Abelian Theorem of Generalized Fourier-Stieltjes Transform

V. D. Sharma 1, P. D. Dolas 2

1HOD Mathematics Department, Arts, Commerce and Science College, Amravati, 444606 (M.S.) India
2Mathematics Department, IBSS College of Engineering, Amravati. 444602 (M.S.) India

Abstract: The integral transform plays an important role in the solution of a wide class of problems of mathematical physics, for instance, boundary value problem for Laplace equation etc. Considerably which are used to solve the boundary value problems of Mathematical Physics and Partial Differential equation etc. In this paper, the generalization of Fourier-Stieltjes transform is presented. Abelian theorem of initial value type and final value type are proved. These results are widely used to solve the boundary value problems.

Keywords: Stieltjes transform, Fourier transform, Fourier-Stieltjes transform, generalized function, Testing function space.

1. Introduction

The conventional Fourier-Stieltjes transform of a complex valued smooth function \( f(t, x) \) is defined by the convergent integral.

\[
F(s, y) = FS\{f(t, x)\} = \int_{0}^{\infty} \int_{0}^{\infty} f(t, x) e^{-ist} (x + y)^{-p} \, dt \, dx \tag{1.1}
\]

Where, \( s \) and \( x \) are positive real numbers. The peculiarity of the transformation (1.1) lies in fact that it involves the integration with respect to parameter. Its extension to the distribution of compact support which involves some complicated analysis has been done by Zemanian [7].

The distributional Fourier-Stieltjes transform is defined as

\[
FS\{f(t, x)\} = F(\sigma, y) = <f(t, x), e^{-ist} (x + y)^{-p}> \tag{1.2}
\]

where for each fixed \( t \) the right hand side of above equation has same as an application of Zemanian to \( f(t, x) \). For some \( s > 0 \) and \( k \) the above equation has the form \( \sigma e^{-\sigma t} \).

Considerably the Abelian theorem is important in solving the boundary value problems of partial differential equation and mathematical physics etc. We know that, Integral transformation is one the well known techniques used for the functions transformation and integral transform method have proved to be the great importance in solving boundary value problems of mathematical physics and partial differential equation[1] etc.

There are various transforms such as Laplace, Fourier, Stieltjes, Mellin etc. We studied several theorems for the integral transform and then extend this result to the distributional generalized sense. This paper provides extension of distributional generalized Fourier-Stieltjes transform to initial and final value theorem.

Different S-types spaces are introduced in[3,4,5] along with some operators on these spaces. The testing spaces \( FS_{S} \) transform given as-

\[
A \text{ function } \mathcal{O}(t, x) \text{ defined on } t(0 \leq t \leq \infty), x(0 \leq x \leq \infty) \text{ is said to be member of } FS_{S}, \text{ if } \mathcal{O}(t, x) \text{ is smooth and for each non-negative integer } p, l, q, \]

\[
y_{k+1} \mathcal{O}(t, x) = \text{ Sup } \{ \mathcal{O}(t+x, x+y)^{-p} \} \text{ for } t \in t_{0} \tag{1.3}
\]

In this paper we have proved some Abelian theorem of initial value type and some Lemmas in section 2. In section 3, we established the Abelian theorem of final value type. The notation and terminology will follow that Zemanian[7].

2. Some Abelian Theorem of Initial value Type

Conditions:

(i) \( f(t, x) = 0 \) for \( -\infty < t < T, \ 0 < x < \infty \)

(ii) There exist real number \( s \) and \( y \) such that \( f(t, x) e^{-ist} (x + y)^{-p} \) is absolutely integrable.

2.1 Theorem:-

For locally integrable function \( f(t, x) \) satisfying above condition with \( T = 1 \) and existence of any complete constant \( A \) and real number \( m \) and \( n \) such that-

\[
\lim_{m \to -1} n > -1 \text{ and } m > -1 \text{ or } n > 0 \text{ or } n > 1
\]

And \( \lim_{m \to -1} \text{ and } \lim_{n \to 1} \text{ and } \lim_{x \to \infty} \text{ and } f(t, x) = A \text{ then } \]

\[
\lim_{x \to \infty} s^{n+1} y^{-n-1} F(s, y) = A
\]

Where, \( C = \frac{T}{s^n y^{n+1}} \)

Proof:-

We extend the result to a distributional \( F_{S} \) Transform-

\[
\int_{0}^{\infty} t^m e^{-ist} \, dt = (-i)^{n+1} T_{m+1} \tag{2.1.1}
\]

for \( m > -1, \ s > 0 \)

\[
\int_{0}^{\infty} x^n (x + y)^{-p} \, dx = T_{p-2-m-1} y^{p-2-m-1}
\]

for \( n > -1, \ T_{p} > n + 1 \)

So that-

\[
| s^{n+1} y^{p-2-m-1} F(s, y) - A | \]

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For any $\xi$, we can find a constant $M$ such that

\[ |s^{m+1} \psi_{\eta-1}| \leq \frac{\xi}{\epsilon^{m+1}} \left( \frac{C_m}{\epsilon^{m+1}} \right) \]

for $s > \epsilon$. Whereas R.H.S. of this inequality approaches zero as $s$ becomes infinite. And also by Widder page number 183, Lemma 2.

