An Elementary Proof of Catalan-Mihailescu and Fermat-Wiles Theorems and Generalization to Beal Conjecture

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Abstract: ***A proof of both Catalan and Fermat theorems is presented and a generalization to Beal conjecture is proposed. For this, we begin with Fermat, Catlan and Fermat-Catalan equations and solve them.***

Keywords: Diophantine ; Fermat ; Catalan ; Fermat-Catalan ; Resolution

1. Resolution of Catalan Equation

Catalan equation is $y^p = x^q + 1$

Let $x^{q-3}y^2 - y^{p-2}x^3 = A$

If

$$A = 0 \Rightarrow x^{q-6} = y^{p-4}$$

$$GCD(x, y) = 1 \Rightarrow p = 4$$

impossible, there is no solution, Ko Chao proved it and

If

$$A^2 = 1; p \geq 3 \Rightarrow x^{q-3}y - y^{p-3}x^3 = \frac{1}{y} \in Z \text{ impossible}$$

thus p=2

We have

$$x^{q-3}y^2 - y^{p-2}x^3 = A = Ay^{p-2}y^2 - Ax^{q-3}x^3$$

$$\Rightarrow (x^{q-3} - Ay^{p-2})y^2 = (y^{p-2} - Ax^{q-3})x^3$$

$$(y^2 + Ax^3)x^{q-3} = (x^3 + Ay^2)y^{p-2}$$

But

$$GCD(x, y) = 1$$

We have then four cases:

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$

$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

or

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$

$$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

or

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$

$$vx^{q-3} = x^3 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

or

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$

$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

$$u, v \in Z$$

First case

$$x^3 = u(-x^{q-3} + Ay^{p-2}); y^2 = u(-y^{p-2} + Ax^{q-3})$$

$$x^{q-3} = v(x^3 + Ay^2); y^{p-2} = v(y^2 + Ax^3)$$

We have

$$y^p = u\ v(-y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3})) = uv(-y^p + A^2x^q + A^2) = uv(A^2 - 1)y^p$$

$$uv(A^2 - 1) = 1$$

Impossible because A,u,v are integers

Second case

$$ux^3 = -x^{q-3} + Ay^{p-2}; uy^2 = -y^{p-2} + Ax^{q-3}$$

$$vx^{q-3} = x^2 + Ay^2; vy^{p-2} = y^2 + Ax^3$$

We have

$$uvy^p = -y^p + A^2x^q - A(x^3y^{p-2} - y^2x^{q-3}) = -y^p + A^2x^q + A^2 = (A^2 - 1)y^p$$

$$uv = A^2 - 1$$

and

$$uv(y^2x^{q-3}-x^3y^{p-2})=uvA=v(-y^{2p-4}+x^{2q-6})A=u(y^4-x^6)A$$
$$\Rightarrow v=y^4-x^6;u=-y^{2p-4}+x^{2q-6}$$
$$uv=A^2-1=(y^4-x^4)(-y^{2p-4}+x^{2q-6})$$
$$\Rightarrow(y^2x^{q-3}-x^3y^{p-2})^2-1=(y^4-x^6)(-y^{2p-4}+x^{2q-6})$$
$$\Rightarrow x^{2q}+y^{2p}-2x^qy^p=1=(x^q-y^p)^2=(2y^p-1)^2=4y^{2p}-4y^p+1$$

Impossible : $\Rightarrow p=2$

Third case

$$x^3=u(-x^{q-3}+Ay^{p-2});y^2=u(-y^{p-2}+Ax^{q-3})$$
$$vx^{q-3}=x^3+Ay^2;vy^{p-2}=y^2+Ax^3$$

We have

$$vy^p=u(-y^p+A^2x^q-A(x^3y^{p-2}-y^2x^{q-3}))=u(-y^p+A^2x^q+A^2)=u(A^2-1)y^p$$
$$v=u(A^2-1)$$

and

$$v(y^2x^{q-3}-x^3y^{p-2})=vA=uv(-y^{2p-4}+x^{2q-6})A=(y^4-x^6)A$$
$$\Rightarrow u(-y^{2p-4}+x^{2q-6})=1\Rightarrow p-2=q-3=0$$

Impossible because u, A are integers

Fourth Case

$$ux^3=-x^{q-3}+Ay^{p-2};uy^2=-y^{p-2}+Ax^{q-3}$$
$$x^{q-3}=v(x^3+Ay^2);y^{p-2}=v(y^2+Ax^3)$$

We have

$$u\,y^p=v(-y^p+A^2x^q-A(x^2y^{p-2}-y^2x^{q-3}))=v(-y^p+A^2x^q+A^2)=v(A^2-1)y^p$$
$$u=v(A^2-1)$$

and

$$u(y^2x^{q-3}-x^3y^{p-2})=uA=(-y^{2p-4}+x^{2q-6})A=uv(y^4-x^6)A$$
$$\Rightarrow v(y^4-x^6)=1$$

Impossible, because v,A are integers !

