Adapted Method for Solving Linear Volterra Integral Equations of the Second kind Using Corrected Simpson's Rule

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Abstract: ***In this paper, we present an adapted method for solving linear Volterra integral equations of the second kind with regular kernel. This method is based on the corrected Simpson's rule. Numerical examples are given to prove the efficiency and accuracy of our method by comparison with known methods.***

Keywords: Volterra Integral Equation, Corrected Simpson's Rule

1. Introduction

We consider the following linear Volterra integral equations of the second kind:

$$u(x) = f(x) + \int_a^x k(x,y)u(y)dy, \quad a \le x \le b. \qquad \text{..........(1.1)}$$

where the function f(x) and the regular kernel k(x,y) are given, and u(x) is the unknown function to be determined.

Many problems for physics and other disciplines lead to integral equations. Several analytical and numerical methods for solving integral equations have been studied by the authors [1,2,5,6,8]. An adapted trapezoidal method presented in [3] for solving Fredholm integral equations and then in [4] for solving Volterra integral equations given by Eq.(1.1). Also, Eq.(1.1) solved by modified adapted trapezoidal methods in [9] and by adapted Simpson's method in [10]. In this paper we introduce an adapted method which is based on the corrected Simpson's rule to solve Eq.(1.1). To do this we assume the functions $\frac{\partial k(x,y)}{\partial x}$, $\frac{\partial k(x,y)}{\partial y}$ and $f'(x)$ must exist. The idea is to approximate the solutions of the above equations in even number of equally spaced points. Then in interval [a, a+2h] we have

$$\int_a^{a+2h} k(x,y)u(y)dy = \frac{h}{15}\big(7k(x,a)u(a) + 16k(x,a+h)u(a+h) + 7k(x,a+2h)u(a+2h)\big) + \frac{h^2}{15}\big(J(x,a)u(a) + k(x,a)u'(a) - J(x,a+2h)u(a+2h) - k(x,a+2h)u'(a+2h)\big) - \frac{h^6}{302400}\big(k(x,\zeta)u(\zeta)\big)^5.$$

where, $J(x,y) = \frac{\partial k(x,y)}{\partial y}$.

This approximation indicates that the error E(h) of integration over two segments by corrected Simpson's rule is proportional to h^6. Also, we note that if the segment width h is halved to h / 2, then

$$E(\frac{h}{2}) \approx -2\frac{(h/2)^6}{302400}\big(k(x,\zeta)u(\zeta)\big)^5 = \frac{1}{32}E(h).$$

2. Adapted Corrected Simpson's Method

Consider the Volterra integral equation given by (1.1). Let the interval [a, b] be finite and partitioned by 2n equally spaced points

$$a = x_0 < x_1 < \cdots < x_{2i-1} < x_{2i} < \cdots < x_{2n} = b.$$

The approximation of Eq.(1.1) in the even nodes (x_{2i}) is given by

$$u(x_{2i}) = f(x_{2i}) + \int_a^{x_{2i}} k(x_{2i},y)u(y)dy$$

$$= f(x_{2i}) + \sum_{j=0}^{i-1}\int_{x_{2j}}^{x_{2j+2}} k(x_{2i},y)u(y)dy$$

which can be rewritten as

$$u_{2i} = f_{2i} + \sum_{j=0}^{i-1}\int_{x_{2j}}^{x_{2j+2}} k(x_{2i},y)u(y)dy.$$

Using corrected Simpson's quadrature rule, the above discrete equation becomes

$$u_{2i} = f_{2i} + \sum_{j=0}^{i-1}\frac{h}{15}\big(7k_{2i,2j}u_{2j} + 16k_{2i,2j+1}u_{2j+1} + 7k_{2i,2j+2}u_{2j+2}\big) + \frac{h^2}{15}\big(J_{2i,0}u_0 + k_{2i,0}u'_0 - J_{2i,2i}u_{2i} - k_{2i,2i}u'_{2i}\big)$$

where $h = x_{i+1} - x_i = \frac{(b-a)}{2n}$.

