International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

Abelian Theorem of Generalized Fourier-Stieltjes Transform

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Abstract: ***The integral transform plays an important role in the solution of a wide class of problems of mathematical physics, for instance, boundary value problem for Laplace equation etc. Considerably which are used to solve the boundary value problems of Mathematical Physics and Partial Differential equation etc. In this paper, the generalization of Fourier- Stieltjes transform is presented. Abelian theorem of initial value type and final value type are proved. These results are widely used to solve the boundary value problems.***

Keywords: Stieltjes transform, Fourier transform, Fourier-Stieltjes transform, generalized function, Testing function space.

1. Introduction

The conventional Fourier-Stieltjes transform of a complex valued smooth function $f(t,x)$ is defined by the convergent integral.

$$F(s,y) = FS\{f(t,x)\}$$
$$= \int_0^\infty \int_0^\infty f(t,x)\, e^{-ist} (x+y)^{-p}\, dt\, dx \qquad (1.1)$$

Where, t and x are positive real numbers. The peculiarity of the transformation (1.1) lies in fact that it involves the integration with respect to parameter. Its extension to the distribution of compact support which involves some complicated analysis has been done by Zemanian [7].

The distributional Fourier-Stieltjes transform is defined as

$$FS\{f(t,x)\} = F(s,y) = \langle f(t,x), e^{-ist} (x+y)^{-p} \rangle \qquad (1.2)$$

where for each fixed $t(0 \le t \le \infty), x(0 \le x \le \infty)$ the right hand side of above equation has same as an application of $f(t,x) \in FS_\alpha^*$ to $e^{-ist}(x+y)^{-p} \in FS_\alpha$ for some $s > 0$ and $k < Re\ p$ and FS_α^* is dual space of FS_α.

Considerably the Abelian theorem is important in solving the boundary value problems of partial differential equation and mathematical physics etc. We know that, Integral transformation is one the well known techniques used for the functions transformation and integral transform method have proved to be the great importance in solving boundary value problems of mathematical physics and partial differential equation[1] etc.

There are various transforms such as Laplace, Fourier, Stieltjes, Mellin etc. We studied several theorems for the integral transform and then extend this result to the distributional generalized sense. This paper provides extension of distributional generalized Fourier-Stieltjes transform to initial and final value theorem.

Different S-types spaces are introduced in[3,4,5] along with some operators on these spaces. The testing spaces FS_α transform given as-

A function $\emptyset(t,x)$ defined on $t(0 \le t \le \infty), x(0 \le x \le \infty)$ is said to be member of FS_α if $\emptyset(t,x)$ is smooth and for each non-negative integer p, l, q,

$$\gamma_{k,p,l,q}\emptyset(t,x) = \sup_{I_1} \left| t^k (1+x)^p D_t^l (xD_x)^q \emptyset(t,x) \right| \le \infty \quad (1.3)$$

In this paper we have proved some Abelian theorem of initial value type and some Lemmas in section 2. In section 3, we established the Abelian theorem of final value type. The notation and terminology will follow that Zemanian[7].

2. Some Abelian Theorem of Initial value Type

Conditions:

(i) $f(t,x) = 0$ for $-\infty < t < T,\ 0 < x < \infty$

(ii) There exist real number s and y such that $f(t,x)e^{-ist}(x+y)^{-p}$ is absolutely integrable.

