2(x))} (cov(\bar{z}_h, s^2{}_{xh}) - cov(\bar{z}_{st}, s^2{}_{xh})) - \frac{v_0}{(v_2+\beta_2(z))} (cov(\bar{z}_h, s^2{}_{zh}) - cov(\bar{z}_{st}, s^2{}_{zh})) \Big] \Big\}$

$$d_h = \frac{(s^2{}_x C_x + \beta_2(x))(s^2{}_z C_z + \beta_2(z))}{(v_1 C_x + \beta_2(x))(v_2 C_z + \beta_2(z))} \left[\omega_h \; 2\omega_h(\bar{Y}_h - \bar{Y}) \; - 2\omega_h(\bar{Y}_h - \bar{Y}) - \frac{v_0\omega_h C_x}{(v_1 C_x+\beta_2(x))} - \frac{2v_0\omega_h C_x(\bar{X}_h - \bar{X})}{(v_1 C_x+\beta_2(x))} \; \frac{2v_0\omega_h C_x(\bar{X}_h - \bar{X})}{(v_1 C_x+\beta_2(x))} - \frac{v_0\omega_h C_z}{(v_2 C_z+\beta_2(z))} - \frac{2v_0\omega_h C_z(\bar{Z}_h - \bar{Z})}{(v_2 C_z+\beta_2(z))} \; \frac{2v_0\omega_h C_z(\bar{Z}_h - \bar{Z})}{(v_2 C_z+\beta_2(z))} \right]$$

Using (A.1) and (A.2), we have

$MSE(s^2{}_{pr_2}) \cong \frac{((s^2{}_x C_x + \beta_2(x))(s^2{}_z C_z + \beta_2(z)))^2}{((v_1 C_x + \beta_2(x))(v_2 C_z + \beta_2(z)))^2} \{H_1 + H_5\}$ -- (C.2)

Where $H_5 = -4 \sum_{h=1}^{k} \frac{v_0\, C_x\omega^2{}_h(\bar{Y}_h - \bar{Y})}{(v_1 C_x+\beta_2(x))} (cov(\bar{y}_h, s^2{}_{xh}) - cov(\bar{y}_{st}, s^2{}_{xh})) - 4 \sum_{h=1}^{k} \frac{v_0\, C_z\omega^2{}_h(\bar{Y}_h - \bar{Y})}{(v_2 C_z+\beta_2(z))} (cov(\bar{y}_h, s^2{}_{zh}) - cov(\bar{y}_{st}, s^2{}_{zh})) - 2 \sum_{h=1}^{k} \frac{C_x v_0\omega^2{}_h}{(v_1 C_x+\beta_2(x))} cov(s^2{}_{xh}, s^2{}_{yh}) - 4 \sum_{h=1}^{k} \frac{v_0 C_x\omega^2{}_h(\bar{X}_h - \bar{X})}{(v_1 C_x+\beta_2(x))} \Big[cov(\bar{x}_h, s^2{}_{yh}) - cov(\bar{x}_{st}, s^2{}_{yh}) - \frac{v_0 C_x}{(v_1 C_x+\beta_2(x))} (cov(\bar{x}_h, s^2{}_{xh}) - cov(\bar{x}_{st}, s^2{}_{xh})) - \frac{v_0 C_z}{(v_2 C_z+\beta_2(z))} (cov(\bar{x}_h, s^2{}_{zh}) - cov(\bar{x}_{st}, s^2{}_{zh})) \Big] - 2 \sum_{h=1}^{k} \frac{v_0 C_z\omega^2{}_h}{(v_2 C_z+\beta_2(z))} cov(s^2{}_{zh}, s^2{}_{yh}) - 8 \sum_{h=1}^{k} \frac{v_0 C_x\omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{X}_h - \bar{X})}{(v_1 C_x+\beta_2(x))} [cov(\bar{y}_h, \bar{x}_h) - cov(\bar{y}_h, \bar{x}_{st}) - cov(\bar{y}_{st}, \bar{x}_h) + cov(\bar{y}_{st}, \bar{x}_{st})] - 8 \sum_{h=1}^{k} \frac{v_0 C_z\omega^2{}_h(\bar{Y}_h - \bar{Y})(\bar{Z}_h - \bar{Z})}{(v_2 C_z+\beta_2(z))} [cov(\bar{y}_h, \bar{z}_h) - cov(\bar{y}_h, \bar{z}_{st}) - cov(\bar{y}_{st}, \bar{z}_h) + cov(\bar{y}_{st}, \bar{z}_{st})] + \sum_{h=1}^{k} \frac{v^2{}_0 C_x{}^2\omega^2{}_h}{(v_1 C_x+\beta_2(x))^2} v(s^2{}_{xh}) +$