For a smaller step h, an approximation to u_{2i} can then be computed by replacing u_{2j+1} by the average $\left(u_{2j} + u_{2j+2}\right)/2$,

$$u_{2i}=f_{2i}+\sum_{j=0}^{i-1}\frac{h}{15}\left(7k_{2i,2j}u_{2j}+16k_{2i,2j+1}\frac{u_{2j}+u_{2j+2}}{2}+7k_{2i,2j+2}u_{2j+2}\right)+\frac{h^2}{15}\left(J_{2i,0}u_0+k_{2i,0}u'_0-J_{2i,2i}u_{2i}-k_{2i,2i}u'_{2i}\right)$$

$$=f_{2i}+\sum_{j=0}^{i-1}\frac{h}{15}\left(7k_{2i,2j}+8k_{2i,2j+1}\right)u_{2j}+\sum_{j=1}^{i}\frac{h}{15}\left(8k_{2i,2j-1}+7k_{2i,2j}\right)u_{2j}+\frac{h^2}{15}\left(J_{2i,0}u_0+k_{2i,0}u'_0-J_{2i,2i}u_{2i}-k_{2i,2i}u'_{2i}\right)$$

$$=f_{2i}+\left(\frac{h}{15}\left(7k_{2i,0}+8k_{2i,1}\right)+\frac{h^2}{15}J_{2i,0}\right)u_0+\sum_{j=1}^{i-1}\frac{h}{15}\left(8k_{2i,2j-1}+14k_{2i,2j}+8k_{2i,2j+1}\right)u_{2j}+\left(\frac{h}{15}\left(8k_{2i,2i-1}+7k_{2i,2i}\right)-\frac{h^2}{15}J_{2i,2i}\right)u_{2i}+\frac{h^2}{15}\left(k_{2i,0}u'_0-k_{2i,2i}u'_{2i}\right),\ i=1,2,\dots,n,$$

...........(2.1)

If we take the derivative from both sides of Eq.(1.1) with respect to x we obtain the equation (2.2)

$$u'(x)=f'(x)+\int_a^x H(x,y)u(y)dy+k(x,x)u(x).$$

...........(2.2)

where $H(x,y)=\frac{\partial k(x,y)}{\partial x}$. We note that if u be a solution of Eq.(1.1) it is a solution of Eq.(2.2) too. Now, for solving Eq.(2.2), we must consider two cases.

Case1: If $\frac{\partial^2 k(x,y)}{\partial x\partial y}$ exist, in this case, we use adapted corrected Simpson's method for solving Eq.(2.2). Therefore, we obtain the following system:

$$u'_0=f'_0+k_{0,0}u_0,$$

$$u'_{2i}=f'_{2i}+\left(\frac{h}{15}\left(7H_{2i,0}+8H_{2i,1}\right)+\frac{h^2}{15}L_{2i,0}\right)u_0+\sum_{j=1}^{i-1}\frac{h}{15}\left(8H_{2i,2j-1}+14H_{2i,2j}+8H_{2i,2j+1}\right)u_{2j}+\left(\frac{h}{15}\left(8H_{2i,2i-1}+7H_{2i,2i}\right)-\frac{h^2}{15}L_{2i,2i}+k_{2i,2i}\right)u_{2i}+\frac{h^2}{15}\left(H_{2i,0}u'_0-H_{2i,2i}u'_{2i}\right),\ (i=1,2,\dots,n).$$

...........(2.3)

By substituting Eq.(2.3) to Eq.(2.1) we obtain following system:

