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Applications of Generalized Two Dimensional Fractional Fourier transform

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Abstract: *As the one dimensional (1–D) Fourier transform can be extended into 1-D fractional Fourier transform (FrFT), we can also generalize the two-dimensional (2-D) Fourier transform. Recently several properties of FrFT have been developed by generalizing the properties of the ordinary Fourier transform (FrFT). In this paper Applications of Generalized two –dimensional fractional Fourier transform is presented.*

Keywords: Fourier transform, Fractional Fourier transform, generalized function, signal processing optics.

1. Introduction

Fractional Fourier transform (FrFT) is a generalization of the ordinary FT. FrFT was first introduced as a way to solve certain classes of ordinary and partial differential equations arising in quantum mechanics [1]. FrFT has established itself as a powerful tool for the analysis of time varying signals, especially in optics [2]. FrFT has found applications in areas of signal processing such as repeated filtering, fractional convolution and correlation, beam forming, optional filter, convolution, filtering and wavelet transforms, time frequency representation. In every area in which Fourier transform and frequency domain concepts are used, then exists the potential for generalization and improved by using the FrFT[3].

Recently FrFT independently discussed by lots of researchers. Ozaktas etal [4] had given applications of FrFT in optics and signal processing. Alieva T [5] has developed fractional transforms in optical Information processing. Bultheel A. et al in [6] had given recent developments in the theory of fractional Fourier & linear canonical transforms. Djurovic [7] etal had established FrFT as a signal processing tool. Salazar F. [8] had given a new introduction to the FrFT and its applications.

In the present work generalization of two dimensional Fractional Fourier transform in presented. Application of two-dimensional Fractional Fourier transform in the form of examples are given. Two dimensional FrFT of some functions are obtained.

2. Preliminaries

2.1 Fourier Transform

The Fourier transform of function x (t) in defined as

$$x(w) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} x(w)\, e^{jwt}\, dt$$

And inverse is given by

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} x(w)\, e^{jwt}\, dw$$

2.2 One dimensional fractional fourier transform

One dimensional fractional Fourier transform with parameter $\propto$ of $f(x)$ difined as

$$FrFT\{f(x)\} = F_{\propto}(u)$$

$$= \int_{-\infty}^{\infty} f(x) k_{\propto}(x,u)dx \text{-----}(1.1)$$

where the kernel

$$k_{\propto}(x,u) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+u^2)\cos\alpha - 2(xu)]} \text{-----}(1.2)$$

3. Distributional Two Dimensional Fractional Fourier Transform

3.1 Conventional Two- dimensional Fractional Fourier transforms

The two dimensional fractional Fourier transform with parameter $\propto$ of $f(x,y)$ denoted by $FRFT\{f(x,y)\}$ performs a linear operation given by the integral transform

$$FRFT\{f(x,y)\} = F_{\propto}\{f(x,y)\}(u,v)$$

$$= F_{\propto}(u,v)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, K_{\propto}(x,y,u,v)dx\, dy, \text{----}(3.1)$$

Where the kernel,

$$K_{\propto}(x,y,u,v) = C_{1\propto} e^{iC_{2\propto}[(x^2+u^2+y^2+v^2)cos\alpha - 2(xu+yv)]},$$

$$C_{1\propto} = \sqrt{\frac{1-icot\propto}{2\pi}},\ C_{2\propto} = \frac{1}{2sin\alpha} \text{----}(3.2)$$

----(3.2)

3.2 The testing function space E

An infinitely differentiable complex values smooth function $\emptyset$ on R^n belongs to $E(R^n)$ if for each compact set $I < S_{a,b}$ where,

$$S_{a,b} = \{x,y: x,y \in R^n, |x| \le a, |y| \le b, a>0, b>0\}, I \in R^n,$$

$$\gamma_{E,p,q}(\emptyset) = \sup_{x,y \in I} \left|D_{x,y}^{p,q}\, \emptyset(x,y)\right| < \infty, \text{ where } p,q = 1,2,3, \ldots.$$

Thus $E(R^n)$ will denote the space of all $\emptyset \in E(R^n)$ with support contained in $S_{a,b}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that f is a fractional Fourier transformable if it is a member of E^*, the dual space of E.

3.3 Distributional Two-dimensional fractional Fourier transform

The two distributional two-dimensional fractional Fourier transform of $f(x,y) \in E(R^n)$ can be defined by,

$$FRFT\{F(X,Y)\} = F_{\propto}(u,v) = \langle f(x,y), K_{\propto}(x,y,u,v)\rangle \; \ldots\ldots\ldots (3.1)$$

When $K_{\propto}(x,y,u,v) = C_{1\propto} e^{iC_{2\propto}[(x^2+u^2+y^2+v^2)cos\alpha - 2(xu+yv)]}$

, $C_{1\propto} = \sqrt{\frac{1-icot\propto}{2\pi}}$, $C_{2\propto} = \frac{1}{2sin\alpha}$ ……… (3.2)

The right hand side of (3.1) has a meaning as the application of $f \in E^*$ to $K_{\propto}(x,y,u,v) \in E$.

