Split Line Domination in Graphs

M. H. Muddebihal¹, U. A. Panfarosh¹, ², Anil R. Sedamkar³

¹Professor, Department of Mathematics, Gulbarga University, Gulbarga – 585106, Karnataka, India
²Associate Professor, Department of Mathematics, Anjuman Arts, Science and Commerce College, Bijapur – 586104, Karnataka, India
³Lecturer, Department of Science, Government Polytechnic, Bijapur – 586101, Karnataka, India

Abstract: A line dominating set \( D \subseteq V(L(G)) \) is a split line dominating set, if the subgraph \( \{ V(L(G)) - D \} \) is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number in \( L(G) \) and is denoted by \( \gamma_{sl}(G) \). In this paper, we introduce the new concept in domination theory. Also, we study the graph theoretic properties of \( \gamma_{sl}(G) \) and many bounds were obtained in terms of elements of \( G \) and its relationships with other domination parameters were found.

Subject Classification Number: AMS 05C69, 05C70.

Keywords: Graph, Line graph, Dominating set, split line dominating set, split line domination number

1. Introduction

In this paper, we follow the notations of [1]. All the graphs considered here are simple and finite. As usual \( p = |V| \) and \( q = |E| \) denote the number of vertices and edges of a graph \( G \) respectively.

In general, we use \( \{ X \} \) to denote the subgraph induced by the set of vertices \( X \) and \( N(v) (N[v]) \) denote the open (closed) neighborhoods of a vertex \( v \).

The notation \( \alpha_0(G) (\alpha_1(G)) \) is the minimum number of vertices (edges) in a vertex (edge) cover of \( G \). The notation \( \beta_0(G) (\beta_1(G)) \) is the maximum cardinality of a vertex (edge) independent set in \( G \). Let \( \deg(v) \) is the degree of vertex \( v \) and as usual \( \delta(G) (\Delta(G)) \) is the minimum (maximum) degree. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. The degree of an edge \( e = uv \) of \( G \) is defined by \( \deg(e) = \deg(u) + \deg(v) - 2 \) and \( \delta'(G) (\Delta'(G)) \) is the minimum (maximum) degree among the edges of \( G \).

A line graph \( L(G) \) is the graph whose vertices correspond to the edges of \( G \) and two vertices in \( L(G) \) are adjacent if and only if the corresponding edges in \( G \) are adjacent. We begin by recalling some standard definitions from domination theory.

A set \( S \subseteq V(G) \) is said to be a dominating set of \( G \), if every vertex in \( V - S \) is adjacent to some vertex in \( S \). The minimum cardinality of vertices in such a set is called the domination number of \( G \) and is denoted by \( \gamma(G) \). A dominating set \( S \) is called the total dominating set, if for every vertex \( v \in V \), there exists a vertex \( u \in S \) for \( u \neq v \) such that \( u \) is adjacent to \( v \). The total domination number of \( G \), denoted by \( \gamma_t(G) \) is the minimum cardinality of total dominating set of \( G \). A dominating set \( S \subseteq V(G) \) is a connected dominating set, if the induced subgraph \( \{ S \} \) has no isolated vertices. The connected domination number, \( \gamma_c(G) \) of \( G \) is the minimum cardinality of a connected dominating set of \( G \). A set \( D \subseteq V(L(G)) \) is said to be line dominating set of \( G \), if every vertex not in \( D \) is adjacent to a vertex in \( D \). The line domination number of \( G \), is denoted by \( \gamma_l(G) \) is the minimum cardinality of a line dominating set. The concept of domination in graphs with its many variations is now well studied in graph theory (see [2] and [3]).

Analogously, a line dominating set \( D \subseteq V(L(G)) \) is a split line dominating set, if the subgraph \( \{ V(L(G)) - D \} \) is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number of \( G \) and is denoted by \( \gamma_{sl}(G) \). In this paper, we introduce the new concept in domination theory. Also we study the graph theoretic properties of \( \gamma_{sl}(G) \) and many bounds were obtained in terms of elements of \( G \) and its relationships with other domination parameters were found. Throughout this paper, we consider the graphs with \( p \geq 4 \) vertices.