$$u_{2i}=f_{2i}+\frac{h^2}{15}k_{2i,0}f'_0-\frac{h^2k_{2i,2i}}{15+h^2H_{2i,2i}}\left(f'_{2i}+\frac{h^2}{15}H_{2i,0}f'_0\right)+\left[\frac{h}{15}\left(7k_{2i,0}+8k_{2i,1}\right)+\frac{h^2}{15}\left(J_{2i,0}+k_{2i,0}k_{0,0}\right)-\frac{h^2k_{2i,2i}}{15+h^2H_{2i,2i}}\left(\frac{h}{15}\left(7H_{2i,0}+8H_{2i,1}\right)+\frac{h^2}{15}\left(L_{2i,0}+H_{2i,0}k_{0,0}\right)\right)\right]u_0+\frac{h}{15}\sum_{j=1}^{i-1}\left[\left(8k_{2i,2j-1}+14k_{2i,2j}+8k_{2i,2j+1}\right)-\frac{h^2k_{2i,2i}}{15+h^2H_{2i,2i}}\left(8H_{2i,2j-1}+14H_{2i,2j}+8H_{2i,2j+1}\right)\right]u_{2j}+\left[\frac{h}{15}\left(8k_{2i,2i-1}+7k_{2i,2i}\right)-\frac{h^2}{15}J_{2i,2i}-\frac{h^2k_{2i,2i}}{15+h^2H_{2i,2i}}\left(\frac{h}{15}\left(8H_{2i,2i-1}+7H_{2i,2i}\right)-\frac{h^2}{15}L_{2i,2i}+k_{2i,2i}\right)\right]u_{2i},\ (i=1,2,\dots,n).$$

...........(2.4)

where $u_0=f_0$. This system can be solved to find the unknowns $\{u_{2i}\}_{i=1}^n$ by any suitable methods.

Case2: If $\frac{\partial^2 k(x,y)}{\partial x\partial y}$ dose not exist, in this case, we use adapted Simpson's method for solving Eq.(2.2). Therefore, we obtain the following system:

$$u'_0=f'_0+k_{0,0}u_0,$$

$$u'_{2i}=f'_{2i}+\frac{h}{3}\left(H_{2i,0}+H_{2i,1}\right)u_0+\frac{2h}{3}\sum_{j=1}^{i-1}\left(H_{2i,2j-1}+H_{2i,2j}+H_{2i,2j+1}\right)u_{2j}+\left(\frac{h}{3}\left(H_{2i,2i-1}+H_{2i,2i}\right)+k_{2i,2i}\right)u_{2i},\ (i=1,2,\dots,n).$$

...........(2.5)

By substituting Eq.(2.5) to Eq.(2.1) we obtain following system:

$$u_{2i} = f_{2i} + \frac{h^2}{15}\left(k_{2i,0}f_0' - k_{2i,2i}f_{2i}'\right) + \left[\frac{h}{15}\left(7k_{2i,0} + 8k_{2i,1}\right) + \frac{h^2}{15}\left(k_{2i,0}k_{0,0} + J_{2i,0} - \frac{hk_{2i,2i}}{3}\left(H_{2i,0} + 2H_{2i,1}\right)\right)\right]u_0 + \frac{h}{15}\sum_{j=1}^{i-1}\left[\left(8k_{2i,2j-1} + 14k_{2i,2j} + 8k_{2i,2j+1}\right) - \frac{2h^2k_{2i,2i}}{3}\left(H_{2i,2j-1} + H_{2i,2j} + H_{2i,2j+1}\right)\right]u_j + \left[\frac{h}{15}\left(8k_{2i,2i-1} + 7k_{2i,2i}\right) - \frac{h^2}{15}\left(J_{2i,2i} + k^2_{2i,2i} + \frac{hk_{2i,2i}}{3}\left(2H_{2i,2i-1} + H_{2i,2i}\right)\right)\right]u_{2i}, (i = 1, 2, \ldots, n).$$

…………(2.6)

where, $u_0 = f_0$. This system can be solved to find the unknowns $\{u_{2i}\}_{i=1}^{n}$ by any suitable methods.

3. Numerical Examples

We present in this section numerical results (error between exact and approximate value of u(x)) for some examples to show the efficiently and accuracy of the adaptive corrected Simpson's method (ACSM) by compares this method with other numerical methods such as adaptive Simpson's method (ASM) and Taylor-series expansion method (TSEM), the computations were carried out using MALAB® 7.6.0

Example 1. Here we solve Eq.(1.1) with $k(x,y) = \sin(x-y)$. To evaluate the accuracy of approximation produced, f(x) is chosen such that the exact solution is $u(x) = x+1$. Numerical results by ASM and ACSM with h=0.1, 0.05 are given in Table 1.

