Operation Transform Formulae for Two Dimensional Fractional Mellin Transform

V. D. Sharma [1], P. B. Deshmukh [2]

[1]HOD Mathematics Department, Arts, Commerce and Science College, Amravati, 444606 (M.S.) India

[2]Mathematics Department, IBSS College of Engineering, Amravati, 444602 (M.S.), India

Abstract: ***The Mellin transform is a basic tool for analyzing the behavior of many important functions in mathematics and mathematical physics. Mellin transform has many applications such as quantum calculus, radar classification of ships, electromagnetic, stress distribution, agriculture, medical stream, statistics, probability, signal processing, optics, pattern recognition, algorithms, correlators, navigation, vowel recognition and cryptographic scheme. The aim of this paper is to provide a generalization of two dimensional fractional Mellin transform. Also derived some operation transform formulae as Derivative, Parsval's Indentity, Modulation and Scaling property for two dimensional fractional Mellin transform***.

Keywords: Fractional Mellin Transform, Generalize function, Mellin Transform, Testing Function space.

1. Introduction

Here, we have generalized the integral transform and the integral transforms provide a powerful technique for solving initial and boundary value problems arising in applied mathematics, mathematical physics and engineering. There are many integral transforms including Fourier, Laplace, Mellin, Hankel, Hilbert, Stieltjes etc. Still new integral transforms are being introduced in mathematical literature [7].

Mellin transform is a natural analytical tool to study the distribution of products and quotients of independent random variables. So by using Mellin transform agricultural land classification is possible [4]. Mellin transforms has application to derive different properties in statistics and probability densities of single continuous random variable and also used in deriving densities for algebraic combination of random variables [6]. Mellin transform method is applied to fractional differential equations with a right-sided derivative and variable potential [2]. Mellin transform also use to establish the means, variances, skewness, and kurtosis of fuzzy numbers and applied them to the random coefficient autoregressive (RCA) time series models [5]. Dirichlet Boundary Value Problem is solved by Mellin Transform [6].

The two dimensional fractional Mellin transform with parameter θ of $f(x,y)$ denoted by $FRMT\{f(x,y)\}$ performs a linear operation, given by the integral transform

$$FRMT\{f(x,y)\} = F_\theta(u,v) = \int_0^\infty\int_0^\infty f(x,y) K_\theta(x,y,u,v)\,dx\,dy \quad (1)$$

where the kernel

$$K_\theta(x,y,u,v) = x^{\frac{2\pi i u}{\sin\theta}-1}\, y^{\frac{2\pi i v}{\sin\theta}-1}\, e^{\frac{\pi i}{\tan\theta}[u^2+v^2+\log x^2+\log y^2]}$$

$$0 < \theta \le \frac{\pi}{2} \quad (2)$$

where, $C_{1\theta} = \frac{2\pi i}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$ (3)

2. Preliminaries

2.1 The Test Function Space E

An infinitely differentiable complex valued function $\phi(x,y)$ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_\alpha$, $K \subset S_b$ where

$S_a = \{x : x \in R, |x| \le a, a > 0\}$

$S_b = \{y : y \in R, |y| \le b, b > 0\}$, $I, K \in R^n$

$$\gamma_{E,q,\lambda}[\phi(x,y)] = \sup_{\substack{x\in I\\ y\in K}} \left| D_{x,y}^{q,\lambda} \phi(x,y) \right| < \infty$$

Thus $E(R^n)$ will denote the space of all $\phi(x,y) \in E(R^n)$ with compact support contained in S_α & S_b.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that $f(x,y)$ is a fractional Mellin transformable if it is a member of E^*, the dual of E.

2.2 Distributional two dimensional fractional Mellin transform (FRMT)

The two dimensional fractional Mellin transform of $f(x,y) \in E^*(R^n)$ can be defined by

$$FRMT\{f(x,y)\} = F_\theta(u,v) = \langle f(x,y), K_\theta(x,y,u,v)\rangle$$

$$K_\theta(x,y,u,v) = x^{\frac{2\pi i u}{\sin\theta}-1}\, y^{\frac{2\pi i v}{\sin\theta}-1}\, e^{\frac{\pi i}{\tan\theta}[u^2+v^2+\log x^2+\log y^2]}$$

Right hand side of equation has a meaning as the application of $f(x,y) \in E^*$ to $K_\theta(x,y,u,v) \in E$.

In this paper we proposed an operation transform formulae such as Derivative, Parsval's Indentity, Modulation and Scaling property for two dimensional fractional Mellin transform.

