On Differential Sandwich Theorems of Analytic Functions Defined by Generalized Integral Operator

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Abstract: ***In this paper, we obtain some applications of first order differential Subordination and super ordination results involving a generalized integral operator for certain normalized analytic functions.***

Keywords: Analytic functions, p–valent, integral operator, subordination and sandwich

2014 Mathematics Subject Classification: 30C45

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ (a_k \geq 0, p \in N = \{1,2,3,\ldots.\}), (1.1)$$

which are analytic and $p-valent$ in the open unit disk $U = \{z: z \in \mathbb{C}, |z| < 1\}$. If f and g are analytic functions in U, we say that f is subordinate to g in U,

written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)), (z \in U).$$

In particular, if the function g is univalent in U, then $f \prec g$ if

$$f(0) = g(0), \text{ and } f(U) \subset g(U) \text{ ([4,13]).}$$

For the function f given by (1.1) and $g \in A(p)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k.$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=P+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

The set of all functions f that are analytic and injective on $\overline{U} / E(f)$, Denote by Q where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\}.$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$ (see [14]).

Let $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$, and h is univalent in U with $q \in Q$. Miller and Mocanu [13] consider the problem of determining conditions on admissible functions ψ such that

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z) \ (1.2)$$

implies $p(z) \prec q(z)$, for all functions $p(z) \in H[a,n]$ that satisfy the differential subordination (1.2),moreover, they found conditions so that q is the smallest function with this property, called the best dominant of the subordination (1.2).

Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$, and $h \in H$ with $q \in H[a,n]$. Recently Miller and Mocanu [14,15] studied the dual problem and determined conditions on ϕ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) \ (1.3)$$

implies $q(z) \prec p(z)$, for all functions $p \in Q$ that satisfy the above super ordination. They also found conditions so that the function q is the largest function with this property, called the best subordinate of the super ordination (1.3).

In [5] Cataş extended the multiplier transformation and defined the operator $I_p^m(\lambda, \ell) f(z)$ on $A(p)$ by the following infinite series

$$I_p^m(\lambda, l) f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p + l + \lambda(k-p)}{p + l}\right]^m a_k z^k,$$

$$(\lambda \geq 0; l \geq 0; p \in N, m \in N_0; z \in U), (1.4)$$

we note that:

$I_p^0(\lambda, l) f(z) = f(z)$, and $I_p^1(1,0) f(z) = \frac{zf'(z)}{p}$.

By specializing the parameters m, λ, ℓ and p, we obtain the following operators studied by various authors:

1) $I_p^m(1, l) f(z) = I_p(m.l) f(z)$ (see [12,21])
2) $I_p^m(1,0) f(z) = D_p^m f(z)$ (see [2.11,18]).
3) $I_1^m(1, l) f(z) = I_l^m f(z)$ (see [6,7]).
4) $I_1^m(1,0) f(z) = D^m f(z) (m \in N_0)$ (see [19]).
5) $I_1^m(\lambda, 0) f(z) = D_\lambda^m f(z)$ (see [1]).
6) $I_1^m(1,1) f(z) = I^m f(z)$ (see [22]).
7) $I_p^m(\lambda, 0) f(z) = D_{\lambda,p}^m f(z)$, where $D_{\lambda,p}^m f(z)$ is defined by

$$D_{\lambda,p}^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p + \lambda(k-p)}{p}\right]^m a_k z^k,$$

Furthermore we define the integral operator $I_p^m(\lambda, \alpha, \delta) f(z), f(z) \in A(p)$ as follows :

$$I_p^0(\lambda, \alpha, \delta) f(z) = f(z)$$

$$I_p^1(\lambda, \alpha, \delta) f(z) = I_p(\lambda, \alpha, \delta) f(z)$$

$$= \left(\frac{p + \alpha\delta}{\lambda}\right) z^{p - \left(\frac{p+\alpha\delta}{\lambda}\right)} \int_0^z t^{\left(\frac{p+\alpha\delta}{\lambda}\right) - (p+1)} f(t) dt$$

$$I_p^2(\lambda, \alpha, \delta) f(z)$$

$$= \left(\frac{p + \alpha\delta}{\lambda}\right) z^{p - \left(\frac{p+\alpha\delta}{\lambda}\right)} \int_0^z t^{\left(\frac{p+\alpha\delta}{\lambda}\right) - (p+1)} I_p^1(\lambda, \alpha, \delta) f(t) dt$$