Table (1): Numerical results for Example 1

x	ASM		ACSM	
	h=0.1	h=0.05	h=0.1	h=0.05
0	0	0	0	0
0.2	4.5715×10^{-7}	2.8532×10^{-8}	1.2938×10^{-10}	2.0197×10^{-12}
0.4	9.4108×10^{-7}	5.8732×10^{-8}	2.6386×10^{-10}	4.1191×10^{-12}
0.6	1.4563×10^{-6}	9.0879×10^{-8}	4.0428×10^{-10}	6.3114×10^{-12}
0.8	2.0072×10^{-6}	1.2525×10^{-7}	5.5151×10^{-10}	8.6093×10^{-12}
1	2.5983×10^{-6}	1.6213×10^{-7}	7.0638×10^{-10}	1.1026×10^{-11}

Example 2. Here we solve Eq.(1.1) with $k(x,y) = e^{-(x-y)}$, $f(x) = 1$ and the exact solution is the same as in Example 1. Numerical results by ASM [10] and ACSM with h=0.1, 0.05 are given in Table 2.

Table (2): Numerical results for Example 2

x	ASM		ACSM	
	h=0.1	h=0.05	h=0.1	h=0.05
0	0	0	0	0
0.2	5.6610×10^{-7}	3.5408×10^{-8}	1.5019×10^{-10}	2.3477×10^{-12}
0.4	1.1544×10^{-6}	7.2204×10^{-8}	3.0462×10^{-10}	4.7611×10^{-12}
0.6	1.7649×10^{-6}	1.1039×10^{-7}	4.6328×10^{-10}	7.2411×10^{-12}
0.8	2.3976×10^{-6}	1.4996×10^{-7}	6.2616×10^{-10}	9.7875×10^{-12}
1	3.0525×10^{-6}	1.9092×10^{-7}	7.9328×10^{-10}	1.2399×10^{-11}

Example 3. In this example we consider Eq. (1.1) with $k(x,y) = e^{-(x-y)}$, $f(x) = e^{-x}(1-x)$ and the exact solution is $u(x) = e^{-x}$. Numerical results (also, by ASM and ACSM) with h=0.1, 0.05 are given in Table 3.

Table (3): Numerical results for Example 3

x	ASM		ACSM	
	h=0.1	h=0.05	h=0.1	h=0.05
0	0	0	0	0
0.2	6.0272×10^{-4}	1.5096×10^{-4}	4.8250×10^{-4}	1.2079×10^{-4}
0.4	1.0962×10^{-3}	2.7456×10^{-4}	8.7754×10^{-4}	2.1969×10^{-4}
0.6	1.5002×10^{-3}	3.7576×10^{-4}	1.2010×10^{-3}	3.0065×10^{-4}
0.8	1.8310×10^{-3}	4.5861×10^{-4}	1.4658×10^{-3}	3.6695×10^{-4}
1	2.1018×10^{-3}	5.2644×10^{-4}	1.6826×10^{-3}	4.2122×10^{-4}

Example 4. In this example we solve Eq.(1.1) with $k(x,y) = y - x$, $f(x) = 1$ and the exact solution is $u(x) = \cos x$. Numerical results by TSEM [7], ASM [10] and ACSM with h=0.1 are given in Table 4.

Table (4): Numerical results for Ex. 4 with h=0.1

x	TSEM	ASM	ACSM
0	0	0	0
0.2	1.8678×10^{-4}	6.5872×10^{-5}	5.2706×10^{-5}
0.4	2.3937×10^{-3}	2.5825×10^{-4}	2.0663×10^{-4}
0.6	7.8078×10^{-3}	5.6170×10^{-4}	4.4943×10^{-4}
0.8	1.0145×10^{-2}	9.5158×10^{-4}	7.6136×10^{-4}
1	5.1522×10^{-3}	1.3954×10^{-3}	1.1165×10^{-3}

4. Conclusions and Recommendations

In this paper we proposed an adaptive corrected Simpson's method for solving the linear Volterra integral equations of the second kind with regular kernels. From numerical examples it can be seen that the proposed numerical method is efficient and accurate to estimate the solution of these equations, also, this method be more efficient in case the exact solution u(x) is a polynomial type.

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