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$$2\sum_{h=1}^{k}\frac{C_xC_z{v^2}_0{\omega^2}_h}{(v_1C_x+\beta_2(x))(v_2C_z+\beta_2(z))}cov({s^2}_{xh},{s^2}_{zh})+4\sum_{h=1}^{k}\frac{{v^2}_0{C_x}^2{\omega^2}_h(\bar{X}_h-\bar{X})^2}{(v_1C_x+\beta_2(x))^2}[v(\bar{x}_h)-2cov(\bar{x}_h,\bar{x}_{st})+v(\bar{x}_{st})]+$$
$$\sum_{h=1}^{k}\frac{{v^2}_0{C_z}^2{\omega^2}_h}{(v_2C_z+\beta_2(z))^2}v({s^2}_{zh})-8\sum_{h=1}^{k}\frac{v_0\,C_xC_z{\omega^2}_h(\bar{X}_h-\bar{X})(\bar{Z}_h-\bar{Z})}{(v_1C_x+\beta_2(x))(v_2C_z+\beta_2(z))}[cov(\bar{x}_h,\bar{z}_{st})-cov(\bar{x}_h,\bar{z}_h)+cov(\bar{x}_{st},\bar{z}_h)-cov(\bar{x}_{st},\bar{z}_{st})]+$$
$$4\sum_{h=1}^{k}\frac{{v^2}_0{C_z}^2{\omega^2}_h(\bar{Z}_h-\bar{Z})^2}{(v_2C_z+\beta_2(z))^2}[v(\bar{z}_h)-2cov(\bar{z}_h,\bar{z}_{st})+v(\bar{z}_{st})]-4\sum_{h=1}^{k}\frac{v_0C_z\,{\omega^2}_h(\bar{Z}_h-\bar{Z})}{(v_2C_z+\beta_2(z))}\Big[cov(\bar{z}_h,{s^2}_{yh})-cov(\bar{z}_{st},{s^2}_{yh})-$$
$$\frac{v_0C_x}{(v_1C_x+\beta_2(x))}\left(cov(\bar{z}_h,{s^2}_{xh})-cov(\bar{z}_{st},{s^2}_{xh})\right)-\frac{v_0C_z}{(v_2C_z+\beta_2(z))}\left(cov(\bar{z}_h,{s^2}_{zh})-cov(\bar{z}_{st},{s^2}_{zh})\right)\Big]$$

Appendix D

$\mu_{rsth}=\frac{1}{N_h}\sum_{h=1}^{N_h}(Y_{hi}-\bar{Y}_h)^r(X_{hi}-\bar{X}_h)^s(Z_{hi}-\bar{Z}_h)^t$, $\lambda_h=\frac{1}{n_h}$, $\omega_h=\frac{n_h}{n}=\frac{N_h}{N}$

$\theta_h(yx)=\frac{\mu_{220h}}{\mu_{200h}\mu_{020h}}$, $\theta_h(yz)=\frac{\mu_{202h}}{\mu_{200h}\mu_{002h}}$, $\theta_h(xz)=\frac{\mu_{022h}}{\mu_{020h}\mu_{020h}}$,

$\beta_2(y_h)=\frac{\mu_{400h}}{{\mu^2}_{200h}}$ - is the population kurtosis of the variate of interest in stratum h.

$\beta_2(x_h)=\frac{\mu_{040h}}{{\mu^2}_{020h}}$ -is the population kurtosis of the first auxiliary variable (X) in stratum h.

$\beta_2(z_h)=\frac{\mu_{004h}}{{\mu^2}_{002h}}$ - is the population kurtosis of the second auxiliary variable (Z) in stratum h.

$\sigma_1^2=v({s^2}_{yh})=\lambda_h{S^4}_{yh}(\beta_2(y_h)-1)$ $\sigma_2^2=v(\bar{y}_h)=\lambda_h{S^2}_{yh}$ $\sigma_3^2=v(\bar{y}_{st})=\sum_{h=1}^{k}{\omega^2}_h\lambda_h{S^2}_{yh}$

$\sigma_4^2=v({s^2}_{xh})=\lambda_h{S^4}_{xh}(\beta_2(x_h)-1)$ $\sigma_5^2=v(\bar{x}_h)=\lambda_h{S^2}_{xh}$ $\sigma_6^2=v(\bar{x}_{st})=\sum_{h=1}^{k}{\omega^2}_h\lambda_h{S^2}_{xh}$

$\sigma_7^2=v({s^2}_{zh})=\lambda_h{S^4}_{zh}(\beta_2(z_h)-1)$ $\sigma_8^2=v(\bar{z}_h)=\lambda_h{S^2}_{zh}$ $\sigma_9^2=v(\bar{z}_{st})=\sum_{h=1}^{k}{\omega^2}_h\lambda_h{S^2}_{zh}$

$\sigma_{12}=\sigma_{21}=cov(\bar{y}_h,{s^2}_{yh})=\lambda_h\mu_{300h}$ $\sigma_{13}=\sigma_{31}=cov(\bar{y}_{st},{s^2}_{yh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{300h}$

$\sigma_{14}=\sigma_{41}=cov({s^2}_{xh},{s^2}_{yh})=\lambda_h{S^2}_{xh}{S^2}_{yh}(\theta_h(yx)-1)$, $\sigma_{15}=\sigma_{51}=cov(\bar{x}_h,{s^2}_{yh})=\lambda_h\mu_{210h}$

