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Introduction to Generalized Simplified Fractional Fourier Transform (SFRFT)

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Abstract: ***Simplified fractional Fourier transform (SFRFT) are equivalent to the fractional Fourier transform (FRFT) for the fractional filter design or fractional correlation. Besides, the SFRFTs can be used in many applications. In the present work, we have generalized Simplified fractional Fourier transform in distributional sense.***

Keywords: Fourier transform, Simplified Fractional Fourier transform, generalized function.

1. Introduction

The fractional Fourier transform (FRFT) has been used for many years and it is useful in many applications. Most applications of the FRFT are based on the design of fractional filters or on fractional correlation. Simplified fractional Fourier transform are special cases of linear canonical transform. They have the same capabilities as the original FRFT. But they are simpler than the original FRFT in terms of digital computation, optical implementation, implementation of gradient index media and implementation of radar system [1].

The simplified FRFT of type 1, type 2, type 3 are special cases of LCT where

$\{a,b,c,d\} = \{cot\alpha, 1, -1, 0\}$, $\{a,b,c,d\} = \{1, tan\alpha, -2cot\alpha, -1\}$, $\{a,b,c,d\} = \left\{cos\varphi, W_x sin\varphi, \frac{-sin\varphi}{W_x}, cos\varphi\right\}$ [2].

In digital implementation and conventional convolution, the simplified FRFT of type 1 is simpler than the FRFT but it has some effects as the FRFT of order α for filter design. Furthermore, the simplified FRFT of type 1 has a main use in optical implementation.

The SFRFT of type 2 has main use in optical implementation for the reduction of optical components. The SFRFT of type 3 has additive and periodic properties that use the GRIN medium to design fractional filter that the total length of the system is independent of the value of φ [3,4]. In the present paper introduction to generalized Simplified fractional FTs are presented.

2. Conventional Simplified Fractional Fourier Transform

2.1 The Simplified Fractional Fourier transform (SFRFT) of type 1:

The Simplified fractional Fourier transform with parameter α of $f(t)$ denoted by $O^{\alpha}_{F(1)}(f(t))$ performs a linear operation given by the integral transform,

$O^{\alpha}_{F(1)}(f(t)) = F_{\alpha}\{f(t)\}(u)$

$= \int_{-\infty}^{\infty} f(t) K_{\alpha}(t,u) dt$ (2.1)

Where the kernel,

$K_{\alpha}(t,u) = (j2\pi)^{-1/2} e^{\left(-jut + \frac{j}{2}t^2 cot\alpha\right)}$ (2.2)

2.2 The Simplified Fractional Fourier transform (SFRFT) of type 2:

The Simplified fractional Fourier transform with parameter θ of $f(t)$ denoted by $O^{\theta}_{F(2)}(f(t))$ is defined as

$O^{\theta}_{F(2)}(f(t)) = F_{\theta}\{f(t)\}(u)$

$= \int_{-\infty}^{\infty} f(t) K_{\theta}(t,u) dt$ (2.3)

Where the kernel,

$K_{\theta}(t,u) = (\frac{cot\theta}{j2\pi})^{1/2} e^{-\frac{j}{2}u^2 cot\theta} e^{-jut + \frac{j}{2}t^2 cot\theta}$ (2.4)

2.3 The Simplified Fractional Fourier transform (SFRFT) of type 3:

The Simplified fractional Fourier transform with parameter $\emptyset$ of $f(t)$ denoted by $O^{\emptyset}_{F(3)}(f(t))$ is defined as

$O^{\emptyset}_{F(3)}(f(t)) = F_{\emptyset}\{f(t)\}(u)$

$= \int_{-\infty}^{\infty} f(t) K_{\emptyset}(t,u) dt$ (2.5)

Where the kernel,

$K_{\emptyset}(t,u) = (\frac{csc\emptyset}{j2\pi W_x})^{1/2} e^{\frac{j}{2W_x}u^2 cot\emptyset} e^{\left(-jut\frac{csc\emptyset}{W_x} + \frac{j}{2W_x}t^2 cot\emptyset\right)}$

(2.6)

where $W_x = \frac{1}{K}\left(\frac{1}{n_x n_0}\right)^{1/2}$, $\emptyset = L\left(\frac{n_x}{n_0}\right)^{1/2}$

3. Testing Function Space

An infinitely differentiable complex valued smooth function φ on R^n belongs to $E(R^n)$ if for each compact set $I < S_a$ where, $S_a = \{t\epsilon R^n, |t| \leq a, a > 0\}, I\epsilon R^n$.