It extended to the complex space as an entire function given by

$$FRFT\{F(X,Y)\} = F_{\propto}(g,h) = \langle f(x,y), K_{\propto}(x,y,g,h)\rangle \; \ldots\ldots\ldots (3.3)$$

Where

$K_{\propto}(x,y,g,h) = C_{1\propto} e^{iC_{2\propto}[(x^2+g^2+y^2+h^2)cos\alpha - 2(xg+yh)]}$

The right hand side of (3.3) is meaningful because for each $g,h \in c^n$, $K_{\propto}(x,y,u,v) \in E$, as a function of x,y.

4. Examples on Generalized 2DFrFt

4.1- Prove that $[2DFrFT(1)](u,v)$

$$= \frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha} e^{\frac{i\pi}{2}} e^{\frac{i}{2}\left[\frac{3+cos2\alpha}{sin2\alpha}\right](u^2+v^2)}$$

Proof:

$2DFrFT\{f(x,y)\}(u,v)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{1\alpha} e^{\frac{i}{2sin\alpha}[(x^2+u^2+y^2+v^2)cos\alpha - 2(xu+yv)]} f(x,y)dxdy$$

$$[2DFrFT(1)](u,v) = C_{1\alpha} e^{\frac{i}{2}(u^2+v^2)cot\alpha} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2)cot\alpha - i(xu+yv)cosec\alpha}\, 1\, dxdy$$

$$\int_{-\infty}^{\infty} e^{iap^2+ibp}\, dp = \frac{e^{\frac{i\pi}{4}}\sqrt{\pi}}{\sqrt{a}} e^{\frac{ib^2}{4a}} \text{ ----(4.1)}$$

where, $a = \frac{cot\alpha}{2}$, $b = -u\, cosec\alpha$

$$= C_{1\alpha} e^{\frac{i}{2}(u^2+v^2)cot\alpha} \left[\frac{e^{\frac{i\pi}{4}}\sqrt{\pi}}{\sqrt{\frac{cot\alpha}{2}}} e^{\frac{i(-u\,cosec\alpha)^2}{4\frac{cot\alpha}{2}}}\right]\left[\frac{e^{\frac{i\pi}{4}}\sqrt{\pi}}{\sqrt{\frac{cot\alpha}{2}}} e^{\frac{i(-v\,cosec\alpha)^2}{4\frac{cot\alpha}{2}}}\right]$$

$$= C_{1\alpha} e^{\frac{i}{2}(u^2+v^2)cot\alpha} \left[\frac{e^{\frac{i\pi}{4}}\pi}{\frac{cot\alpha}{2}}\right] e^{\frac{iu^2cosec^2\alpha + iv^2cosec^2\alpha}{2cot\alpha}}$$

$$= \frac{\sqrt{1-icot\alpha}}{cot\alpha}\sqrt{2\pi} e^{\frac{i\pi}{2}} e^{\frac{i}{2}u^2\left[\frac{cos\alpha}{sin\alpha}+\frac{1}{sin\alpha cos\alpha}\right]} e^{\frac{i}{2}v^2\left[\frac{cos\alpha}{sin\alpha}+\frac{1}{sin\alpha cos\alpha}\right]}$$

$$= \frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha} e^{\frac{i\pi}{2}} e^{\frac{i}{2}u^2\left[\frac{1+cos^2\alpha}{sin\alpha cos\alpha}\right]} e^{\frac{i}{2}v^2\left[\frac{1+cos^2\alpha}{sin\alpha cos\alpha}\right]}$$

$$= \frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha} e^{\frac{i\pi}{2}} e^{\frac{i}{2}u^2\left[\frac{3+cos2\alpha}{sin2\alpha}\right]} e^{\frac{i}{2}v^2\left[\frac{3+cos2\alpha}{sin2\alpha}\right]}$$

$$= \frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha} e^{\frac{i\pi}{2}} e^{\frac{i}{2}\left[\frac{3+cos2\alpha}{sin2\alpha}\right](u^2+v^2)}$$

$[2DFrFT(1)](u,v)$

$$= \frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha} e^{\frac{i\pi}{2}} e^{\frac{i}{2}\left[\frac{3+cos2\alpha}{sin2\alpha}\right](u^2+v^2)}$$