2. Results

Initially, we give the split line domination number for some standard graphs, which are straight forward in the following Theorem.
Theorem 1:
\[ \gamma_{st}(C_p) = \begin{cases} \frac{p}{3} & \text{for } p \equiv 0 \pmod{3} \\ \frac{p}{3} & \text{otherwise.} \end{cases} \]

b. For any path \( P_p \) with \( p \geq 4 \) vertices,
\[ \gamma_{st}(P_p) = n, \text{ for } p = 3n + 1, n = 1, 2, 3, \ldots, \]
\[ = \frac{p}{3} \text{ for } p \equiv 0 \pmod{3}. \]
\[ = \frac{p}{3} \text{ otherwise.} \]

Theorem 2: A split line dominating set \( D \subseteq V(L(G)) \) is minimal if and only if for each vertex \( x \in D \), one of the following conditions holds:

a. There exists a vertex \( y \in V(L(G)) - D \) such that \( N(y) \cap D = \{x\} \).

b. \( x \) is an isolated vertex in \( \{D\} \).

c. \( \{V(L(G)) - D\} \cup \{x\} \) is connected.

Proof: Suppose \( D \) is a minimal split line dominating set of \( G \) and there exists a vertex \( x \in D \) such that \( D - \{x\} \) is a split line dominating set of \( G \) and there exists a vertex \( y \in D - \{x\} \) such that \( y \) dominates \( x \). That is \( y \in N(x) \). Therefore, \( x \) does not satisfy (a) and (b), hence it must satisfy (c). Then there exists a vertex \( y \in V(L(G)) - D \) such that \( N(y) \cap D = \{x\} \). Since \( D - \{x\} \) is a split line dominating set of \( G \), then there exists a vertex \( z \in D - \{x\} \) such that \( z \in N(y) \). Therefore \( w \in N(y) \cap D \), where \( w \neq x \), a contradiction to the fact that \( N(y) \cap D = \{x\} \). Clearly, \( D \) is a minimal split line dominating set of \( G \).

The following Theorem characterizes the split line domination and domination number of graphs.

Theorem 3: For any connected graph \( G \), \( \gamma_{sl}(G) = \gamma_1(G) \) if \( L(G) \) contains the set of end vertices.

Proof: Let \( v \in V(L(G)) \) be an end vertex and there exists a support vertex \( u \in N(v) \). Further, let \( D \) be a split line dominating set of \( G \). Suppose \( u \in D \), then \( D \) is a \( \gamma_{sl} \) set of \( G \). Suppose \( u \notin D \), then \( v \in D \) and hence \( D - \{v\} \cup \{u\} \) forms a minimal \( \gamma_{sl} \) set of \( G \). Repeating this process for all end vertices in \( L(G) \), we obtain a \( \gamma_{sl} \) set of \( G \) containing all the end vertices and \( \gamma_{sl}(G) = \gamma_1(G) \).

The following Theorem relates the split line domination and domination number in terms of vertices of \( G \).

Theorem 4: For any connected \((p,q)\)-graph \( G \),
\[ \gamma_{sl}(G) + \gamma(G) \leq p. \]

Proof: Let \( C = \{v_1, v_2, \ldots, v_n\} \subseteq V(G) \) be the set of all non end vertices in \( G \). Further, let \( S \subseteq C \) be the set of vertices with \( \text{diam}(u_i, v_j) \geq 3 \), \( \forall u_i, v_j \in S \), \( 1 \leq i \leq k \). Clearly, \( N[S] = V(G) \) and \( S \) forms a \( \gamma \) set of \( G \). Suppose \( \text{diam}(u_i, v_j) < 3 \). Then there exists at least one vertex \( x \in V(G) - S \) such that, either \( x \in N(v) \) or \( x \notin N(v) \), \( v \in S \) and \( v \in S \cup \{x\} \). Then \( S \cup \{x\} \) forms a minimal dominating set of \( G \). Now in \( L(G) \), let \( F = \{u_1, u_2, \ldots, u_n\} \subseteq V(L(G)) \) be the set of vertices corresponding to the edges which are incident to the vertices of \( S \) in \( G \). Further, let \( D \subseteq F \) be the minimal set of vertices which covers all the vertices in \( L(G) \), also making the subgraph \( V(L(G)) - D \) contains at least two components. Clearly, \( D \) forms a minimal split line dominating set of \( G \). Hence, it follows that \( |D| \cup |S \cup \{x\}| \leq |V(G)| \) and gives \( \gamma_{sl}(G) + \gamma(G) \leq p \).

The following Theorem relates the split line domination and total domination number of \( G \).