$\gamma_{E,p}(\varphi) = \sup_{t\epsilon I} |D_t^p \varphi(t)| < \infty$, where $p = 1,2,3, ----$

Thus $E(R^n)$ will denote the space of all $\varphi\epsilon E(R^n)$ with support contained in S_a.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that f is a Simplified fractional Fourier transformable if it is a member of E^*, the dual space of E.

4. Distributional Simplified fractional Fourier transform (SFRFT) of type 1:

The distributional Simplified fractional Fourier transform of $f(t)\epsilon E(R^n)$ can be defined by

$O^{\alpha}_{F(1)}(f(t)) = F_{\alpha}\{f(t)\}(u) = \langle f(t), K_{\alpha}(t,u)\rangle$ (4.1)

where, $K_{\alpha}(t,u) = (j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)}$ (4.2)

RHS of equation (4.1) has a meaning as the applications of $f\epsilon E^*$ to $K_{\alpha}(t,u)\epsilon E$.

5. Distributional Simplified fractional Fourier

5.1-Conversion of Simplified fractional Fourier transform to Simplified Fourier transform

Proof: The generalized simplified fractional Fourier transform is

$$O^{\alpha}_{F(1)}(f(t)) = F_{\alpha}\{f(t)\}(u)$$
$$= \int_{-\infty}^{\infty} f(t)(j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)} dt$$
$$= (j2\pi)^{-1/2} \int_{-\infty}^{\infty} f(t) e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)} dt$$
$$= (j2\pi)^{-1/2} \int_{-\infty}^{\infty} \tilde{f}(t) e^{(-jut)} dt$$

where, $\tilde{f}(t) = f(t)e^{\left(\frac{j}{2}t^2 cot\alpha\right)}$

$= (j2\pi)^{-1/2} F\{\tilde{f}(t)\}$

$F_{\alpha}\{f(t)\}(u) = (j2\pi)^{-1/2} F\{\tilde{f}(t)\}$,

where, $F\{\tilde{f}(t)\}$ is simplified Fourier transform.

Thus if $f(t)$ is any signal then

$O^{\alpha}_{F(1)}(f(t)) = (j2\pi)^{-1/2} F\{\tilde{f}(t)\}$

5.2-Generalized Simplified fractional Fourier transform reduces to conventional Simplified Fourier transform if $\theta = \frac{\pi}{2}$.

Proof: The generalized simplified fractional Fourier transform is

$$O^{\alpha}_{F(1)}(f(t)) = F_{\alpha}\{f(t)\}(u)$$
$$= \int_{-\infty}^{\infty} f(t)(j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)} dt$$

putting $\alpha = \frac{\pi}{2}$

$$= \int_{-\infty}^{\infty} f(t)(j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\frac{\pi}{2}\right)} dt$$
$$= (j2\pi)^{-1/2} \int_{-\infty}^{\infty} f(t) e^{(-jut)} dt$$

$= F\{f(t)\}(u)$, which is simplified Fourier transform of $f(t)$.

$\therefore O^{\alpha}_{F(1)}(f(t)) = F_{\alpha}\{f(t)\}(u) = F\{f(t)\}(u)$.