The only solution is A=1 and p=2 this case has been studied by Ko Chao.

2. The Fermat Equation

Fermat equation is $y^n=x^n\pm z^n=x^n+az^n$

Let $x^{n-2}y^2-y^{n-2}x^2=Aa$

If $A=0\Rightarrow x^{n-4}=y^{n-4}$ but $GCD(x,y)=1\Rightarrow n=4$ impossible, there is no solution and

If $A^2=z^{2n};n\geq 3\Rightarrow x^{n-3}y-y^{n-2}=\dfrac{Aaz^n}{x}\in Z$

impossible $\Rightarrow n=2$

We have

$$(x^{n-2}y^2-y^{n-2}x^2)z^n=Aaz^n=Ay^{n-2}y^2-Ax^{n-2}x^2$$
$$\Rightarrow(x^{n-2}z^n-Ay^{n-2})y^2=(y^{n-2}z^n-Ax^{n-2})x^2$$
$$(y^2z^n+Ax^2)x^{n-2}=(x^2z^n+Ay^2)y^{n-2}$$

GCD(x,y)=1

We have then four cases:

$$x^2=u(-x^{n-2}z^n+Ay^{n-2});y^2=u(-y^{n-2}z^n+Ax^{n-2})$$
$$x^{n-2}=v(x^2z^n+Ay^2);y^{n-2}=v(y^2z^n+Ax^2)$$

or

$$ux^2=-x^{n-2}z^n+Ay^{n-2};uy^2=-y^{n-2}z^n+Ax^{n-2}$$
$$vx^{n-2}=x^2z^n+Ay^2;vy^{n-2}=y^2z^n+Ax^2$$

or

$$x^2=u(-x^{n-2}z^n+Ay^{n-2});y^2=u(-y^{n-2}z^n+Ax^{n-2})$$
$$vx^{n-2}=x^2z^n+Ay^2;vy^{n-2}=y^2z^n+Ax^2$$

or

$$ux^2=-x^{n-2}z^n+Ay^{n-2};uy^2=-y^{n-2}z^n+Ax^{n-2}$$
$$x^{n-2}=v(x^2z^n+Ay^2);y^{n-2}=v(y^2z^n+Ax^2)$$

With $u,v\in Z$

First case

$$x^2=u(-x^{n-2}z^n+Ay^{n-2});y^2=u(-y^{n-2}z^n+Ax^{n-2})$$
$$x^{n-2}=v(x^2z^n+Ay^2);y^{n-2}=v(y^2z^n+Ax^2)$$

We have

$$y^n=uv(-y^nz^{2n}+A^2x^n-Az^n(x^2y^{n-2}-y^2x^{n-2}))=uv(-y^nz^{2n}+A^2x^n+A^2az^n)=uv(A^2-z^{2n})y^n$$
$$uv(A^2-z^{2n})=1$$

Impossible because A,u,v are integers

Second case

or

$$ux^2=-x^{n-2}z^n+Ay^{n-2};uy^2=-y^{n-2}z^n+Ax^{n-2}$$
$$vx^{n-2}=x^2z^n+Ay^2;vy^{n-2}=y^2z^n+Ax^2$$

We have

$$uvy^n=-y^nz^{2n}+A^2x^n-Az^n(x^2y^{n-2}-y^2x^{n-2})=-y^nz^{2n}+A^2x^n+A^2az^n=(A^2-z^{2n})y^n$$
$$uv=A^2-z^{2n}$$