Theorem 5: For any connected graph \( G \),
\[ \gamma_{sl}(G) + \gamma_1(G) \leq \alpha_0(G) + \beta_0(G) + 1. \]

Proof: Let \( C = \{v_1, v_2, \ldots, v_n\} \subseteq V(G) \) be the minimal set of vertices with \( \text{dist}(u, v) \geq 2 \) for all \( u, v \in C \), covers all the edges in \( G \). Clearly, \( |C| = \alpha_0(G) \). Further, if any vertex \( x \in C, N(x) \in V(G) - C \). Then \( C \) itself is an independent vertex set. Otherwise, \( C_1 \cup C_2 \) where \( C_1 \subseteq C \) and \( C_2 \subseteq V(G) - C \), forms a maximum independent set of vertices \( |C_1 \cup C_2| = \beta_0(G) \). Now, let \( S = C' \cup C'' \), where \( C' \subseteq C \) and \( C'' \subseteq V(G) - C \), be the minimal set of vertices with \( N[S] = V(G) \) and \( \text{deg}(x) \geq 1 \), \( \forall x \in S \) in the sub graph \( S \). Clearly, \( S \) forms a minimal total dominating set in \( G \). Now by the definition of line graph, let \( F = \{u_1, u_2, \ldots, u_n\} \subseteq V(L(G)) \) be the set of vertices corresponding to the edges which are incident with the vertices of \( S \) in \( G \). Let there exists a set \( D \subseteq F \) of vertices

Volume 3 Issue 8, August 2014

Paper ID: 02015294
www.ijsr.net

Impact Factor (2012): 3.358

International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064

www.ijsr.net

Licensed Under Creative Commons Attribution CC BY
which are minimally independent and covers all the vertices in line graph. Clearly, $\gamma^*_d$ itself is a $\gamma_d$ - set of $G$. Therefore, it follows that $|D||C|\subseteq |C|\cup C_1\cup C_2|$ and hence $\gamma_d(G)+\gamma_c(G)\leq \alpha_0(G)+\beta_0(G)+1$.

The following Theorem relates the split line domination, connected domination and domination number of $G$.

**Theorem 6:** For any connected graph $G$, $\gamma_d(G)+\gamma_c(G)\leq diam(G)+\gamma(G)+\alpha_0(G)$.

**Proof:** Let $C \subseteq V(G)$ be the minimal set of vertices which covers all the edges in $G$ with $|C|=\alpha_0(G)$. Further, there exists an edge set $J \subseteq J$, where $J$ is the set of edges which are incident with the vertices of $C$, constituting the longest path in $G$ such that $|J|=diam(G)$. Let $S=\{v_1,v_2,\ldots,v_k\} \subseteq C$ be the minimal set of vertices which covers all the vertices in $G$. Clearly, $S$ forms a minimal dominating set of $G$. Suppose the subgraph $(S)$ is connected, then $S$ itself is a $\gamma_c$ - set. Otherwise, there exists at least one vertex $x \in V(G)-S$ such that $S_1=S \cup \{x\}$ forms a minimal connected dominating set of $G$. Now, in $L(G)$, let $F=\{u_1,u_2,\ldots,u_k\} \subseteq V(L(G))$ be the set of vertices such that $\{u_j\}=\{e_j\} \in E(G)$, $1 \leq j \leq k$, which $\{e_j\}$ are incident with the vertices of $S$. Further, let $D \subseteq F$ be the set of vertices with $N[D]=\gamma_d(G)$ and if the subgraph $V[L(G)]-D$ contains more than one component. Then $D$ forms a split line dominating set of $G$. Otherwise, there exists at least one vertex $\{u\} \in V(L(G))-D$ such that $V[L(G)]-D-\{u\}$ yields more than one component. Clearly, $D \cup \{u\}$ forms a minimal $\gamma_d$ - set of $G$. Therefore, it follows that $|D|\cup |u\cup S_1|\subseteq |F|\cup |S|\cup |C|$ and hence $\gamma_d(G)+\gamma_c(G)\leq diam(G)+\gamma(G)+\alpha_0(G)$.

In the following Theorems we give lower bounds to split line domination number of graphs.

**Theorem 7:** If every non end vertex of a tree $T$ is adjacent to at least one end vertex with $T$ containing at least two cut vertices, then $\gamma_d(T)\leq e-1$, where $c$ is the number of cut vertices in $T$.

**Proof:** Let $F=\{v_1,v_2,\ldots,v_m\} \subseteq V(T)$ be the set of all cut vertices in $T$ with $|F|=c$. Further, let $A=\{e_1,e_2,\ldots,e_x\}$ be the set of edges which are incident with the vertices of $F$. Now by the definition of line graph, suppose $D=\{u_1,u_2,\ldots,u_c\} \subseteq A$ be the set of vertices which covers all the vertices in $L(T)$. Clearly, $D$ forms a minimal split line dominating set of $L(T)$. Therefore, it follows that $|D|\leq |F|-1$ and hence $\gamma_d(T)\leq e-1$.