3. Results

3.1 Differential Property

Prove that $FRMT\{f'(x,y)\}(u,v)$

$$= \frac{-2\pi i}{\sin\theta}\{\left[\frac{\log x \cos\theta}{x}\right] FRMT[f(x,y)](u,v) + (u - \frac{\sin\theta}{2\pi i}) FRMT\left[\frac{1}{x} f(x,y)\right](u,v)\}$$

Proof:

$FRMT\{f'(x,y)\}(u,v)$

$$= \int_0^\infty \int_0^\infty f'(x,y) x^{\frac{2\pi iu}{\sin\theta}-1} y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+v^2+\log^2 x+\log^2 y)} dxdy$$

$$= [\int_0^\infty y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 y)}] \int_0^\infty f'(x,y) x^{\frac{2\pi iu}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)}] dxdy$$

$$= [\int_0^\infty y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 y)}] \{[\int_0^\infty x^{\frac{2\pi iu}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)}][f'(x,y)]dx\}dy$$

$$= [\int_0^\infty y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 y)}] \{\left[x^{\frac{2\pi iu}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)} f(x,y)\right]_0^\infty - \int_0^\infty [x^{\frac{2\pi iu}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)} \left(\frac{\pi i}{\tan\theta} 2\log x (\frac{1}{x})\right) + e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)} x^{\frac{2\pi iu}{\sin\theta}-2} (\frac{2\pi iu}{\sin\theta} - 1)] f(x,y) dx\}dy$$

$$= [y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 y)}]\{-\int_0^\infty f(x,y) e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)} x^{\frac{2\pi iu}{\sin\theta}-1} \frac{2\pi i}{x}[\log x \frac{\cos\theta}{\sin\theta} + \frac{u}{\sin\theta} - \frac{1}{2\pi i}] dx\}dy$$

$$= [y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(v^2+\log^2 y)}] \{-\int_0^\infty f(x,y) e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)} x^{\frac{2\pi iu}{\sin\theta}-1} \frac{2\pi i}{x\sin\theta}(\log x \cos\theta) dx - \int_0^\infty f(x,y) e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)} x^{\frac{2\pi iu}{\sin\theta}-1} \frac{2\pi i}{x\sin\theta} (u - \frac{\sin\theta}{2\pi i}) dx\}dy$$

$$= -\int_0^\infty \int_0^\infty f(x,y) x^{\frac{2\pi iu}{\sin\theta}-1} y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+v^2+\log^2 x+\log^2 y)} [\frac{2\pi i}{x\sin\theta}(\log x \cos\theta)] dxdy - \int_0^\infty \int_0^\infty f(x,y) x^{\frac{2\pi iu}{\sin\theta}-1} y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+v^2+\log^2 x+\log^2 y)} [\frac{2\pi i}{x\sin\theta}(u - \frac{\sin\theta}{2\pi i})] dxdy$$

$$= \frac{-2\pi i}{\sin\theta}\{\left[\frac{\log x \cos\theta}{x}\right] FRMT[f(x,y)](u,v) + (u - \frac{\sin\theta}{2\pi i}) FRMT\left[\frac{1}{x} f(x,y)\right](u,v)\}$$

3.2 Parseval's Indentity for two dimensional fractional Mellin Transform:

If $FRMT[f(x,y)](u,v) = F_\theta(u,v)$ &

$FRMT[g(x,y)](u,v) = G_\theta(u,v)$ then

i] $\int_0^\infty \int_0^\infty f(x,y)\overline{g(x,y)} dxdy$

$$= \int_0^\infty \int_0^\infty \frac{-1}{\sin^2\theta} \overline{G_\theta(u,v)} dudv\, FRMT\{xyf(x,y)\}(u,v)$$

ii] $\int_0^\infty \int_0^\infty |f(x,y)|^2 dxdy$

$$= \frac{-1}{\sin^2\theta} \int_0^\infty \int_0^\infty |F_\theta(u,v)|^2\, dudv$$

Proof:

$FRMT[f(x,y)](u,v) = F_\theta(u,v)$

$$= \int_0^\infty \int_0^\infty f(x,y) x^{\frac{2\pi iu}{\sin\theta}-1} y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+v^2+\log^2 x+\log^2 y)} dxdy$$

$$= e^{\frac{\pi i}{\tan\theta}(u^2+v^2)} \int_0^\infty \int_0^\infty f(x,y) x^{\frac{2\pi iu}{\sin\theta}-1} y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(\log^2 x+\log^2 y)} dxdy$$