and, in general

$$I_p^m(\lambda, \alpha, \delta) f(z)$$

$$= \left(\frac{p + \alpha\delta}{\lambda}\right) z^{p - \left(\frac{p+\alpha\delta}{\lambda}\right)} \int_0^z t^{\left(\frac{p+\alpha\delta}{\lambda}\right) - (p+1)} I_p^{m-1}(\lambda, \alpha, \delta) f(t) dt$$

$$(f(z) \in A(p); m \in N_0; z \in U) \ (1.5)$$

We see that for $f(z) \in A(p)$, we have that

$$I_p^m(\lambda,\alpha,\delta)f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{p+\alpha\delta}{p+\alpha\delta+\lambda(k-p)}\right)^m a_k z^k, (m \in N_0)\,.(1.6)$$

From (1.6), it easy to verify that

$$\lambda z\left(I_p^{m+2}f(z)\right)' = (\alpha\delta+p)\left(I_p^{m+1}f(z)\right) - (\alpha\delta+p(1-\lambda))\left(I_p^{m+2}f(z)\right)\,.(1.7)$$

We note that:

1) $I^m(\lambda,0,0)f(z) = I_\lambda^{-m}f(z)$ (see [18])
2) $I_1^\alpha(1,1,1)f(z) = I^\alpha f(z)$ (see [10]).
3) $I_p^m(1,1,1)f(z) = I_p^m f(z)$ (see [20]).
4) $I_1^m(1,1,1)f(z) = D^m f(z)$ (see [17]).
5) $I_1^m(1,1,1)f(z) = I^m f(z)$ (see [9]).
6) $I_1^m(1,0,0)f(z) = I^m f(z)$ (see [19]).

Also we note that :

1- $I_p^m(1,0,0)f(z) = J_p^m f(z)$

$$= \left\{f(z): J_p^m f(z) = z^p + \sum_{k=n+p}^{\infty}\left(\frac{p}{k}\right)^m a_k z^k, m \in N_0, z \in U\right\}.$$

2- $I_p^m(1,l,1)f(z) = J_p^m(l)f(z)$

$$= \left\{f(z): J_p^m(l)f(z) = z^p + \sum_{k=n+p}^{\infty}\left(\frac{p+l}{k+l}\right)^m a_k z^k, m \in N_0, l > 0, z \in U\right\}.$$

3- $I_p^m(\lambda,0,0)f(z) = J_{p,\lambda}^m f(z)$

$$= \left\{f(z): J_{p,\lambda}^m f(z) = z^p + \sum_{k=n+p}^{\infty}\left(\frac{p}{k+\lambda(k-p)}\right)^m a_k z^k, m \in N_0, \lambda \geq 0, z \in U\right\}.$$

In this paper, we shall determine some properties on the admissible functions defined with operator $I_p^m(\lambda,\alpha,\delta)$.

2. Preliminaries

In order to prove our results, we shall make use of the following known results.

Lemma (2.1)[8]: Let q be univalent in $U, \zeta \in \mathbb{C}^* \setminus \{0\}$ and suppose that

$$Re\left\{1+\frac{z\acute{\acute{q}}(z)}{\acute{q}(z)}\right\} > \max\left\{0, -Re\left(\frac{1}{\zeta}\right)\right\}.(2.1)$$

If $p(z)$ is analytic in U, with $p(0) = q(0)$ and

$$p(z) + \zeta z\acute{p}(z) \prec q(z) + \zeta z\acute{q}(z),(2.2)$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant.

Lemma (2.2)[13] :Let the function $q(z)$ be univalent in the unit disk, and let θ, φ be analytic in domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = z\acute{q}(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

1- Q is star like univalent in U.

2- $Re\left\{\frac{z\acute{h}(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is analytic with $p(0) = q(0), p(U) \subseteq D$ and

$$\theta(p(z)) + z\acute{p}(z)\varphi(p(z)) \prec \theta(q(z)) + z\acute{q}(z)\varphi(q(z)),(2.3)$$

then $p \prec q$, and $q(z)$ is the best dominant.

Lemma (2.3)[3] :Let $q(z)$ be convex in $U, q(0) = a$ and $\zeta \in \mathbb{C}, Re(\zeta) > 0$.

