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Representation Theorem for the Distributional Fourier-Laplace Transform

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Abstract: ***There are many integral transforms including Fourier, Laplace, Mellin, Hankel, Whittaker, Stieltjes, Hilbert, Hartely etc. but the origin of integral transform is Fourier and Laplace transforms. So these two transforms are very important. And this Fourier and Laplace transforms have many applications in various fields like science, physics, mathematics, engineering, geophysics, medical, chemistry, electrical and mechanical separately. In our research we tried to join the Fourier and Laplace transforms and work on it and this resultant Fourier-Laplace transform also may have various applications in some fields of science and technology. The aim of the present paper is to provide the generalization of Fourier-Laplace transform in the distributional sense and giving representation theorem for the distributional Fourier-Laplace transform.***

Keywords: Fourier transform, Laplace transform, Fourier- Laplace transform, generalized function, Testing function space.

1. Introduction

The Fourier transform is a mathematical procedure which transforms a function from the time domain to the frequency domain. Fourier transform is a mathematical method using the trigonometric functions to transform a time domain spectrum into a frequency domain spectrum. Fourier transform can be used to convert from the series of numbers to sound [5]. Fourier transform is also used in signal processing, cell phones, in the measurement of heart rate variability (HRV), image processing. It is also of fundamental importance in quantum mechanics.

The Laplace transform is a mathematical tool based on integration that has a number of applications. It particular, it can simplify the solving of many differential equations. The Laplace transform is just one of many “integral transforms” in general use. It is used for the analysis of HVAC (Heating, Ventilation and Air Conditioning) control systems, which are used in all modern buildings and constructions.

Besides solving differential equations, Fourier and Laplace transform are important tools in analyzing signals and the transfer of signals by systems. Hence, the Fourier and Laplace transforms play a predominant role in the theory of signals and systems. Mechanical networks consisting of springs, masses and dampers, for the production of shock absorbers for example, processes to analyze chemical components, optical systems, and computer programs to process digitized sounds or images, can all be considered as system for which one can use Fourier and Laplace transform as well [1].

Many authors studied on various integral transforms separately. However there is much scope in extending double transformation to a certain class of generalized functions. B. N. Bhosale and M.S. Choudhary [2] and S. M. Khairnar et.al. [6] has discussed double transform. Motivated by this we have also defined a new combination of integral transforms in distributional generalized sense namely Fourier-Laplace transform. Along with the definition its analyticity theorem is proved in [8], [9]. To generalize the Fourier-Laplace transform and for proving the Representation theorem we need certain testing function spaces, which are already mentioned in our previous papers [8], [9]. But here also it is very necessary to mention these testing function spaces.

These testing function spaces are as follows:

1.1 The Space $FL_{a,\alpha}^{\beta}$

This space is given by

$$FL_{a,\alpha}^{\beta} = \{\phi : \phi \in E_+ \, / \, \rho_{a,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C A^k k^{k\alpha} B^l l^{l\beta} \} \quad (1.1)$$

where, $k, l, q = 0,1,2,3,..........$, and the constants A, B depends on the testing function ϕ.

1.2 The Space $FL_{a,\gamma}$

It is given by

$$FL_{a,\gamma} = \{\phi : \phi \in E_+ \, / \, \xi_{a,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C_{lk} A^q q^{q\gamma} \} \quad (1.2)$$

where, $k, l, q = 0,1,2,3,..........$, and the constants depends on the testing function ϕ.

Motivated by the above work, we have generalized Fourier-Laplace transform in the distributional sense in this paper. Representation theorem for the distributional Fourier-

Laplace transform is also presented. This paper is summarized as follows:
In section 2, we have given the definitions. Definition of Distributional generalized Fourier-Laplace transform is derived in section 3. In section 4, we have proved Representation theorem. Lastly the conclusions are given in section 5.
Notations and terminologies are as per Zemanian [11], [12].