By using inversion formula for two dimensional fractional Mellin Transform

$$g(x,y) = \frac{-1}{\sin^2\theta} \int_0^\infty \int_0^\infty x^{\frac{-2\pi iu}{\sin\theta}} y^{\frac{-2\pi iv}{\sin\theta}} e^{\frac{-\pi i}{\tan\theta}(u^2+v^2+\log^2 x+\log^2 y)} G_\theta(u,v) dudv$$

$$= \frac{-1}{\sin^2\theta} \int_0^\infty \int_0^\infty x^{\frac{-2\pi iu}{\sin\theta}} y^{\frac{-2\pi iv}{\sin\theta}} e^{\frac{-\pi i}{\tan\theta}(u^2+v^2)} e^{\frac{-\pi i}{\tan\theta}(\log^2 x+\log^2 y)} G_\theta(u,v) dudv$$

$$\overline{g(x,y)} = \frac{-1}{\sin^2\theta} e^{\frac{\pi i}{\tan\theta}(\log^2 x+\log^2 y)} \int_0^\infty \int_0^\infty x^{\frac{2\pi iu}{\sin\theta}} y^{\frac{2\pi iv}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(u^2+v^2)} \overline{G_\theta(u,v)} dudv$$

Consider,

$$\int_0^\infty \int_0^\infty f(x,y)\overline{g(x,y)}dxdy$$

$$= \int_0^\infty \int_0^\infty \frac{-1}{sin^2\theta} \overline{G_\theta(u,v)}dudv$$

$$[\int_0^\infty \int_0^\infty x^{\frac{2\pi iu}{sin\theta}} y^{\frac{2\pi iv}{sin\theta}} e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}$$

$$f(x,y)dxdy]$$

$$= \int_0^\infty \int_0^\infty \frac{-1}{sin^2\theta} \overline{G_\theta(u,v)}dudv$$

$$[\int_0^\infty \int_0^\infty xy\, x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}$$

$$f(x,y)dxdy]$$

$$= \frac{-1}{sin^2\theta}\int_0^\infty \int_0^\infty F_\theta\{xyf(x,y)\}(u,v)\, \overline{G_\theta(u,v)}dudv$$

Putting

$$f(x,y) = g(x,y)$$

$$F_\theta(u,v) = G_\theta(u,v)$$

$$\overline{F_\theta(u,v)} = \overline{G_\theta(u,v)}$$

By using above result

$$\int_0^\infty \int_0^\infty f(x,y)\overline{f(x,y)}dxdy$$

$$= \frac{-1}{sin^2\theta}\int_0^\infty \int_0^\infty F_\theta(u,v)\, \overline{F_\theta(u,v)}dudv$$

$$\int_0^\infty \int_0^\infty |f(x,y)|^2 dxdy$$

$$= \frac{-1}{sin^2\theta}\int_0^\infty \int_0^\infty |F_\theta(u,v)|^2\, dudv$$

3.3 Modulation property:

(i) Prove that:

$$FRMT[f(x,y)\, cos(ax+by)](u,v) = \frac{1}{2}\{FRMT[e^{i(ax+by)}f(x,y)](u,v) + FRMT[e^{-i(ax+by)}f(x,y)](u,v)\}$$

Proof: $FRMT[f(x,y)cos\,(ax+by)](u,v)$

$$= \int_0^\infty \int_0^\infty x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1}$$

$$e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}$$

$$cos\,(ax+by)f(x,y)dxdy$$

$$= \tfrac{1}{2}\{[\int_0^\infty \int_0^\infty x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1}$$

$$e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}e^{i(ax+by)}$$

$$f(x,y)dxdy]$$

$$+[\int_0^\infty \int_0^\infty x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1}$$

$$e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}e^{-i(ax+by)}$$

$$f(x,y)dxdy]$$

$$= \frac{1}{2}\{FRMT[e^{i(ax+by)}f(x,y)](u,v)$$

$$+FRMT[e^{-i(ax+by)}f(x,y)](u,v)\}$$

(ii) Prove that:

$$FRMT[f(x,y)\cos(ax+by)](u,v) = \frac{1}{2i}\{FRMT[e^{i(ax+by)}f(x,y)](u,v) - FRMT[e^{-i(ax+by)}f(x,y)](u,v)\}$$

Proof: $FRMT[f(x,y)cos\,(ax+by)](u,v)$

$$= \int_0^\infty \int_0^\infty x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1}$$

$$e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}$$

$$sin\,(ax+by)f(x,y)dxdy$$

$$= \frac{1}{2i}\{[\int_0^\infty \int_0^\infty x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1}$$

$$e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)} e^{i(ax+by)}$$

$$f(x,y)dxdy] - [\int_0^\infty \int_0^\infty x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1}$$

$$e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)} e^{-i(ax+by)} f(x,y)dxdy]$$

$$= \frac{1}{2i}\{FRMT[e^{i(ax+by)}f(x,y)](u,v)$$

$$-FRMT\left[e^{-i(ax+by)}f(x,y)\right](u,v)\}$$

3.4 Scaling property :

Prove that: $FRMT[f(ax,by)](u,v)$
$= a^{-\frac{2\pi iu}{\sin\theta}} b^{-\frac{2\pi iv}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(\log^2 a+\log^2 b)}$
$FRMT\left\{e^{\frac{-2\pi i}{\tan\theta}(\log p\log a+\log q\log b)} f(p,q)\right\}(u,v)$