2.1 Theorem:-

For locally integrable function $f(t,x)$ satisfying above condition with $T = 0$ and existence of any complete constant A and real number m and n such that-

i] $m > -1$ ii] $n > -1$ iii] $Re\ p > n+1$

And $\lim_{t,x\to 0^+} \frac{(-i)^{m+n}\,\Gamma(m+n)}{C\, t^m\, x^n} f(t,x) = A$ then

$$\lim_{\substack{s\to\infty \\ y\to 0^+}} s^{m+1} y^{p-n-1} F(s,y) = A$$

Where, $C = \frac{\Gamma p}{\Gamma p-n-1\, \Gamma n+1}$

Proof:-

We extend the result to a distributional FS Transform-

i] $\int_0^\infty t^m e^{-ist} dt = \frac{(-i)^{m+1}\ \Gamma m+1}{s^{m+1}}$

for $m > -1,\ s > 0$ (2.1.1)

ii] Also by Widder-

$\int_0^\infty x^n (x+y)^{-p} dx = \frac{\Gamma p-n-1\, \Gamma n+1}{\Gamma p\ y^{p-n-1}}$

for $n > -1$, $Re\ p > n+1$, (2.1.2)

So that-

$\left| s^{m+1} y^{p-n-1} F(s,y) - A \right|$

$$= s^{m+1} y^{p-n-1} \int_0^\infty \int_0^\infty e^{-ist}(x+y)^{-p} \left| f(t,x) - \frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\Gamma(m+1)} \right| dt\,dx$$

$$\le s^{m+1} y^{p-n-1} \{\int_0^T \int_0^X e^{-ist}(x+y)^{-p} \left| f(t,x) - \frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\Gamma(m+1)} \right| dt\,dx + \int_T^\infty \int_0^X e^{-ist}(x+y)^{-p} \left| f(t,x) - \frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\Gamma(m+1)} \right| dt\,dx + \int_0^T \int_X^\infty e^{-ist}(x+y)^{-p} \left| f(t,x) - \frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\Gamma(m+1)} \right| dt\,dx + \int_T^\infty \int_X^\infty e^{-ist}(x+y)^{-p} \left| f(t,x) - \frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\Gamma(m+1)} \right| dt\,dx\}$$

(2.1.3)

By Widder Page No.181,

For any ϵ, we can find a constant M Such that –

$$\therefore s^{m+1} \int_T^\infty e^{-ist} \left| f(x,t) - \frac{C\,x^n\,A\,t^m}{(-i)^{m+1}\Gamma(m+1)} \right| dt < \frac{C\,x^n\,M s^{m+1}}{(-i)^{m+1}(s-\epsilon)e^{i(s-\epsilon)T}}$$ for $s > \epsilon$ Whereas R.H.S. of this inequality approaches zero as s becomes infinite.

And also by Widder page number 183, Lemma 2.

$$\lim_{y\to 0^+} \int_X^\infty \frac{\alpha(x)}{(x+y)^p}\,dx \to 0$$

Therefore by using above inequalities

$$\le \lim_{\substack{s\to\infty \\ y\to 0^+}} s^{m+1} y^{p-n-1} \int_0^T \int_0^X \frac{c\,x^n\,t^m}{(-i)^{m+1}\Gamma(m+1)} \left| f(t,x) \frac{(-i)^{m+1}\Gamma(m+1)}{c\,x^n\,t^m} - A \right| e^{-ist}(x+y)^{-p}\,dt\,dx$$

$$\le \sup_{\substack{0\le t\le T \\ 0\le x\le X}} \left| f(t,x) \frac{(-i)^{m+1}\Gamma(m+1)}{C\,x^n\,t^m} - A \right|$$

Since, T and X are arbitrary

$$\lim_{\substack{s\to\infty \\ y\to 0^+}} |s^{m+1} y^{p-n-1} F(s,y) - A| \le \lim_{\substack{t\to 0^+ \\ x\to 0^+}} \left| f(t,x) \frac{(-i)^{m+1}\Gamma(m+1)}{c\,x^n\,t^m} - A \right|$$

From which the result follows.

2.2 Lemma:

If $f(t,x) \in FS_\alpha$ with its support in $t_f \le t \le \infty$ and $x_f \le x \le \infty$. Where $t_f > 0$ and $x_f > 0$ then $|F(s,y)| \le M\,t^k\,x^p$ where, M is sufficiently large constant.