Therefore by using above inequalities

\[ \lim_{y \to 0} \int_0^\infty \frac{a(x)}{(x + y)^p} \, dx \to 0 \]

From which the result follows.

2.2 Lemma:

If $f(t, x) \in FS_n$ with its support in $t_f \leq t \leq \infty$ and $x_f \leq x \leq \infty$. Where $t_f > C$ and $x_f > 1$ then $|F(s, y)| \leq M t^{k} x^{q}$ where, $M$ is sufficiently large constant.

Proof:-

We know by Zemanian page number 243 and extend the result to a distributional $e^{sF}$ transform-

\[ \int_0^\infty t^m \epsilon^{-1} \left( \frac{C_m}{\epsilon^{m+1}} \right) \]

for $m > -1$, $s > C$ (3.1)

Also by Widder-

\[ \int_0^\infty t^m \epsilon^{-1} \left( \frac{C_m}{\epsilon^{m+1}} \right) \]

for $m > -1$, $s > C$ (3.2)

So that-

\[ |s^{m+1} \psi_{\eta-1} F(s, y) - A | \leq \frac{C_m}{\epsilon^{m+1}} \left( \frac{C_m}{\epsilon^{m+1}} \right) \]

for $m > -1$, $s > C$ (3.3)

Moreover the distributional $e^{sF}$ transform $F_t$ of $f_t$ equals the ordinary generalized $e^{sF}$ transform of $f_t$, so that $F(s, y) = F(s, y) + F_t(s, y)$. Therefore (i) and (ii) proves theorem.

3. Some Abelian Theorems of the Final Value Type

Theorem:

For a locally integrable function $f(t, x)$ satisfying condition with $T ^{m+1}$ and existence of any complex constant $A$ and real number $m$ and $n$ such that-

i) $m > -1$  
ii) $n > -1$  
iii) $Re \ p > n + 1$

and $\lim_{x \to \infty} s^{n+1} \psi_{\eta-1} F(s, y) = A$ then

\[ \lim_{x \to \infty} s^{n+1} \psi_{\eta-1} F(s, y) = A \]

Where, $C = \frac{R_p}{\gamma p - n - 1 m + 1}$

Proof:-

We know by Zemanian page number 243 and extend the result to a distributional $e^{sF}$ transform-

\[ \int_0^\infty t^m \epsilon^{-1} \left( \frac{C_m}{\epsilon^{m+1}} \right) \]

for $m > -1$, $s > C$ (3.1)

Also by Widder-

\[ \int_0^\infty t^m \epsilon^{-1} \left( \frac{C_m}{\epsilon^{m+1}} \right) \]

for $m > -1$, $s > C$ (3.2)

So that-

\[ |s^{m+1} \psi_{\eta-1} F(s, y) - A | \leq \frac{C_m}{\epsilon^{m+1}} \left( \frac{C_m}{\epsilon^{m+1}} \right) \]

for $m > -1$, $s > C$ (3.3)
As $\epsilon \to 0^+$ above integral approaches zero.

Also,

$$\lim_{y \to \infty} \frac{\alpha(x)}{(x + y)^p} \alpha x \to 0$$

As $y$ approaches above integral becomes zero. Therefore (3.3) implies that-

$$\lim_{y \to \infty} \int_0^x (x + y)^{-p} \left( f(t, x) - \frac{A x^n t^m}{c x^n t^m + 1} \right) \, dt \, dx$$

Taking limits on $\epsilon \to 0^+$ and $y \to \infty$

$$\lim_{y \to \infty} \int_0^x (x + y)^{-p} \left( f(t, x) - \frac{A x^n t^m}{c x^n t^m + 1} \right) \, dt \, dx$$

Since $T$ and $X$ are arbitrary

$$\lim_{y \to \infty} \left| \int_{x \to \infty} (x + y)^{-p} \left( f(t, x) - \frac{A x^n t^m}{c x^n t^m + 1} \right) \, dt \, dx \right|$$

Hence proved

4. Conclusion

This paper provides extension of distributional generalized Fourier-Stieltjes transform, Also Abelian theorem of initial value type and final value type are proved. This Abelian theorem can be used to solve boundary value problems.

References


Author Profile

Dr. V. D. Sharma is currently working as an Assistant professor in the department of Mathematics, Arts, Commerce and Science College, Kiran Nagar, Amravati-444606 (M.S.) India. She has got 18 years of teaching and research experience. She has obtained her Ph.D. degree in 2007 from SGB Amravati University Amravati. Her field of interest is Integral Transforms. Six research students are working under her supervision. She has published more than 60 research articles.

P. D. Dolas is an Assistant professor in the department of Mathematics, IBSS College of Engineering, Amravati, 444602 (M.S.) India. He has obtained his master degree in 2010 from Sant Gadge Baba Amravati University, Amravati. He has 4 years of teaching experience. He has 4 researches articles in journals to her credit.