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Generalization Sum of Boundedness of Toeplitz Operators

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Abstract: ***In this paper, we investigate the product sum of bounded of Toeplitz product $T_{f_j}T_{g_j}$, with f_j and g_j in weighted Bergman space. Where f_j and g_j are square integrable analytic functions.***

1. Introduction

The study of this problem was initiated by Sarason [4] .In the context of hardy space H^2 of the unit circle, often he had obtained of functions f and g in H^2 such that the product T_fT_g is bounded on H^2 [5,6].The Poisson kernel plays the role of the Berezin transform .Treil showed that a condition analogous is necessary [4],while the second author proved that a condition analogous is sufficient [1].the above results are analogous to [7],and generalize the results [8,9].However, the proofs require new tools to establish the necessary condition, as well as consideration of higher derivatives,and an inner product formula involving higher derivatives. Recently Park [10] has proved a necessary and close - to - sufficient condition for Toeplitz product on the Bergman space of the ball. Also condition B_2 was introduced by Bekoll and Bonami in [11].

In this paper we will generalize a necessary condition for boundedness of sum of the Toeplitz product $T_{f_j}T_{g_j}$. Instead of product of two Toeplitz operators are bounded in weighted Bergman space we will show that the product sum of Toeplitz operators are bounded in the same space and also we will prove that B_2 is a necessary condition for the Bergman projection to be bounded on $L^2(udA)$.

2. Main Results

We show that the following condition for sum of boundedness of Toeplitz product $T_{f_j}T_{g_j}$ in weighted Bergman space . Consider the product $f_j\otimes g_j$ on $A_\alpha^2(\mathbb{B}_n)$ defined by $f_j\otimes g_j h_j$ .It is easily proved that sum of $f_j\otimes g_j$ is bounded on $A_\alpha^2(\mathbb{B}_n$ with norm equal to $\sum_{j=1}^n\|f_j\otimes g_j\| = \sum_{j=1}^n\|f_j\|_\alpha\|g_j\|_\alpha$. We will use Berezin transform :writing $K_\omega^{(\alpha)}$ for normalize reproducing kernel , we define the Berezin transform of bounded linear operator S on $A_\alpha^2(\mathbb{B}_n)$. To be the function $B_\alpha[S]$ defined on $\mathbb{B}_n$ .By $B_\alpha[S](\omega) = \langle SK_\omega^{(\alpha)}, K_\omega^{(\alpha)}\rangle$ for $\omega\in\mathbb{B}_n$, the boundedness of S implies that the function $B_\alpha[S]$ is bounded on $\mathbb{B}_n$ .The Berezin transform is injective , for $B_\alpha[S](\omega) = 0$ for all $\omega\in\mathbb{B}_n$, implies that $S = 0$,the zero product on $A_\alpha^2(\mathbb{B}_n)$ see[1].We will also make use of following continuity condition of Berezin transform : $S_N\to S$ in operator norm , then $B_\alpha[S](\omega) = \lim_{N\to\infty}B_\alpha[S](\omega)$ for $\omega\in\mathbb{B}_n$ .The above statement is an immediate consequence of the following inequality:
$|B_\alpha[S](\omega) - B_\alpha[S_N](\omega)| \le \|S - S_N\|$.

Theorem 1. let $1<\alpha<\infty$,and f_j, g_j in $A_\alpha^2(\mathbb{B}_n)$ .If $T_{f_j}T_{g_j}$ is bounded on $A_\alpha^2(\mathbb{B}_n)$,then

$$\sum_{j=1}^n \sup_{\omega\in B_n} B_\alpha\left[|f_j|^2\right](\omega)B_\alpha\left[|g_j|^2\right](\omega) < \infty$$

Proof: Suppose that f_j, g_j are analytic on $A_\alpha^2(\mathbb{B}_n)$ such that the densely defined Toeplitz product $T_{f_j}T_{g_j}$ is bounded on $A_\alpha^2(\mathbb{B}_n)$ .We see that there exists a finite constant C_α such that

$$\sum_{j=1}^n\|f_j\otimes g_j\| \le C_\alpha\sum_{j=1}^n\left\|T_{f_j}T_{g_j}\right\| \quad (1)$$

Thus $\sum_{j=1}^n\|f_j\|_2\|g_j\|_2 \le C_\alpha\sum_{j=1}^n\left\|T_{f_j}T_{g_j}\right\|$