If $p \in H[a,1]$ and $p(z) + \gamma z\acute{q}(z)$ is univalent in U then

$$q(z) + \zeta z\acute{q}(z) \prec p(z) + \zeta z\acute{p}(z),(2.4)$$

implies $q(z) \prec p(z)$, and $q(z)$ is the best subordinant.

Lemma (2.4)[4] : Let $q(z)$ be convex univalent in the unit disk U and let θ, φ be analytic in a domain D containing $q(U)$. Suppose that

1- $Re\left\{\frac{\acute{\theta}(q(z))}{\varphi(q(z))}\right\} > 0$, for $z \in U$.

2- $z\acute{q}(z)\varphi(q(z))$ is star like univalent in U.

If $p(z) \in H[q(0),1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + z\acute{p}(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + z\acute{q}(z)\varphi(q(z)) \prec \theta(p(z)) + z\acute{p}(z)\varphi(p(z)),(2.5)$$

then $q(z) \prec p(z)$, and $q(z)$ is the best subordinant.

3. Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda > 0, \alpha, \delta \geq 0; p \in N, m \in N_0 = N \cup \{0\}; z \in U$ and the powers are understood as principle values.

Theorem (3.1): Let $q(z)$ be univalent in U with $q(0) = 1, \beta \in \mathbb{C}^*, \gamma > 0$ and suppose that

$$Re\left\{1+\frac{z\acute{\acute{q}}(z)}{\acute{q}(z)}\right\} > max\left\{0, -Re\left(\frac{\gamma(\alpha\delta+p)}{\beta\lambda}\right)\right\},(3.1)$$

If $f \in A(p)$ satisfies the subordination

$$(1-\beta)\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma + \beta\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma \frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)} \prec q(z) + \frac{\beta\lambda}{\gamma(\alpha\delta+p)}z\acute{q}(z),(3.2)$$

then

$$\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma \prec q(z)$$

and $q(z)$ is the best dominant.

Proof : If we consider the analytic function

$$\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma, \sigma > 0, z \in U\ (3.3)$$

Differentiating (3.3) logarithmically with respect to z and using the identity (1.7) in the resulting equation, we have

$$\frac{z\acute{p}(z)}{p(z)} = \frac{\sigma(\delta\alpha+p)}{\lambda}\left(\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)} - 1\right),(3.4)$$

that is

$$\frac{\lambda}{\sigma(\delta\alpha+p)}z\acute{p}(z) = \left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma\left(\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)} - 1\right)$$

Thus, the subordination (3.2) is equivalent to

$$p(z) + \frac{\beta\lambda}{\sigma(\delta\alpha+p)}z\acute{p}(z) \prec q(z) + \frac{\beta\lambda}{\sigma(\delta\alpha+p)}z\acute{q}(z).\ (3.5)$$

Applying lemma (2.1), with $\zeta = \frac{\beta\lambda}{\sigma(\delta\alpha+p)}$, the proof of Theorem (1.1) is completed.

Taking the convex function $(z) = \frac{1+Az}{1+Bz}$, in the Theorem (1.1), we have the following corollary.

Corollary (3.1): Let $A, B \in \mathbb{C}, A \neq B, |B| < 1, Re(\beta) > 0$ and $\gamma > 0$. If $f(z) \in A(p)$ satisfies the subordination

$$(1-\beta)\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma + \beta\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma \frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)} \prec \frac{1+Az}{1+Bz} + \frac{\beta\lambda}{\sigma(1+p)}\frac{(A-B)z}{(1+Bz)^2}$$

Then

$$\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $m=0$ in Theorem (3.1), we obtain the following result:

Corollary (3.2): Let $q(z)$ be univalent in U, with $q(0)=1, \beta \in \mathbb{C}^*, \sigma > 0$, and suppose that (3.1) holds. If $f(z) \in A(p)$ satisfies the subordination

$$(1-\beta)\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma} + \beta\left(\frac{I_p^2 f(z)}{z^p}\right)^{\gamma}\frac{I_p^1 f(z)}{I_p^2 f(z)} \prec q(z) + \frac{\beta\lambda}{\sigma(\alpha\delta+p)} z\acute{q}(z),$$

then

$$\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma} \prec q(z).$$

and $q(z)$ is the best dominant.