$\sigma_{16}=\sigma_{61}=cov(\bar{x}_{st},{s^2}_{yh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{210h}$, $\sigma_{18}=\sigma_{81}=cov(\bar{z}_h,{s^2}_{yh})=\lambda_h\mu_{201h}$

$\sigma_{17}=\sigma_{71}=cov({s^2}_{yh},{s^2}_{zh})=\lambda_h{S^2}_{zh}{S^2}_{yh}(\theta_h(yz)-1)$, $\sigma_{24}=\sigma_{42}=cov(\bar{y}_h,{s^2}_{xh})=\lambda_h\mu_{120h}$

$\sigma_{19}=\sigma_{91}=cov(\bar{y}_{st},{s^2}_{yh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{201h}$, $\sigma_{23}=\sigma_{32}=cov(\bar{y}_h,\bar{y}_{st})=\sum_{h=1}^{k}\omega_h\lambda_h{S^2}_{yh}$

$\sigma_{25}=\sigma_{52}=cov(\bar{y}_h,\bar{x}_h)=\lambda_hS_{yxh}$, $\sigma_{27}=\sigma_{72}=cov(\bar{y}_h,{s^2}_{zh})=\lambda_h\mu_{102h}$

$\sigma_{26}=\sigma_{62}=\sigma_{35}=\sigma_{53}=cov(\bar{y}_{st},\bar{x}_h)=cov(\bar{y}_h,\bar{x}_{st})=\sum_{h=1}^{k}\omega_h\lambda_hS_{yxh}$

$\sigma_{28}=\sigma_{82}=cov(\bar{y}_h,\bar{z}_h)=\lambda_hS_{yzh}$, $\sigma_{29}=\sigma_{92}=\sigma_{38}=\sigma_{83}=cov(\bar{y}_h,\bar{z}_{st})=\sum_{h=1}^{k}\omega_h\lambda_hS_{yzh}$

$\sigma_{34}=\sigma_{43}=cov(\bar{y}_{st},{s^2}_{xh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{120h}$, $\sigma_{36}=\sigma_{63}=cov(\bar{y}_{st},\bar{x}_{st})=\sum_{h=1}^{k}{\omega^2}_h\lambda_hS_{yxh}$

$\sigma_{37}=\sigma_{73}=cov(\bar{y}_{st},{s^2}_{zh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{102h}$, $\sigma_{39}=\sigma_{93}=cov(\bar{y}_{st},\bar{z}_{st})=\sum_{h=1}^{k}{\omega^2}_h\lambda_hS_{yzh}$

$\sigma_{45}=\sigma_{54}=cov(\bar{x}_h,{s^2}_{xh})=\lambda_h\mu_{030h}$, $\sigma_{46}=\sigma_{64}=cov(\bar{x}_{st},{s^2}_{xh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{030h}$

$\sigma_{47}=\sigma_{74}=cov({s^2}_{xh},{s^2}_{zh})=\lambda_h{S^2}_{zh}{S^2}_{xh}(\theta_h(zx)-1)$, $\sigma_{48}=\sigma_{84}=cov(\bar{z}_h,{s^2}_{xh})=\lambda_h\mu_{021h}$

$\sigma_{49}=\sigma_{94}=cov(\bar{z}_{st},{s^2}_{xh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{021h}$, $\sigma_{56}=\sigma_{65}=cov(\bar{x}_h,\bar{x}_{st})=\sum_{h=1}^{k}\omega_h\lambda_h{S^2}_{xh}$

$\sigma_{57}=\sigma_{75}=cov(\bar{x}_h,{s^2}_{zh})=\lambda_h\mu_{012h}$ $\sigma_{58}=\sigma_{85}=cov(\bar{x}_h,\bar{z}_h)=\lambda_hS_{xzh}$

$\sigma_{59}=\sigma_{95}=\sigma_{68}=\sigma_{86}=cov(\bar{x}_h,\bar{z}_{st})=cov(\bar{z}_h,\bar{x}_{st})=\sum_{h=1}^{k}\omega_h\lambda_hS_{xzh}$

$\sigma_{67}=\sigma_{76}=cov(\bar{x}_{st},{s^2}_{zh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{012h}$, $\sigma_{69}=\sigma_{96}=cov(\bar{x}_{st},\bar{z}_{st})=\sum_{h=1}^{k}{\omega^2}_h\lambda_hS_{xzh}$

$\sigma_{78}=\sigma_{87}=cov(\bar{z}_h,{s^2}_{zh})=\lambda_h\mu_{003h}$, $\sigma_{79}=\sigma_{97}=cov(\bar{z}_{st},{s^2}_{zh})=\sum_{h=1}^{k}\omega_h\lambda_h\mu_{003h}$

$\sigma_{89}=\sigma_{98}=cov(\bar{z}_h,\bar{z}_{st})=\sum_{h=1}^{k}\omega_h\lambda_h{S^2}_{zh}$

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