4.2- Prove that: $[2DFrFT\; \delta(x-a, y-b)](u,v) = C_{1\alpha} e^{\frac{i}{2sin\alpha}[(a^2+u^2+b^2+v^2)cos\alpha - 2(au+bv)]}$

$[2DFrFT\,\{f(x,y)\}](u,v)$

$$= C_{1\alpha} e^{\frac{i}{2}(u^2+v^2)cot\alpha} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2)cot\alpha - i(xu+yv)cosec\alpha}\; f(x,y)\, dxdy$$

$$[2DFrFT\; \delta(x-a,y-b)](u,v) = C_{1\alpha} e^{\frac{i}{2}(u^2+v^2)cot\alpha} e^{\frac{i}{2}(a^2+b^2)cot\alpha - i(au-bv)cosec\alpha} = C_{1\alpha} e^{\frac{i}{2sin\alpha}(a^2+u^2+b^2+v^2)cos\alpha - 2(au-bv)}$$

4.3- Prove that: $\left[2DFrFT\; e^{i(ax^2+by^2)}\right](u,v) = \sqrt{\frac{2\pi(1-cot\emptyset)}{(cot\alpha+2a)(cot\alpha+2b)}} e^{\frac{i}{2}(u^2+v^2)cot\alpha} e^{icosec^2\alpha\left(\frac{u^2}{2cot\alpha+4a}+\frac{v^2}{2cot\alpha+4b}\right)}$

Proof:

$$[2DFrFT\,\{f(x,y)\}](u,v) = C_{1\alpha} e^{\frac{i}{2}(u^2+v^2)cot\alpha} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2)cot\alpha - i(xu+yv)cosec\alpha}\; f(x,y)\, dxdy$$

$$\left[2DFrFT\; e^{i(ax^2+by^2)}\right](u,v) = C_{1\alpha} e^{\frac{i}{2}(u^2+v^2)cot\alpha} \int_{-\infty}^{\infty} e^{ix^2\left[\frac{cot\alpha}{2}+a\right] - iuxcosec\alpha} dx \int_{-\infty}^{\infty} e^{iy^2\left[\frac{cot\alpha}{2}+b\right] - ivycosec\alpha} dy$$

Taking $a = \left[\frac{cot\alpha}{2}+a\right]$, $b = -ucosec\alpha$ etc

Therefore using equation (4.1)

$\left[2DFrFT\; e^{i(ax^2+by^2)}\right](u,v)$

$$= \sqrt{\frac{1-icot\emptyset}{2\pi}} e^{\frac{i}{2}(u^2+v^2)cot\alpha} \frac{e^{\frac{i\pi}{4}}\sqrt{\pi}}{\sqrt{\frac{cot\alpha}{2}+a}} e^{\frac{i(-u\,cosec\alpha)^2}{4\left(\frac{cot\alpha}{2}+a\right)}} \frac{e^{\frac{i\pi}{4}}\sqrt{\pi}}{\sqrt{\frac{cot\alpha}{2}+b}} e^{\frac{i(-u\,cosec\alpha)^2}{4\left(\frac{cot\alpha}{2}+b\right)}}$$

$$= \sqrt{\frac{1-icot\emptyset}{2\pi}} \frac{2\pi e^{\frac{i\pi}{2}}}{\sqrt{(cot\alpha+2a)}\sqrt{(cot\alpha+2b)}} e^{\frac{i}{2}(u^2+v^2)cot\alpha} e^{icosec^2\alpha\left(\frac{u^2}{2cot\alpha+4a}+\frac{v^2}{2cot\alpha+4b}\right)}$$

$$= \sqrt{\frac{2\pi(1-cot\emptyset)}{(cot\alpha+2a)(cot\alpha+2b)}} e^{\frac{i}{2}(u^2+v^2)cot\alpha}$$

$e^{icosec^2\alpha(\frac{u^2}{2cot\alpha+4a}+\frac{v^2}{2cot\alpha+4b})}$

4.4- Prove that: $\left[2DFrFT\ e^{i(ax+by)}\right](u,v)$

$$=\frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha}e^{\frac{i\pi}{2}}e^{\frac{i}{2cot\alpha}}$$

$$e^{\{(a^2+b^2)-2cosec\alpha(au+bv)\}+\left[\frac{3+cos2\alpha}{sin2\alpha}\right](u^2+v^2)}$$