Taking $\alpha=\lambda=1$ in the Theorem (3.1), we have the following result.

Corollary (3.3) : Let $q(z)$ be univalent in U, with $q(0)=1, \beta \in \mathbb{C}^*, \sigma > 0$, and suppose that (3.1) holds. If $f(z) \in A(p)$ satisfies the subordination

$$(1-\beta)\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} + \beta\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma}\frac{I_p^{m+1} f(z)}{I_p^2 f(z)} \prec q(z) + \frac{\beta}{\sigma(\delta+p)} z\acute{q}(z),$$

then

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} \prec q(z).$$

and $q(z)$ is the best dominant.

Theorem (3.2) : Let $q(z)$ be univalent in U, with $q(0)=1$ and $q(z) \neq 0$ for all $z \in U$, let $\lambda, \sigma \in \mathbb{C}^*, f \in A(p)$ and suppose that f and q satisfy the next conditions:

$$\frac{I_p^{m+2} f(z)}{z^p} \neq 0, \quad (3.6)$$

and

$$Re\left\{1+\frac{z\acute{\acute{q}}(z)}{\acute{q}(z)} - \frac{z\acute{q}(z)}{q(z)}\right\} > 0, (z \in U) \quad (3.7)$$

If

$$\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} \prec 1 + \frac{\lambda z q\acute{(z)}}{\sigma(\alpha\delta+p)q(z)}, \quad (3.8)$$

then

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} \prec q(z)$$

and $q(z)$ is the best dominant of (3.6).

Proof: Let

$$p(z) = \left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma}, z \in U \quad (3.9)$$

According to (3.4) the function $p(z)$ is analytic in U, and differentiating (3.9) logarithmically with respect to z, we obtain

$$\frac{z\acute{p}(z)}{p(z)} = \frac{\sigma(\delta\alpha+p)}{\lambda}\left(\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} - 1\right), \quad (3.10)$$

In order to prove our result we will use Lemma (2.2). In this lemma consider

$$\theta(w) = 1 \text{ and } \varphi(w) = \frac{\lambda}{\sigma(\alpha\delta+p)w}$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = z\acute{q}(z)\varphi(q(z)) = \frac{\lambda z\acute{q}(z)}{\sigma(\alpha\delta+p)q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \frac{\lambda z\acute{q}(z)}{\gamma\sigma(\alpha\delta+p)q(z)}$$

from (3.7), we see that $Q(z)$ is a starlike function in U. We also have

$$Re\left\{\frac{z\acute{h}(z)}{Q(z)}\right\} = Re\left\{1+\frac{z\acute{\acute{q}}(z)}{\acute{q}(z)} - \frac{z\acute{q}(z)}{q(z)}\right\} > 0, (z \in U)$$

and then, by using Lemma (2.2) we deduce that the subordination (3.6) implies

$$p(z) \prec q(z)$$

and the function $q(z)$ is the best dominant of (3.8).

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem (3.2), it is easy to check that the assumption (3.5) holds, hence we obtain the next result.

Corollary (3.4) : Let $\sigma \in \mathbb{C}^*$. Let $f(z) \in A(p)$ and suppose that

$$\frac{I_p^{m+2} f(z)}{z^p} \neq 0, (z \in U).$$

If

$$\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} \prec 1 + \frac{\lambda z(A-B)}{\sigma(\alpha\delta+p)(1+Az)(1+Bz)}$$

then

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem (3.2), it is easy to check that the assumption (3.5) holds, hence we obtain the next result.

Corollary (3.5) : Let $\sigma \in \mathbb{C}^*$, $f(z) \in A(p)$ and suppose that

$$\frac{I_p^{m+2} f(z)}{z^p} \neq 0, (z \in U).$$

If

$$\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} \prec 1 + \frac{2\lambda z}{\sigma(\alpha\delta+p)(1-z)(1+z)}$$

then

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} \prec \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant.