Proof:-

Let $g(t,x)$ be a smooth function on $0 \le t \le \infty$ and $0 \le x \le \infty$ such that $g(t,x) = 1$ on $[t_f,\infty)$ and $[x_f,\infty)$ and $g(t,x) = 0$ on $(0,T)$ and $(0,X)$ where, $T < t_f$ and $X < x_f$. As a distribution of slow growth satisfies a boundedness property of distribution, there exist a positive constant K and a non negative integer such ρ that

$$|F(s,y)| \le K \max_{0\le t\le \rho} \sup_{0\le t,X<\infty} |t^k(1+x)^p D_t^l (xD_x)^q \varnothing(t,x)|$$

$$\le K \max_{0\le t\le\rho} \sup_{0\le t,X<\infty} |t^k(1+x)^p D_t^l (xD_x)^q g(t,x) f(t,x)|$$

$$\le K \max_{0\le t\le\rho} \sup_{\substack{0\le t\le\infty \\ 0\le X\le\infty}} |t^k(1+x)^p\, g^{l+q}(t,x) f(t,x)|$$

Where $g^{l+q}(t,x)$ gives the lth derivative of $g(t,x)$ w.r.to t and derivative of $g(t,x)$ w.r.to x.

$$|F(s,y)| \le K \max_{0\le t\le\rho} \sup_{\substack{0\le t\le\infty \\ 0\le X\le\infty}} |t^k(1+x)^p\, g^{l+q}(t,x) f(t,x)|$$

$$\le K \max_{0\le t\le\rho} \sup_{\substack{0\le t\le\infty \\ 0\le X\le\infty}} t^k\, x^{p+q}$$

As $f(t,x)$ is a member of FS_α and $k, q = 0,1,2 \ldots\ldots$

$$|F(s,y)| \le K \max_{0\le t\le\rho} \sup_{\substack{0\le t\le\infty \\ 0\le X\le\infty}} t^k\, x^p$$

$$|F(s,y)| \le M\,t^k x^p$$

Where, M is sufficiently large constant.

2.3 Lemma

If $f(t,x)$ is decomposed into $f(t,x) = f_1(t,x) + f_2(t,x)$, where $f_1(t,x)$ is the ordinary FS transformable function satisfying the hypothesis theorem (3.1) and $f_2(t,x)$ is a regular distribution satisfying the above lemma then

$$\lim_{\substack{s\to\infty \\ y\to 0^+}} s^{m+1} y^{p-n-1} F(s,y) = A$$

Proof:-

It follows from above lemma that-

i] $\lim_{\substack{s\to\infty \\ y\to 0^+}} s^{m+1} y^{p-n-1} F(s,y) = 0$

And from theorem (3.1), we have-

ii] $\lim_{\substack{s\to\infty \\ y\to 0^+}} s^{m+1} y^{p-n-1} F(s,y) = A$

Moreover the distributional FS transform F_1 of f_1 equals the ordinary generalized FS transform of f_1, so that $F(s,y) = F_1(s,y) + F_2(s,y)$.Therefore (i) and (ii) proves theorem.

3. Some Abelian Theorems of the Final Value Type

Theorem:

For a locally integrable function $f(t,x)$ satisfying condition with $T = 0$ and existence of any complex constant A and real number m and n such that-

i] $m > -1$ ii] $n > -1$ iii] $Re\ p > n+1$

and $\lim_{t,x\to\infty} \frac{(-i)^{m+n}\Gamma(m+n)}{C\,t^m\,x^n} f(t,x) = A$ then

$\lim_{\substack{s\to 0^+ \\ y\to\infty}} s^{m+1} y^{p-n-1} F(s,y) = A$ where, $C = \frac{\Gamma p}{\Gamma p-n-1\,\Gamma n+1}$