Proof-

$$[2DFrFT\ \{f(x,y)\}](u,v)\ =C_{1\alpha}e^{\frac{i}{2}(u^2+v^2)cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{i}{2}(x^2+y^2)cot\alpha-i(xu+yv)cosec\alpha}f(x,y)\,dxdy$$

$$\left[2DFrFT\ e^{i(ax+by)}\right](u,v)$$

$$=C_{1\alpha}e^{\frac{i}{2}(u^2+v^2)cot\alpha}\int_{-\infty}^{\infty}e^{ix^2cot\alpha-iuxcosec\alpha+iax}dx$$

$$\int_{-\infty}^{\infty}e^{iy^2cot\alpha-ivycosec\alpha+iby}dy$$

Taking $a=\frac{cot\alpha}{2}$, $b=a-ucosec\alpha$ etc

Therefore using equation (4.1)

$$\left[2DFrFT\ e^{i(ax+by)}\right](u,v)\ =\sqrt{\frac{1-icot\emptyset}{2\pi}}\,e^{\frac{i}{2}(u^2+v^2)cot\alpha}\frac{e^{\frac{i\pi}{4}}\sqrt{\pi}}{\sqrt{\frac{cot\alpha}{2}}}e^{\frac{i(a-u\,cosec\alpha)^2}{4\left(\frac{cot\alpha}{2}\right)}}\frac{e^{\frac{i\pi}{4}}\sqrt{\pi}}{\sqrt{\frac{cot\alpha}{2}}}e^{\frac{i(b-u\,cosec\alpha)^2}{4\left(\frac{cot\alpha}{2}\right)}}$$

$$=\frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha}e^{\frac{i\pi}{2}}e^{\frac{i}{2}\left[u^2cot\alpha+\frac{(a-u\,cosec\alpha)^2}{cot\alpha}\right]}$$

$$e^{\frac{i}{2}\left[v^2cot\alpha+\frac{(b-v\,cosec\alpha)^2}{cot\alpha}\right]}$$

$$=\frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha}e^{\frac{i\pi}{2}}e^{\frac{i}{2}\left[u^2cot\alpha+\frac{a^2-2au\,cosec\alpha}{cot\alpha}+\frac{u^2\,cosec^2\alpha}{cot\alpha}\right]}$$

$$e^{\frac{i}{2}\left[v^2cot\alpha+\frac{b^2-2bv\,cosec\alpha}{cot\alpha}+\frac{v^2\,cosec^2\alpha}{cot\alpha}\right]}$$

$$=\frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha}e^{\frac{i\pi}{2}}e^{\frac{i}{2}\left[\frac{a^2-2au\,cosec\alpha}{cot\alpha}+\frac{u^2\,(3+cos2\alpha)}{sin2\alpha}\right]}$$

$$e^{\frac{i}{2}\left[\frac{b^2-2bv\,cosec\alpha}{cot\alpha}+\frac{v^2\,(3+cos2\alpha)}{sin2\alpha}\right]}$$

$$=\frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha}$$

$$e^{\frac{i\pi}{2}}e^{\frac{i}{2cot\alpha}\left[\frac{(a^2+b^2)-2\,cosec\alpha(au+bv)}{cot\alpha}\right]+\frac{(3+cos2\alpha)(u^2+v^2)}{sin2\alpha}}$$

$$\left[2DFrFT\ e^{i(ax+by)}\right](u,v)$$

$$=\frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha}$$

$$e^{\frac{i\pi}{2}}e^{\frac{i}{2cot\alpha}\left[\frac{(a^2+b^2)-2\,cosec\alpha(au+bv)}{cot\alpha}\right]+\left[\frac{(3+cos2\alpha)}{sin2\alpha}\right](u^2+v^2)}$$

Generalized 2DFrFT

S.N.	$f(x,y)$	$\{2DFrFTf(x,y)\}(u,v)$
1	1	$\frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha}e^{i\pi/2}$ $e^{\frac{i}{2}(3cosec2\alpha+cot2\alpha)(u^2+v^2)}$
2	$\delta(x-a,y-b)$	$C_{1\alpha}e^{\frac{i}{2sin\alpha}[(a^2+u^2+b^2+v^2)cos\alpha-2(au+bv)]}$
3	$e^{i(ax^2+by^2)}$	$\sqrt{\frac{2\pi(1-cot\emptyset)}{(cot\alpha+2a)(cot\alpha+2b)}}e^{\frac{i}{2}(u^2+v^2)cot\alpha}$ $e^{icosec^2\alpha(\frac{u^2}{2cot\alpha+4a}+\frac{v^2}{2cot\alpha+4b})}$
4	$e^{i(ax+by)}$	$\frac{\sqrt{2\pi(1-icot\alpha)}}{cot\alpha}$ $e^{\frac{i\pi}{2}}e^{\frac{i}{2cot\alpha}}e^{\{(a^2+b^2)-2cosec\alpha(au+bv)\}}$ $+\left[\frac{3+cos2\alpha}{sin2\alpha}\right](u^2+v^2)$

5. Conclusion

In the present work generalization of two dimensional Fractional Fourier transform is presented. Some applications of 2DFrFt is obtained.

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