Theorem (3.3): Let $q(z)$ be univalent in U, with $q(0)=1$, let $\sigma \in \mathbb{C}^*$,and let$\psi, \nu, \eta \in \mathbb{C}$ with $\nu+\eta \neq 0$. Let$f \in A(p)$ and suppose that f and q satisfy the next conditions:

$$\frac{\nu I_p^{m+1} f(z) + \eta I_p^{m+2} f(z)}{(\nu+\eta)z^p} \neq 0, (z \in U) \quad (3.11)$$

and

$$Re\left\{1+\frac{z\acute{\acute{q}}(z)}{\acute{q}(z)}\right\} > max\{0, -Re(\psi)\}, (z \in U) \quad (3.12)$$

If

$$\Psi(z) = \psi\left[\frac{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right]^\sigma + \sigma\left[\left(\frac{\nu z(I_p^{m+1}f(z))' + \nu z(I_p^{m+2}f(z))'}{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)} - p\right)\right] \quad (3.13)$$

and

$$\Psi(z) \prec \psi q(z) + \frac{zq'(z)}{q(z)}, \quad (3.14)$$

then

$$\left[\frac{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right]^\sigma \prec q(z)$$

and $q(z)$ is the best dominant of (3.11).

Proof : Let

$$p(z) = \left[\frac{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right]^\sigma, z \in U \quad (3.15)$$

According to (3.8) the function $p(z)$ is analytic in U, and differentiating (3.15) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)} = \sigma\left[\frac{\nu z(I_p^{m+1}f(z))' + \nu z(I_p^{m+2}f(z))'}{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)} - p\right], \quad (3.16)$$

and hence

$$zp'(z) = \sigma\left[\frac{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right]^\sigma \left[\frac{\nu z(I_p^{m+1}f(z))' + \nu z(I_p^{m+2}f(z))'}{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)} - p\right]$$

In order to prove our result, we will use Lemma (2.2). In this lemma consider

$$\theta(w) = \psi w \text{ and } \varphi(w) = \frac{1}{w}$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma\left[\frac{\nu z(I_p^{m+1}f(z))' + \nu z(I_p^{m+2}f(z))'}{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)} - p\right]$$

and

$$h(z) = \theta(q(z)) + Q(z) = \psi\left[\frac{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right]^\sigma + \sigma\left[\left(\frac{\nu z(I_p^{m+1}f(z))' + \nu z(I_p^{m+2}f(z))'}{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)} - p\right)\right]$$

from (3.11), we see that $Q(z)$ is a starlike function in U. We also have

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\psi + 1 + \frac{zq'(z)}{q'(z)}\right\} > 0, (z \in U)$$

and then, by using Lemma (2.2) we deduce that the subordination (3.14) implies

$$p(z) \prec q(z).$$

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem (3.3) and according to(3.4), the condition (3.12) becomes

$$max\{0, -Re(\psi)\} \le \frac{1-|B|}{1+|B|}.$$

Hence, for the special case $\nu = 1$ and $\eta = 0$, we obtain the following result.

Corollary (3.6) : Let $\psi \in \mathbb{C}$ with

$$max\{0, -Re(\psi)\} \le \frac{1-|B|}{1+|B|}.$$

Let $f(z) \in A(p)$ and suppose that

$$\frac{I_p^{m+1}f(z)}{z^p} \neq 0, (z \in U).$$

If

$$\psi\left[\frac{\nu I_p^{m+1}f(z) +}{z^p}\right]^\sigma + \sigma\left[\left(\frac{z(I_p^{m+1}f(z))'}{I_p^{m+1}f(z)} - p\right)\right] \prec \psi\frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)}$$

then

$$\left(\frac{I_p^{m+1}f(z)}{z^p}\right)^\gamma \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

Taking $p = \nu = m = 1, \eta = 0$ and $q(z) = \frac{1+z}{1-z}$ in Theorem (3.3), we obtain the next result.

Corollary (3.7) : Let $f(z) \in A(p)$ and suppose that

$$\frac{I_p^2 f(z)}{z^p} \neq 0, (z \in U).$$

and $\sigma \in \mathbb{C}^*$. If

$$\psi\left[\frac{I^2 f(z)}{z}\right]^\sigma + \sigma\left[\left(\frac{z(I^2 f(z))'}{I^2 f(z)} - 1\right)\right] \prec \psi\frac{1+z}{1-z} + \frac{2z}{(1+z)(1-z)}$$

then

$$\left(\frac{I^2 f(z)}{z}\right)^\gamma \prec \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant.