6. Properties of Kernel of Simplified fractional Fourier transform of type 1

6.1 To prove $K_{(-\alpha)}(t,u) = K^*_{\alpha}(-t,u)$

Proof:- Consider,

$$K_{\alpha}(t,u) = (j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)}$$
$$K_{-\alpha}(t,u) = (j2\pi)^{-1/2} e^{\left[-jut+\frac{j}{2}t^2 cot(-\alpha)\right]}$$
$$= (j2\pi)^{-1/2} e^{\left[-jut-\frac{j}{2}t^2 cot\alpha\right]}$$

$= Ae^{\left[-jut-\frac{j}{2}t^2 cot\alpha\right]}$, where, $A = (2\pi j)^{-1/2}$

$= K^*_{\alpha}(-t,u)$

6.2 To prove $K_{\alpha}(-t,u) = e^{2j}K_{\alpha}(t,u)$

Proof:- Consider,

$$K_{\alpha}(t,u) = (j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)}$$
$$K_{\alpha}(-t,u) = (j2\pi)^{-1/2} e^{\left(jut+\frac{j}{2}t^2 cot\alpha\right)}$$
$$= e^{2j}(j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)}$$
$$= e^{2j}K_{\alpha}(t,u)$$
$$K_{\alpha}(-t,u) = e^{2j}K_{\alpha}(t,u)$$

6.3 To prove $K_{\alpha}(t,0) = e^{ujt}K_{\alpha}(t,u)$

Proof:- Consider,

$$K_{\alpha}(t,u) = (j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)}$$
$$\therefore K_{\alpha}(t,0) = (j2\pi)^{-1/2} e^{\left(\frac{j}{2}t^2 cot\alpha\right)}$$
$$= (j2\pi)^{-1/2} e^{\left(\frac{j}{2}t^2 cot\alpha\right)} e^{-jut+jut}$$
$$= e^{jut}K_{\alpha}(t,u)$$

7. Linearity Property

If $O^{\alpha}_{F(1)}(f(t))$ is generalized Simplified fractional Fourier transform of $f(t)$ and $O^{\alpha}_{F(1)}(g(t))$ generalized Simplified fractional Fourier transform of $g(t)$ then

$$O^{\alpha}_{F(1)}[C_1 f(t) + C_2 g(t)](u)$$
$$= C_1 O^{\alpha}_{F(1)}[f(t)](u) + C_1 O^{\alpha}_{F(1)}[g(t)](u)$$

Proof:

Consider,

$$O^{\alpha}_{F(1)}[C_1 f(t) + C_2 g(t)](u)$$
$$= \int_{-\infty}^{\infty} (j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)} [C_1 f(t) + C_2 g(t)] dt$$
$$= C_1 \int_{-\infty}^{\infty} (j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)} f(t) dt$$
$$+ C_2 \int_{-\infty}^{\infty} (j2\pi)^{-1/2} e^{\left(-jut+\frac{j}{2}t^2 cot\alpha\right)} g(t) dt$$
$$= C_1 O^{\alpha}_{F(1)}[f(t)](u) + C_1 O^{\alpha}_{F(1)}[g(t)](u)$$

8. Conclusion

In the present work generalization of simplified fractional Fourier transform is presented. They have great potential for replacing the original FRFTs in many applications.

References

[1] Zemanian A.H., Generalized Integral transform, Interscience Publication, New York, 1968.

[2] B.N.Bhosale, Integral transformation of Generalized functions. Discovery publishing hour, 2005.

[3] Ervin S., Djurovic I. et.al., fractional Fourier transform as a signal processing tool; an review of recent developments, signal processing (2011), 1351-1369.

[4] V. Namias, The Fractional order Fourier transform and its applications to quantum mechanics, J. Inst. Maths appl. 25 241-265, 1980.

[5] R. N. Bracewll, The Fourier Transform and its Application, 2nd ed., McGraw-Hill, New York, 1986.

[6] H.M. Ozaktes, A. Zalevsky, M.A. Katay, the FRFT with Application in optics and signal processing, John Wiky, Chichester New York, USA, 2001.

[7] V.D. Sharma, Generalized Two-Dimensional Fractional Fourier Transform and Inversion Formula, Journal of Fractional Calculus and Applications, Vol. 4(2) July 2013, pp. 191- 195.

[8] V.D. Sharma, Operational calculus on generalized two-Dimensional Fractional Fourier transform, International Journal of Engineering and Innovative Technology (IJEIT) Volume 3, Issue 2, August 2013 ,253-256

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