2. Definitions

The Fourier transform with parameter s of $f(t)$ denoted by $F[f(t)] = F(s)$ and is given by

$$F[f(t)] = F(s) = \int_{-\infty}^{\infty} e^{-ist} f(t)dt, \text{ for parameter } s > 0.$$
(2.1)

The Laplace transform with parameter p of $f(x)$ denoted by $L[f(x)] = F(p)$ and is given by

$$L[f(x)] = F(p) = \int_0^{\infty} e^{-px} f(x)dx,$$
For parameter $p > 0$. (2.2)

The Conventional Fourier-Laplace transform is defined as

$$FL\{f(t,x)\} = F(s,p) = \int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)K(t,x)dtdx,$$
(2.3)

where, $K(t,x) = e^{-i(st-ipx)}$.

3. Distributional Generalized Fourier-Laplace Transforms (FLT)

For $f(t,x) \in FL_{a,\alpha}^{*\beta}$, where $FL_{a,\alpha}^{*\beta}$ is the dual space of $FL_{a,\alpha}^{\beta}$. The distributional Fourier-Laplace transform is a function of $f(t,x)$ and is defined as

$$FL\{f(t,x)\} = F(s,p) = \left\langle f(t,x), e^{-i(st-ipx)}\right\rangle, \quad (3.1)$$

where, for each fixed t $(0 < t < \infty)$, x $(0 < x < \infty)$, $s > 0$ and $p > 0$, the right hand side of (3.1) has a sense as an application of $f(t,x) \in FL_{a,\alpha}^{*\beta}$ to $e^{-i(st-ipx)} \in FL_{a,\alpha}^{\beta}$.

4. Representation Theorem

Let $f(t,x)$ be an arbitrary element of $FL_{a,\alpha}^{*\beta}$ and $\phi(t,x)$ be an element of $D(I)$, the space of infinitely differentiable function with compact support on I. Then there exists a bounded measurable functions $g_{m,n}(t,x)$ defined over I such that

$$\langle f,\phi\rangle = \left\langle \sum_{m=0}^{r+1}\sum_{n=0}^{\nu+1}(-1)^{m+n} t^k e^{ax}\frac{\partial^{m+n}}{\partial t^m \partial x^n} g_{m,n}(t,x), \phi(t,x)\right\rangle$$

where k is a fixed real number and r and ν are appropriate non-negative integers satisfying $m \le r+1$ and $n \le \nu+1$.

Proof:- Let $\{\gamma_{a,k,l,q}\}_{l,q=0}^{\infty}$ be the sequence of seminorms.

Let $f(t,x)$ and $\phi(t,x)$ be arbitrary elements of $FL_{a,\alpha}^{*\beta}$ and $D(I)$ respectively. Then by boundedness property of generalized function by Zemanian [12], pp.52, we have for an appropriate constant C and a non-negative integer r and ν satisfying $|l| \le r$ and $|q| \le \nu$

$$|\langle f,\phi\rangle| \le C \max_{\substack{|l|\le r\\ |q|\le \nu}} \gamma_{a,k,l,q}\phi(t,x) \le C \max_{\substack{|l|\le r\\ |q|\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<\infty}} \left|t^k e^{ax} D_t^l D_x^q \phi(t,x)\right|$$

$$\le C \max_{\substack{|l|\le r\\ |q|\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<\infty}} t^k e^{ax}\left|\sum_{m=0}^{l}\sum_{n=0}^{q} B_n \frac{\partial^{m+n}}{\partial t^m \partial x^n}\phi(t,x)\right|$$

$$\le C' \max_{\substack{|l|\le r\\ |q|\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<\infty}} t^k e^{ax} \max_{\substack{m\le l\\ n\le q}} \frac{\partial^{m+n}}{\partial t^m \partial x^n}\phi(t,x)$$

where, C' is a constant which depends only on m,n and hence l,q, so

$$|\langle f,\phi\rangle| \le C'' \max_{\substack{|m|\le r\\ |n|\le \nu}} \sup_{\substack{0<t<\infty\\ 0<x<\infty}} t^k e^{ax}\frac{\partial^{m+n}}{\partial t^m \partial x^n}\phi(t,x)$$

(4.1)

Now let us set

$$\phi_{r,\nu}(t,x) = t^k e^{ax}\phi(t,x), \quad m \le r, n \le \nu$$

Then clearly $\phi_{r,\nu}(t,x) \in D(I)$.