Proof: $FRMT[f(ax,by)](u,v)$

$$= \int_0^\infty \int_0^\infty f(ax,by)K_\theta(x,y,u,v)dxdy$$

$$= \int_0^\infty \int_0^\infty f(ax,by) x^{\frac{2\pi iu}{\sin\theta}-1} y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+v^2+\log^2 x+\log^2 y)}dxdy$$

Putting
$ax = p, by = q$
$dx = \frac{1}{a}dp, dy = \frac{1}{b}dq$
$FRMT[f(ax,by)](u,v)$

$$= \int_0^\infty \int_0^\infty f(p,q)\left(\frac{p}{a}\right)^{\frac{2\pi iu}{\sin\theta}-1}\left(\frac{q}{b}\right)^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}(u^2+v^2+\log^2\frac{p}{a}+\log^2\frac{q}{b})}\frac{1}{ab}dpdq$$

$$= \frac{1}{ab}\int_0^\infty \int_0^\infty f(p,q)\frac{1}{a^{\frac{2\pi iu}{\sin\theta}-1} b^{\frac{2\pi iv}{\sin\theta}-1}} e^{\frac{\pi i}{\tan\theta}(u^2+v^2)} e^{\frac{\pi i}{\tan\theta}(\log^2 p+\log^2 q)} e^{\frac{\pi i}{\tan\theta}(-2\log p\log a-2\log q\log b)} e^{\frac{\pi i}{\tan\theta}(\log^2 a+\log^2 b)}dpdq$$

$$= a^{-\frac{2\pi iu}{\sin\theta}} b^{-\frac{2\pi iv}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(\log^2 a+\log^2 b)} \int_0^\infty \int_0^\infty f(p,q) e^{\frac{\pi i}{\tan\theta}(u^2+v^2+\log^2 p+\log^2 q)} e^{\frac{-2\pi i}{\tan\theta}(\log p\log a+\log q\log b)}dpdq$$

$$= a^{-\frac{2\pi iu}{\sin\theta}} b^{-\frac{2\pi iv}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(\log^2 a+\log^2 b)} FRMT\left\{e^{\frac{-2\pi i}{\tan\theta}(\log p\log a+\log q\log b)} f(p,q)\right\}(u,v)$$

4. Conclusion

In this paper we have defined Distributional two dimensional fractional Mellin transform with compact support. This paper is mainly focused on some operation transform formulae such as Derivative, Parsval's Identity, Modulation and Scaling property for two dimensional fractional Mellin transform is proved.

References

[1] A.H. Zemanian, Generalized Integral Transformation, Interscience Publisher, New York,1968.
[2] Daniel Dziembowski, "On Mellin transform application to solution of fractional differential equations", Scientific Research of the Institute of Mathematics and Computer Science.
[3] George .N. Emenogu, "Solution of Dirichlet Boundary Value Problem by Mellin Transform", IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-3008, p-ISSN :2319-7676. Volume 9, Issue 6 (Jan. 2014), PP 01-06.
[4] M. Mahdian , S. Homayouni , M. A. Fazel , F. Mohammadimanesh, "Agricultural land classification based on statistical analysis of full polarimetric SAR data", International Archives of the Photogrammetry, Remote Sensing and Spatial Information Sciences, Volume XL-1/W3, 2013, SMPR 2013, 5 – 8 October 2013, Tehran, Iran.
[5] S. S. Appadoo, A. Thavaneswaran and S.Mandal, "Mellin's transform and application to some time series models", Hindawi Publishing Corporation, ISRN Applied Mathematics Volume 2014, Article ID 976023, 12 pages.
[6] S. M. Khairnar, R. M. Pise and J. N. Salunkhe, "Study of the Mellin integral transform with applications in statistics and probability", Scholars Research Library, Archives of Applied Science Research, 2012, 4 (3):1294-1310.
[7] V.D. Sharma, P.B.Deshmukh: "Inversion theorem of two dimensional fractional Mellin transform", International Journal of Applied Mathematics & Mechanics, Vol.-3, No.-1 (2014), pp 33-39.

Author Profile

Dr. V. D. Sharma is currently working as an Assistant professor in the department of Mathematics, Arts, Commerce and Science College, Kiran Nagar, Amravati-444606 (M.S.) India. She has got 18 years of teaching and research experience. She has obtained her Ph.D. degree in 2007 from SGB Amravati University Amravati. Her field of interest is Integral Transforms. Six research students are working under her supervision. She has published more than 60 research articles.

P. B. Deshmukh is an Assistant professor in the department of Mathematics, IBSS College of Engineering, Amravati 444602 (M.S.) India. She has obtained her master degree in 2006 from Sant Gadage Baba Amravati University, Amravati. She has got 7 years of teaching experience. She has 6 research articles in journals to her credit.