Applied to f_j, g_j that $T_{f_j o\varphi_\omega}U_\omega^{(\alpha)} = U_\omega^{(\alpha)}T_{f_j}$,

$$\sum_{j=1}^n T_{f_j o\varphi_\omega}T_{g_j o\varphi_\omega} = \sum_{j=1}^n\left(T_{f_j o\varphi_\omega}U_\omega^{(\alpha)}\right)U_\omega^{(\alpha)}\left(T_{g_j o\varphi_\omega}U_\omega^{(\alpha)}\right)U_\omega^{(\alpha)} = \sum_{j=1}^n\left(U_\omega^{(\alpha)}T_{f_j}\right)U_\omega^{(\alpha)}\left(U_\omega^{(\alpha)}T_{g_j}\right)U_\omega^{(\alpha)} = \sum_{j=1}^n U_\omega^{(\alpha)}\left(T_{f_j}T_{g_j}\right)U_\omega^{(\alpha)} \quad (2)$$

For all $\omega\in A_\alpha^2(\mathbb{B}_n)$. Inequality (1) applied $f_j o\varphi_\omega$ and $g_j o\varphi_\omega$ gives

$$\sum_{j=1}^n\|f_j o\varphi_\omega\|\|g_j o\varphi_\omega\| \le C_\alpha\sum_{j=1}^n\left\|T_{f_j o\varphi_\omega}T_{g_j o\varphi_\omega}\right\| = C_\alpha\sum_{j=1}^n\left\|T_{f_j}T_{g_j}\right\|$$

$$\sum_{j=1}^n B_\alpha\left[|f_j|^2\right](\omega)B_\alpha\left[|g_j|^2\right](\omega) \le C_\alpha\sum_{j=1}^n\left\|T_{f_j}T_{g_j}\right\|$$

For all $\omega\in A_\alpha^2(\mathbb{B}_n)$ .So, for $f_j, g_j\in A_\alpha^2(\mathbb{B}_n)$, a necessary condition for the sum of Toeplitz product $T_{f_j}T_{g_j}$ to be bounded on $A_\alpha^2(\mathbb{B}_n)$ is

$$\sum_{j=1}^n \sup_{\omega\in B_n} B_\alpha\left[|f_j|^2\right](\omega)B_\alpha\left[|g_j|^2\right](\omega) < \infty$$

We will need the two generalized estimates conditioned in the following lemmas.

Lemma 2.Let $1 < \alpha < \infty$ for $f_j \in L^2(B_n, V_\alpha)$ and $h \in H^\infty(\mathbb{B}_n)$ we have

$\sum_{j=1}^n \left|\left(T_{f_j} h_j\right)(\omega)\right| \le \sum_{j=1}^n \frac{B_\alpha\left[|f_j|^2\right](\omega)^{\frac{1}{2}}}{(1-|\omega|^2)^{\frac{n+\alpha+1}{2}}} \|h_j\|_\alpha$, For all $\omega \in \mathbb{B}_n$.

proof. By Cauchy-Schwarz's inequality,

$$\sum_{j=1}^n \left|\left(T_{f_j} h_j\right)(\omega)\right|^2 \le \sum_{j=1}^n \left(\int_{\mathbb{B}_n} \frac{|f_j(z)||h_j(z)|}{|1-\langle\varphi_\omega, z, \omega\rangle|^{n+\alpha+1}}\right)^2$$

$$\le \int_{\mathbb{B}_n} \sum_{j=1}^n \frac{|f_j(z)|^2}{|1-\langle\varphi_\omega, z, \omega\rangle|^{n+\alpha+1}} dv_\alpha \int_{\mathbb{B}_n} \sum_{j=1}^n |h_j(z)|^2 dv_\alpha$$

$$= \sum_{j=1}^n \frac{B_\alpha\left[|f_j|^2\right](\omega)}{(1-|\omega|^2)^{n+\alpha+1}} \|h_j\|_\alpha^2$$

Lemma 3.let $-1 < \alpha < \infty$,and let f_j be in $A_\alpha^2(\mathbb{B}_n)$. If for $\varepsilon > 0$.

$$\sum_{j=1}^n sup_{\omega \in \mathbb{B}_n} B_\alpha\left[|f_j|^{2+\omega}\right](\omega) < \infty,$$

Then the operator sum of T_{f_j} is bounded on $A_\alpha^2(\mathbb{B}_n)$ .By Hölder's inequality,

$$\left(\int_{\mathbb{B}_n} |f_j|^2 dv_\alpha\right)^{\frac{1}{2}} \le \left(\int_{\mathbb{B}_n} |f_j|^{2+\omega} dv_\alpha\right)^{\frac{1}{2+\omega}}$$

Applying this to the function $f_j o \varphi_\omega$, it follows that

$$\sum_{j=1}^n B_\alpha\left[|f_j|^2\right](\omega)^{\frac{1}{2}} \le \sum_{j=1}^n B_\alpha\left[|f_j|^{2+\omega}\right](\omega)^{\frac{1}{(2+\omega)}}$$

and thus

$$\sum_{j=1}^n \left(B_\alpha\left[|f_j|^2\right](\omega)\right)^{\frac{1}{2}} \le \sum_{j=1}^n \left(B_\alpha\left[|f_j|^{2+\omega}\right](\omega)\right)^{\frac{1}{(2+\omega)}}$$