But

$$uv(y^2x^{n-2}-x^2y^{n-2})=auvA=v(-y^{2n-4}+x^{2n-4})A=u(y^4-x^4)A$$
$$\Rightarrow au=-y^{2n-4}+x^{2n-4};av=y^4-x^4$$
$$x<y$$
$$a(A-z^n)=(y^2+x^2)(x^{n-2}-y^{n-2})<0$$
$$a(A+z^n)=y^2x^{n-2}-x^2y^{n-2}+y^n-x^n=(y^2-x^2)(x^{n-2}+y^{n-2})>0$$
$$uv=A^2-z^{2n}<0$$
$$a>0$$
$$A=a(y^2x^{n-2}-x^2y^{n-2})<0$$
$$v=a(y^4-x^4)>0$$
$$0<auvA=v(x^{2n-4}-y^{2n-4})<0$$
$$x>y$$
$$a(A-z^n)=(y^2+x^2)(x^{n-2}-y^{n-2})>0$$
$$a(A+z^n)=y^2x^{n-2}-x^2y^{n-2}+y^n-x^n=(y^2-x^2)(x^{n-2}+y^{n-2})<0$$
$$uv=A^2-z^{2n}<0$$
$$a<0$$
$$A=a(y^2x^{n-2}-x^2y^{n-2})<0$$
$$v=a(y^4-x^4)>0$$
$$0<auvA=v(x^{2n-4}-y^{2n-4})<0$$

It is impossible because $\Rightarrow n=2$

Third Case

$$or$$
$$x^2=u(-x^{n-2}z^n+Ay^{n-2});y^2=u(-y^{n-2}z^n+Ax^{n-2})$$
$$vx^{n-2}=x^2z^n+Ay^2;vy^{n-2}=y^2z^n+Ax^2$$

We have

$$vy^n=u(-y^nz^{2n}+A^2x^n-Az^n(x^2y^{n-2}-y^2x^{n-2}))=u(-y^nz^{2n}+A^2x^n+A^2az^n)=u(A^2-z^{2n})y^n$$
$$v=u(A^2-z^{2n})$$

and

$$v(y^2x^{n-2}-x^2y^{n-2})=vA=uv(-y^{2n-4}+x^{2n-4})A=(y^4-x^4)A$$
$$\Rightarrow u(-y^{2n-4}+x^{2n-4})=1\Rightarrow u=\infty;n=2$$

Impossible because u, A are integers

Fourth Case

$$ux^2=-x^{n-2}z^n+Ay^{n-2};uy^2=-y^{n-2}z^n+Ax^{n-2}$$
$$x^{n-2}=v(x^2z^n+Ay^2);y^{n-2}=v(y^2z^n+Ax^2)$$

We have

$$uy^n=v(-y^nz^{2n}+A^2x^n-Az^n(x^2y^{n-2}-y^2x^{n-2}))=v(-y^nz^{2n}+A^2x^n+A^2az^n)=v(A^2-z^{2n})y^n$$
$$u=v(A^2-z^{2n})$$

and

$$u(y^2x^{n-2}-x^2y^{n-2})=uA=(-y^{2n-4}+x^{2n-4})A=uv(y^4-x^4)A$$
$$\Rightarrow v(x^4-y^4)=1\Rightarrow v=1;x^4=y^4+1$$

Impossible! Because v, A are integers!

The only solution is A=1 and n=2

3. The Fermat-Catalan equation

The equation now is $y^p=x^q\pm z^c=x^q+az^c$

Let $x^{q-w}y^2-y^{p-2}x^w=aA$

If

$$A=0\Rightarrow x^{q-2w}=y^{p-4}$$
$$GCD(x,y)=1\Rightarrow p=4$$

It means that p=2 is the prime solution!
and

$$A^2=z^{2c};p\geq 3\Rightarrow x^{q-w}y-y^{p-3}x^w=\frac{\pm z^c}{y}\in Z$$

Impossible because GCD(y,z)=1, thus p=2
We have

$$(x^{q-w}y^2-y^{p-2}x^w)z^c=aAz^c=Ay^{p-2}y^2-Ax^{q-w}x^w$$
$$\Rightarrow (x^{q-w}z^c-A\ \ ^{p-2}y)y^2=(y^{p-2}z^c-Ax^{q-w})x^w$$
$$(y^2z^c+Ax^w)x^{q-w}=(x^wz^c+Ay^2)y^{p-2}$$
$$GCD(x,y)=1$$

We have then four cases:
First Case

$$x^w=u(-x^{q-w}z^c+Ay^{p-2});y^2=u(-y^{p-2}z^c+Ax^{q-w})$$
$$x^{q-w}=v(x^wz^c+Ay^2);y^{p-2}=v(y^2z^c+Ax^w)$$

We have

$$y^p=uv(-y^pz^{2c}+A^2x^q-Az^c(x^wy^{p-2}-y^2x^{q-w}))=uv(-y^pz^{2c}+A^2x^q+A^2az^c)=uv(A^2-z^{2c})y^p$$
$$uv(A^2-z^{2c})=1$$