Also $\phi(t,x) = e^{-ax}t^{-k}\phi_{r,\nu}(t,x)$ (4.2)

On differentiating (4.2) partially with respect to t and x successively we get,

$$\frac{\partial^2\phi}{\partial t\partial x} = t^{-k}e^{-ax}\left[\frac{ak}{t}\phi_{r,\nu} - a\frac{\partial\phi_{r,\nu}}{\partial t} - \frac{k}{t}\frac{\partial\phi_{r,\nu}}{\partial x} + \frac{\partial^2\phi_{r,\nu}}{\partial t\partial x}\right]$$

Let us suppose that in I, $Sup\phi = Sup\phi_{r,\nu} = [A,B]$.

Then since $t^{-k}e^{-ax} > 0$

$$\left|\frac{\partial^2\phi}{\partial t\partial x}\right| \le t^{-k}e^{-ax}\left\{\frac{|ak|}{A}\left|\phi_{r,\nu}\right| + a\left|\frac{\partial\phi_{r,\nu}}{\partial t}\right| + \frac{|k|}{A}\left|\frac{\partial\phi_{r,\nu}}{\partial x}\right| + \left|\frac{\partial^2\phi_{r,\nu}}{\partial t\partial x}\right|\right\}$$

$$\le C''' t^{-k}e^{-ax}\left\{\left|\phi_{r,\nu}\right| + \left|\frac{\partial\phi_{r,\nu}}{\partial t}\right| + \left|\frac{\partial\phi_{r,\nu}}{\partial x}\right| + \left|\frac{\partial^2\phi_{r,\nu}}{\partial t\partial x}\right|\right\}$$

where $C''' = \max\left[\frac{|ak|}{A}, a, \frac{|k|}{A}, 1\right]$

If C^{iv} is a constant which depends on a and k then

$$\left|\frac{\partial^2 \phi}{\partial t \partial x}\right| \leq C^{iv} t^{-k} e^{-ax} \left|\frac{\partial^2 \phi_{r,v}}{\partial t \partial x}\right|$$

Hence by induction we prove that in I, for obvious constant C^{v}.

$$\left|\frac{\partial^{m+n} \phi}{\partial t^m \partial x^n}\right| \leq C^{v} t^{-k} e^{-ax} \sum_{\substack{c \leq m \\ d \leq n}} \left|\frac{\partial^{c+d}}{\partial t^c \partial x^d}\right| \phi_{r,v}$$

Substituting this into (4.1)

$$\left|\langle f, \phi \rangle\right| \leq C^{vi} \max_{\substack{m \leq r \\ n \leq v}} \sup_{\substack{0 < t < \infty \\ 0 < x < \infty}} \left|\frac{\partial^{c+d}}{\partial t^c \partial x^d} \phi_{r,v}(t,x)\right|$$

where, $c \leq m$ and $d \leq n$ (4.3)

Now we can write

$$\sup_{\substack{0 < t < \infty \\ 0 < x < \infty}} \left|\phi(t,x)\right|$$

$$\leq \sup_{\substack{0 < t < \infty \\ 0 < x < \infty}} \left|\int_t^\infty \int_x^\infty \frac{\partial^2}{\partial t \partial x} \phi(t,x) dt dx\right| \leq \left\|\frac{\partial^2}{\partial t \partial x} \phi(t,x)\right\|_{L'XL'}$$

(4.4) Hence from (4.3)

$$\left|\langle f, \phi \rangle\right| \leq C^{vi} \max_{\substack{m \leq r+1 \\ n \leq v+1}} \sup_{\substack{0 < t < \infty \\ 0 < x < \infty}} \left\|\frac{\partial^{m+n}}{\partial t^m \partial x^n} \phi_{r,v}(t,x)\right\|_{L'XL'}$$