4. Superordination and Sandwich Results

Theorem (4.1): Let $q(z)$ be a convex in U with $q(0) = 1, \beta \in \mathbb{C}, Re(\beta) > 0, \gamma > 0.$ If $f(z) \in A(p)$ such that $\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma \in H[q(0), 1] \cap Q$ and $(1-\beta)\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma + \beta\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma \frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)}$ is univalent in U, and satisfies the superordination

$$q(z) + \frac{\beta\lambda}{\gamma(\alpha\delta + p)}zq'(z) \prec (1-\beta)\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma + \beta\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma \frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)}, \quad (4.1)$$

then

$$q(z) \prec \left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma$$

and $q(z)$ is the best subordinant.

Proof :If we consider the analytic function

$$\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^\sigma, z \in U \quad (4.2)$$

Differentiating (4.2) logarithmically with respect to z and using the identity (1.7) in the resulting equation, we have

$$\frac{zp'(z)}{p(z)} = \frac{\sigma(\delta\alpha + p)}{\lambda}\left(\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)} - 1\right)$$

that is

$$\frac{\lambda}{\sigma(\delta\alpha+p)} zp'(z) = \left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}\left(\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)}-1\right)$$

Thus, the subordination (4.1) is equivalent to

$$q(z)+\frac{\beta\lambda}{\sigma(\delta\alpha+p)}zq'(z) \prec p(z)+\frac{\beta\lambda}{\sigma(\delta\alpha+p)}zp'(z).$$

Applying Lemma (2.3), with $\zeta=\frac{\beta\lambda}{\sigma(\delta\alpha+p)}$, the proof of Theorem (4.1) is completed.

Taking $m=0$ in Theorem (4.1), we obtain the following result:

Corollary (4.1): Let $q(z)$ be convex in U, with $q(0)=1, \beta\in\mathbb{C}, Re(\beta)>0, \sigma\in\mathbb{C}^*$, and suppose that (3.1) holds. If $f(z)\in A(p)$ such that $\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma}\in H[q(0),1]\cap Q$and

$$(1-\beta)\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma}+\beta\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma}\frac{I_p f(z)}{I_p^2 f(z)}$$

Is univalent in U and satisfies the superordination

$$q(z)+\frac{\beta\lambda}{\sigma(\alpha\delta+p)}zq'(z) \prec (1-\beta)\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma}+\beta\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma}\frac{I_p^1 f(z)}{I_p^2 f(z)},$$

then

$$q(z)\prec\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma}.$$

and $q(z)$ is the best superordinant.

Taking $\alpha=\lambda=1$ in the Theorem (4.1), we have the following result.

Corollary (4.2): Let $q(z)$ be convex in U, with $q(0)=1, \beta\in\mathbb{C}, Re(\beta)>0, \sigma\in\mathbb{C}^*$, and suppose that (3.1) holds. If $f(z)\in A(p)$ such that $\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}\in H[q(0),1]\cap Q$and

$$(1-\beta)\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}+\beta\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)}$$

Is univalent in U and satisfies the superordinant

$$q(z)+\frac{\beta}{\sigma(\delta+p)}zq'(z) \prec (1-\beta)\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}+\beta\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)},$$

$$q(z)\prec\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma}.$$

and $q(z)$ is the best superordination.

Theorem (4.2) : Let $q(z)$ be convex in U, with $q(0)=1$,let $\sigma\in\mathbb{C}^*$ and let $\psi,\nu,\eta\in\mathbb{C}$ with $\nu+\eta\neq0$ and $Re(\psi)>0$. Let $f\in A(p)$ and suppose that $f$satisfies the next conditions:

$$\frac{\nu I_p^{m+1}f(z)+\eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\neq 0,(z\in U)\ (4.3)$$

and

$$\left(\frac{\nu I_p^{m+1}f(z)+\eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right)^{\sigma}\in H[q(0),1]\cap Q,(4.4)$$

If the function $\Psi(z)$given by (3.13) is univalent in$U$ and,

$$\psi q(z)+\frac{zq'(z)}{q(z)}\prec\Psi(z),(4.5)$$

then

$$q(z)\prec\left(\frac{\nu I_p^{m+1}f(z)+\eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right)^{\sigma}$$

and $q(z)$ is the best subordinate of (4.5).