**Theorem 8:** For any connected $(p,q)$ - graph $G$, $\gamma_d(G)\leq \frac{p}{2}$.

**Proof:** Let $D=\{v_1,v_2,\ldots,v_n\} \subseteq V(L(G))$ be the minimal split line dominating set of $G$. Suppose $|V[L(G)]-D|=0$. Then the result follows immediately. Further, if $|V[L(G)]-D|\geq 2$, then $V[L(G)]-D$ contains at least two vertices such that $2n<p$. Clearly, it follows that $\gamma_d(G)=n<\frac{p}{2}$.

**Theorem 9:** For any connected $(p,q)$ - tree $T$, $\gamma_d(T)\leq q-\Delta(T)$.

**Proof:** Let $A=\{v_1,v_2,\ldots,v_q\} \subseteq V(L(T))$ be the set of all support vertices. Suppose there exists a set of vertices $A_1=\{u_1,u_2,\ldots,u_m\} \subseteq V(L(T))-A$ such that $dist(u_j,v_j)\geq 2$, $\forall u_j \in A_1$, $v_j \in A$, $1 \leq i \leq m$, $1 \leq j \leq n$. Then, clearly $S=A \cup A_1$ forms a split line dominating set of $T$. Otherwise, if $A \not\subseteq V(L(T))$, then select the set of vertices $S=A_1$ such that $N[S]=V[L(T)]$ and the subgraph $V[L(T)]-S$ is disconnected. Clearly, in any case $S$ forms a minimal split line dominating set of $T$. Since for any tree $T$, there exists at least one edge $e \in E(T)$ with $deg(e)=\Delta(T)$, we obtain $|S|\leq |E(T)|-\Delta(T)$. Therefore, $\gamma_d(T)\leq q-\Delta(T)$.

**Theorem 10:** For any connected unicyclic graph $G=(V,E)$, $\gamma_d(G)\leq q-\Delta(G)+1$, if one of the following conditions hold:


b. $G=C_3(u_1,u_2,\ldots,u_n)$, $deg(u_i)\geq 3$,
\[\text{deg}(u_2)=\text{deg}(u_3)=2\], $diam(u_1,w)\leq 2$ for all vertices $w$ not on $C_3$ and $\text{deg}(w)\geq 3$ for at most one vertex $w$ not on $C_3$.

c. $G=C_3$, $deg(u_1)\geq 3$, $deg(u_2)\geq 3$, $deg(u_3)\geq 2$, all vertices not on $C_3$ adjacent to $u_1$ have degree at most 2 and all vertices whose distance from $u_1$ is 2 are end vertices.

d. $G=C_3$, $deg(u_1)\geq 3$, $deg(u_2)\geq 3$, $deg(u_3)\geq 3$ and all vertices not on $C_3$ are end vertices.

e. $G=C_4$, either exactly one vertex of $C_4$ or two vertices of $C_4$ have degree at least 3 and all vertices not on $C_3$ are end vertices.

**Proof:** Assume $\gamma_d(G)=q-\Delta(G)+1$. Let $A$ denote the set of all end vertices of $L(G)$ with $|A|=m$. Since
\( V(L(G)) - (A \cup \{v_i\}) \) is a split line dominating set for any vertex \( v_i \) of \( C \), \( \gamma_d(G) \leq q - m \) so that \( m \leq \Delta'(G) \). Let \( e \) be an edge of maximum degree \( \Delta'(G) \). Analogously in \( L(G), e = u \in V(L(G)) \) such that \( |v| = \Delta(L(G)) \). If \( u \) is not on \( C \), then \( m = \Delta'(G) \) and there exists vertices \( v_1 \) and \( v_2 \) on \( C \) such that \( V(L(G)) - (A \cup \{v_1, v_2\}) \) is a split line dominating set of cardinality \( q - \Delta'(G) \), which is a contradiction. Hence \( u \) lies on \( C \) and \( m \geq \Delta'(G) - 1 \), we now consider the following cases.

**Case 1:** \( m = \Delta'(G) - 1 \). In this case, all vertices other than \( u \) and \( v \) have degree either one or two. Hence \( C = C_3 \) or \( C_4 \) and \( G \) is isomorphic to one of the graphs described in (a) to (e).

**Case 2:** \( m = \Delta'(G) \). In this case, there exists a unique vertex \( u \) on \( C \) such that \( V(L(G)) - (A \cup \{u\}) \) is a minimum split line dominating set of \( G \). It follows that \( C = C_3 \) and \( G \) is isomorphic to the graph described in (d).

**References**