Impossible because A,u,v are integers
Second Case

$$ux^w=-x^{q-w}z^c+Ay^{p-2};uy^2=-y^{p-2}z^c+Ax^{q-w}$$
$$vx^{q-w}=x^wz^c+Ay^2;vy^{p-2}=y^2z^c+Ax^w$$

We have

$$uvy^p=-y^pz^{2c}+A^2x^q-Az^c(x^wy^{p-2}-y^2x^{q-w})=-y^pz^{2c}+A^2x^q+A^2az^c)=(A^2-z^{2c})y^p$$
$$uv=A^2-z^{2c}$$

and

$$uv(y^2x^{q-w}-x^wy^{p-2})=auvA=v(-y^{2p-4}+x^{2p-2w})A=u(y^4-x^{2w})A$$
$$\Rightarrow au=-y^{2p-4}+x^{2q-2w};av=y^4-x^{2w}$$
$$1<\frac{y^4}{x^{2w}}<\frac{y^p}{x^q}$$
$$a(A-z^c)=(y^2+x^w)(x^{q-w}-y^{p-2})<0$$
$$a(A+z^c)=y^2x^{q-w}-x^wy^{p-2}+y^p-x^q=(y^2-x^w)(x^{q-w}+y^{p-2})>0$$
$$uv=A^2-z^{2c}<0$$
$$y^p>x^q\Rightarrow a>0$$
$$A=a(y^2x^{q-w}-x^wy^{p-2})<0$$
$$v=a(y^4-x^{2w})>0$$
$$0<auvA=v(x^{2q-2w}-y^{2p-4})<0$$

and if

$$1>\frac{y^4}{x^{2w}}>\frac{y^p}{x^q}$$
$$a(A-z^c)=(y^2+x^w)(x^{q-w}-y^{p-2})>0$$
$$a(A+z^c)=y^2x^{q-w}-x^wy^{p-2}+y^p-x^q=(y^2-x^w)(x^{q-w}+y^{p-2})<0$$
$$uv=A^2-z^{2c}<0$$
$$y^p<x^q\Rightarrow a<0$$
$$A=a(y^2x^{q-w}-x^wy^{p-2})<0$$
$$v=a(y^4-x^{2w})>0$$
$$0<auvA=v(x^{2q-2w}-y^{2p-4})<0$$

Impossible because : $\Rightarrow p=2$

Third Case

$$x^w=u(-x^{q-w}z^c+Ay^{p-2});y^2=u(-y^{p-2}z^c+Ax^{q-w})$$
$$vx^{q-w}=x^wz^c+Ay^2;vy^{p-2}=y^2z^c+Ax^w$$

We have

$$vy^p=u(-y^pz^{2c}+A^2x^q-Az^c(x^wy^{p-2}-y^2x^{q-w}))=u(-y^pz^{2c}+A^2x^q+A^2az^c)=u(A^2-z^{2c})y^p$$
$$v=u(A^2-z^{2c})$$

and

$$v(y^2x^{q-w}-x^wy^{p-2})=vA=uv(-y^{2p-4}+x^{2q-2w})A=(y^4-x^{2w})A$$
$$\Rightarrow u(-y^{2p-4}+x^{2q-2w})=1\Rightarrow p=2$$

Impossible because u, A are integers

Fourth Case

$$ux^w=-x^{q-w}z^c+Ay^{p-2};uy^2=-y^{p-2}z^c+Ax^{q-w}$$
$$x^{q-w}=v(x^wz^c+Ay^2);y^{p-2}=v(y^2z^c+Ax^w)$$

We have

$$uy^p=v(-y^pz^{2c}+A^2x^q-Az^c(x^wy^{p-2}-y^2x^{q-w}))=v(-y^pz^{2c}+A^2x^q+A^2az^c)=v(A^2-z^{2c})y^p$$
$$u=v(A^2-z^{2c})$$

and

$$u(y^2x^{q-w}-x^wy^{p-2})=uA=-y^{2p-4}+x^{2q-2w}A=uv(y^4-x^{2w})A$$
$$\Rightarrow v(y^4-x^{2w})=1$$

Impossible, because v, A are integers! In the Fermat-Catalan equation, one of the exponents must be equal to 2! The Beal conjecture has been proved!

In fact, in the three precedent equations studied here, one of the exponent greater or equal to 2 must be minimum, which means that it must be 2!

4. Conclusion

We have solved both three equations by the same method and proved two theorems and one conjecture.

References

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