Proof:-

We know by Zemanian page number 243 and extend the result to a distributional FS transform-

$$\int_0^\infty t^m e^{-ist}\,dt = \frac{(-i)^{m+1}\Gamma m+1}{s^{m+1}}$$

for $m > -1$, $s > 0$ (3.1)

ii] Also by Widder-

$$\therefore \int_0^\infty x^n (x+y)^{-p}\,dx = \frac{\Gamma p-n-1\,\Gamma n+1}{\Gamma p\; y^{p-n-1}},$$

for $n > -1$, $Re\ p > n+1$. (3.2)

So that-

$$|s^{m+1} y^{p-n-1} F(s,y) - A|$$

$$|s^{m+1} y^{p-n-1} F(s,y) - A|$$

$$\le s^{m+1} y^{p-n-1} \{\int_0^T \int_0^X e^{-ist}(x+y)^{-p} \left| f(t,x) - \frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\Gamma(m+1)} \right| dt\,dx + \int_T^\infty \int_0^X e^{-ist}(x+y)^{-p} \left| f(t,x) - \frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\Gamma(m+1)} \right| dt\,dx$$

$$+\int_0^T\int_X^\infty e^{-ist}(x+y)^{-p}\left|f(t,x)-\frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\,\Gamma(m+1)}\right|dt\,dx$$
$$+\int_T^\infty\int_X^\infty e^{-ist}(x+y)^{-p}\left|f(t,x)-\frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\,\Gamma(m+1)}\right|dt\,dx\}$$
(3.3)

Here,

$$\lim_{s\to 0^+} s^{m+1}\int_0^T e^{-ist}\left|f(x,t)-\frac{C\,x^n\,A\,t^m}{(-i)^{m+1}\,\Gamma(m+1)}\right|dt \to 0$$

As $s \to 0^+$ above integral approaches zero

Also,

$$\lim_{y\to\infty}\int_0^X \frac{\alpha(x)}{(x+y)^p}\,dx \to 0$$

As y approaches above integral becomes zero.

Therefore (3.3) implies that-

$$\left\|s^{m+1}\,y^{p-n-1}F(s,y)-A\right\|$$
$$\le s^{m+1}\,y^{p-n-1}$$
$$\int_T^\infty\int_X^\infty e^{-ist}(x+y)^{-p}\left|f(t,x)-\frac{A\,C\,x^n\,t^m}{(-i)^{m+1}\,\Gamma(m+1)}\right|dt\,dx$$

Taking limits on $s \to 0^+$ $y \to \infty$

$$\lim_{\substack{s\to 0^+\\ y\to\infty}}\left\|s^{m+1}\,y^{p-n-1}F(s,y)-A\right\|$$
$$\le \lim_{\substack{s\to 0^+\\ y\to\infty}} s^{m+1}\,y^{p-n-1}$$
$$\int_T^\infty\int_X^\infty \frac{c\,x^n\,t^m}{(-i)^{m+1}\,\Gamma(m+1)}$$
$$\left|f(t,x)\frac{(-i)^{m+1}\,\Gamma(m+1)}{c\,x^n\,t^m}-A\right|e^{-ist}(x+y)^{-p}\,dt\,dx$$
$$\le \sup_{\substack{T\le t\le\infty\\ X\le x\le\infty}}\left|f(t,x)\frac{(-i)^{m+1}\,\Gamma(m+1)}{C\,x^n\,t^m}-A\right|$$

Since T and X are arbitrary

$$\lim_{\substack{s\to 0^+\\ y\to\infty}}\left|s^{m+1}\,y^{p-n-1}F(s,y)-A\right|$$
$$\le \lim_{t,x\to\infty}\left|f(t,x)\frac{(-i)^{m+1}\,\Gamma(m+1)}{c\,x^n\,t^m}-A\right|$$

Hence proved

4. Conclusion

This paper provides extension of distributional generalized Fourier-Stieltjes transform, Also Abelian theorem of intial value type and final value type are proved. This Abelian theorem can be used to solve boundary value problems.

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