Proof : Let

$$p(z)=\left(\frac{\nu I_p^{m+1}f(z)+\eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right)^{\sigma},z\in U\ (4.6)$$

According to (4.3) the function $p(z)$ is analytic in U, and differentiating (4.6) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)}=\sigma\left[\frac{\nu z(I_p^{m+1}f(z))'+\nu z(I_p^{m+2}f(z))'}{\nu I_p^{m+1}f(z)+\eta I_p^{m+2}f(z)}-p\right],(4.7)$$

In order to prove our result we will use Lemma (2.4). In this lemma consider

$$\theta(w)=\psi w \text{ and } \varphi(w)=\frac{1}{w}$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w)\neq0$ is analytic in $\mathbb{C}^*$.

We see that

$$Q(z)=zq'(z)\varphi(q(z)) = \sigma\left[\frac{\nu z(I_p^{m+1}f(z))'+\nu z(I_p^{m+2}f(z))'}{\nu I_p^{m+1}f(z)+\eta I_p^{m+2}f(z)}-p\right]$$

and

$$h(z)=\theta(q(z))+Q(z) = \psi\left[\frac{\nu I_p^{m+1}f(z)+\eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right]^{\sigma} + \sigma\left[\left(\frac{\nu z(I_p^{m+1}f(z))'+\nu z(I_p^{m+2}f(z))'}{\nu I_p^{m+1}f(z)+\eta I_p^{m+2}f(z)} - p\right)\right]$$

from (3.11), we see that $Q(z)$ is a starlike function in U. From, we also have

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\}=Re\left\{\psi+1+\frac{zq''(z)}{q'(z)}\right\}>0,(z\in U)$$

and then, by using Lemma (2.4) we deduce that the subordination (4.5) implies

$$q(z)\prec p(z)$$

the proof of Theorem (4.2) is completed.

Combining Theorem (3.1) with Theorem (4.1) and Theorem (3.3) with Theorem (4.2), we obtain, respectively the following two sandwich results.

Theorem (4.3) :Let q_1,q_2 are two convex functions in U with$q_1(0)=q_2(0)=1$ and q_2 satisfies (3.1) , $\beta\in\mathbb{C}, Re(\beta)>0$ and $Re(\sigma)>0$. If $f(z)\in A(p)$ such that

$$\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}\in H[q(0),1]\cap Q,$$

and $\Phi(1-\beta)\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}+\beta\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)}$ is univalent in U, and satisfies

$$q_1(z)+\frac{\beta\lambda}{\gamma(\alpha\delta+p)}zq_1'(z) \prec (1-\beta)\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}+\beta\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)} \prec q_2(z)+\frac{\beta\lambda}{\gamma(\alpha\delta+p)}zq_2'(z),(4.8)$$

then

$$q_1(z) \prec \left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma} \prec q_2(z)$$

and q_1, q_2 are, respectively, the best subordinant and the best dominant of (4.8).

Theorem (4.4) :Let q_1, q_2 are two convex in U, with $q_1(0) = q_2(0) = 1$, let $\sigma \in \mathbb{C}^*$ and $\psi, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$ and $Re(\psi) > 0$. Let $f \in A(p)$ and suppose that f satisfies the next conditions:

$$\frac{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)}{(\nu+\eta)z^p} \neq 0, (z \in U)$$

and

$$\left(\frac{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right)^{\sigma} \in H[q(0),1] \cap Q,$$

If the function $\Psi(z)$ given by (3.13) is univalent in U and,

$$\psi q_1(z) + \frac{z\acute{q}_1(z)}{q_1(z)} \prec \Psi(z) \prec \psi q_2(z) + \frac{z\acute{q}_2(z)}{q_2(z)}, (4.9)$$

then

$$q_1(z) \prec \left(\frac{\nu I_p^{m+1}f(z) + \eta I_p^{m+2}f(z)}{(\nu+\eta)z^p}\right)^{\sigma} \prec q_2(z)$$

and $q_1(z), q_2(z)$ are, respectively, the best subordinate and the best dominant of (4.9).

Remark 1: Combining Corollaries (3.2), (4.1) and (3.3), (4.2), we obtain the corresponding Sandwich results for the operators I_p and I_p^{m+1}, respectively.

Remark 2: Taking $p = \lambda = 1$ and $l = 0$ in Theorems (3.1),(4.1) and (4.3), respectively, we obtain the results obtained by Cotirla [8, Theorems (3.1),(3.4) and (3.7), respectively].

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