(4)

for all $\omega \in \mathbb{B}_n$,so condition (4) implies necessary condition (2)

Proof. Assume that for $\omega > 0$,$\check{M}$ is positive constant such that $\sum_{j=1}^n B_\alpha\left[|f_j|^{2+\omega}\right](\omega) \le \check{M}^{2+\omega}$

for all $\omega \in \mathbb{B}_n$ .By (3) we also have

$B_\alpha\left[|f_j|^2\right](\omega) \le \check{M}^2$ (3)

For all $\omega \in \mathbb{B}_n$ .Let $h_1, h_2, \dots, h_j$ be bounded analytic functions on $\mathbb{B}_n$ .It follows from Lemma 2.

3. Bounded sum of Toeplitz operators

Theorem 4. Suppose that u satisfies the B_2 condition and that the Toeplitz operator $T_{f_j}: A^2(u) \to A^2(u)$ is bounded. Then the Berezin transform f_j is bounded.

proof. First notice that for every finite sum $S_m := \sum_{n=1}^m \lambda_n K_u(., a_n)$, $a_n \in \mathbb{D}$, The following holds

$$\langle\sum_{j=1}^n T_{f_j}\left(\sum_{n=1}^m \lambda_n K_u(., a_n)\right), \left(\sum_{n=1}^m \lambda_n K_u(., a_n)\right)\rangle_{A^2(u)} = \sum_{j=1}^n \sum_{\iota=1}^m T_{f_j}\left(\sum_{n=1}^m \lambda_n K_u(., a_n)\right)(a_\iota)$$

$$= \sum_{\iota=1}^n \lambda_\iota \int_{\mathbb{D}} K_u(a_\iota, z) \sum_{n=1}^m \lambda_n k_u(z, a_n) dv(z) = \left\|\sum_{n=1}^m \lambda_n k_u(., a_n)\right\|_{L^2(v)}^2 \lesssim \left\|\sum_{n=1}^m \lambda_n k_u(., a_n)\right\|_{A^2(u)}^2$$

So, if $\{S_m\}$ is a Cauchy sequence in $A^2(u)$, it is also in $L^2(v)$ and also converges in $L^2(v)$. We have that if $S_m \to h_j$ in the $A^2(u)$- norm , then $S_m \to h_j$ in $L^2(v)$ and consequently for any function $q_j^E \in A^2(u)$,

$\langle k_j, S_m\rangle_{L^2(v)} \to \langle E_j, h_j\rangle_{L^2(v)}$, moreover

$\langle\sum_{j=1}^n T_{f_j} E_j, h_j\rangle_{A^2(u)} = \lim_{m\to\infty} \langle\sum_{j=1}^n T_{f_j} E_j, S_m\rangle_{A^2(u)} = \langle\sum_{j=1}^n E_j, S_m\rangle_{L^2(v)}$

therefore

$\langle\sum_{j=1}^n T_{f_j} E_j, h_j\rangle_{A^2(u)} = \langle\sum_{j=1}^n E_j, h_j\rangle_{L^2(v)}$

Hence for $a \in \mathbb{D}$,

$\langle\sum_{j=1}^n T_{f_j} B_a, B_a\rangle_{A^2(u)} = \langle B_a, B_a\rangle_{L^2(v)} = \tilde{v}(a)\|B_a\|_{A^2(u)}^2$

and consequently , $\tilde{v}(a) \left\|T_{f_j}\right\|$.

We will need the following lemma due to constant:

Lemma 5. Suppose u satisfies condition B_2 ,then

$\|B_a\|_{A^2(u)}^2 \sim \frac{\mu(D_r(a))}{(1-|a|)^4}$

Proposition 6. If $\tilde{v}$ is bounded on $\mathbb{D}$ then v is a μ –Carleson measure , where $\mu = udA$.

proof. Fix $0 < r < 1$,if $\tilde{v}$ is bounded then there exists a constant $C < 0$ such that for every $a \in \mathbb{D}$

$\frac{1}{\|B_a\|_{A^2(u)}^2} \int_{D_r(a)} \frac{1}{|1-\bar{a}z|^4} dv(z) \le C$,

Consequently

$$v(D_r(a)) \lesssim C(1-|a|^2)^4 \|B_a\|_{A^2(u)}^2 \sim \frac{(1-|a|)^4 \mu(D_r(a))}{(1-|a|)^4} \lesssim (D_r(a)).$$

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