Let the product space $L' \times L'$ be denoted by $(L')^2$. We consider the linear one-to-one mapping $\tau : \phi \to \left\{\frac{\partial^{m+n}}{\partial t^m \partial x^n} \phi\right\}_{\substack{m \leq r+1 \\ n \leq v+1}}$ of $D(I)$ into $(L')^2$. In view of (4.4) we see that the linear functional $\tau : \phi_{r,v} \to \langle f, \phi \rangle$ is continuous on $\tau D(I)$ for the topology induced by (L'). Hence by Hahn-Banach theorem, it can be a continuous linear functional in the whole of $(L')^2$. But the dual of $(L')^2$ is isomorphic with $(L^\infty)^2$ [10] pp.214 and 259, therefore there exist two L^∞ functions $g_{m,n}(m \leq r+1, n \leq v+1)$ such that,

$$\langle f, \phi \rangle = \sum_{\substack{m \leq r+1 \\ n \leq v+1}} \left\langle g_{m,n}, \frac{\partial^{m+n}}{\partial t^m \partial x^n} \phi_{r,v}(t,x) \right\rangle$$

By (4.2), we have

$$\langle f, \phi \rangle = \sum_{\substack{m \leq r+1 \\ n \leq v+1}} \left\langle g_{m,n}, \frac{\partial^{m+n}}{\partial t^m \partial x^n} t^k e^{ax} \phi(t,x) \right\rangle$$

Now by using property of differentiation of a distribution and property of multiplication of a distribution by an infinitely smooth function,

$$\langle f, \phi \rangle = \sum_{\substack{m \leq r+1 \\ n \leq v+1}} \left\langle (-1)^{m+n} t^k e^{ax} \frac{\partial^{m+n}}{\partial t^m \partial x^n} g_{m,n}(t,x), \phi(t,x) \right\rangle$$

where $g_{m,n}(t,x)$ are bounded measurable functions defined over $I = (0, \infty)$. Therefore

$$f(t,x) = \sum_{\substack{m \leq r+1 \\ n \leq v+1}} (-1)^{m+n} t^k e^{ax} \frac{\partial^{m+n}}{\partial t^m \partial x^n} g_{m,n}(t,x)$$

5. Conclusion

Since Fourier and Laplace transforms has found numerous applications in various fields. We tried to develop a new type of transform, Fourier-Laplace transform on the same lines. In this paper Fourier-Laplace transform is generalized in the distributional sense. Testing function spaces using Gelfand Shilov technique are already developed in our previous papers. The main aim of this paper was to prove Representation Theorem for the distributional Fourier-Laplace transform and we proved it.

References

[1] R. J. Beerends, H. G. ter Morsche, J. C. van den Berg, and E. M. van de Vrie, Fourier and Laplace Transforms, Cambridge University Press, 2003.
[2] B. N. Bhosale and M. S. Chaudhary, "Fourier-Hankel Transform of Distribution of compact support," J. Indian Acad. Math, 24(1), pp. 169-190, 2002.
[3] L. Debnath and D. Bhatta, Integral Transforms and their Applications, Chapman and Hall/CRC Taylor and Francis Group Boca Raton London, New York, 2007.
[4] P. Fitzsimmons and T. Mc EL Roy, On Joint Fourier-Laplace Transforms, Communication in statistics-Theory and Methods, 39: 1883-1885 Taylor and Francis Group, LLC, 2010.
[5] A. Gupta, "Fourier Transform and Its Application in Cell Phones," International Journal of Scientific and Research Publications, Volume 3, Issue 1, pp. 1-2 January 2013.
[6] S. M. Khairnar, R.M. Pise, and J. N. Salunke, "Applications of the Laplace-Mellin integral transform to differrential equations," International Journal of Scientific and Research Publications, 2(5) pp. 1-8, (2012).
[7] R. S. Pathak, A Course in Distribution Theory and Applications, CRC Press 2001.
[8] V. D. Sharma and A. N. Rangari, "Analyticity of distributional Fourier-Laplace transform," International J. of math. Sci. and Engg.Appls. 5(V), pp. 57-62, (2011).
[9] V. D. Sharma and A. N. Rangari, "Generalized Fourier-Laplace Transform and Its Analytical

Structure," International Journal of Applied Mathematics and Mechanics, Vol. 3, No.1, pp. 41-46, 2014.

[10] F. Treves, Topological vector space, distribution and kernels, Academic press, New York, (1967).

[11] A. H. Zemanian, Distribution theory and transform analysis, McGraw Hill, New York, 1965.

[12] A. H. Zemanian, Generalized integral transform, Inter science publisher, New York, 1968.

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