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Split Line Domination in Graphs

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Abstract: ***A line dominating set*** $D \subseteq V(L(G))$ ***is a split line dominating set, if the subgraph*** $\langle V(L(G)) - D \rangle$ ***is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number in*** $L(G)$ ***and is denoted by*** $\gamma_{sl}(G)$***. In this paper, we introduce the new concept in domination theory. Also, we study the graph theoretic properties of*** $\gamma_{sl}(G)$ ***and many bounds were obtained in terms of elements of*** G ***and its relationships with other domination parameters were found.***

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1. Introduction

In this paper, we follow the notations of [1]. All the graphs considered here are simple and finite. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G respectively.

In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ $(N[v])$ denote the open (closed) neighborhoods of a vertex v.

The notation $\alpha_0(G)$ $(\alpha_1(G))$ is the minimum number of vertices (edges) in a vertex (edge) cover of G. The notation $\beta_0(G)$ $(\beta_1(G))$ is the maximum cardinality of a vertex (edge) independent set in G. Let $\deg(v)$ is the degree of vertex v and as usual $\delta(G)$ $(\Delta(G))$ is the minimum (maximum) degree. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. The degree of an edge $e = uv$ of G is defined by $\deg(e) = \deg(u) + \deg(v) - 2$ and $\delta'(G)$ $(\Delta'(G))$ is the minimum (maximum) degree among the edges of G.

A line graph $L(G)$ is the graph whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. We begin by recalling some standard definitions from domination theory.

A set $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V - S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set S is called the total dominating set, if for every vertex $v \in V$, there exists a vertex $u \in S$, $u \neq v$ such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. A dominating set $S \subseteq V(G)$ is a connected dominating set, if the induced subgraph $\langle S \rangle$ has no isolated vertices. The connected domination number, $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set of G. A set $D \subseteq V(L(G))$ is said to be line dominating set of G, if every vertex not in D is adjacent to a vertex in D. The line domination number of G, is denoted by $\gamma_l(G)$ is the minimum cardinality of a line dominating set. The concept of domination in graphs with its many variations is now well studied in graph theory (see [2] and [3]).

Analogously, a line dominating set $D \subseteq V(L(G))$ is a split line dominating set, if the subgraph $\langle V(L(G)) - D \rangle$ is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number of G and is denoted by $\gamma_{sl}(G)$. In this paper, we introduce the new concept in domination theory. Also we study the graph theoretic properties of $\gamma_{sl}(G)$ and many bounds were obtained in terms of elements of G and its relationships with other domination parameters were found. Throughout this paper, we consider the graphs with $p \geq 4$ vertices.

2. Results

Initially, we give the split line domination number for some standard graphs, which are straight forward in the following Theorem.

Theorem 1:

a. For any cycle C_p with $p \geq 4$ vertices,

$$\gamma_{sl}(C_p) = \frac{p}{3} \text{ for } p \equiv 0 (\bmod 3).$$
$$= \left\lceil \frac{p}{3} \right\rceil \text{ otherwise.}$$

b. For any path P_p with $p \geq 4$ vertices,

$$\gamma_{sl}(P_p) = n, \text{ for } p = 3n+1, n = 1,2,3,...,$$
$$= \frac{p}{3} \text{ for } p \equiv 0 (\bmod 3).$$
$$= \left\lceil \frac{p}{3} \right\rceil \text{ otherwise.}$$

Theorem 2: A split line dominating set $D \subseteq V(L(G))$ is minimal if and only if for each vertex $x \in D$, one of the following condition holds:

a. There exists a vertex $y \in V(L(G)) - D$ such that $N(y) \cap D = \{x\}$.

b. x is an isolated vertex in $\langle D \rangle$.

c. $\langle (V(L(G)) - D) \cup \{x\} \rangle$ is connected.

Proof: Suppose D is a minimal split line dominating set of G and there exists a vertex $x \in D$ such that x does not hold any of the above conditions. Then for some vertex v, the set $D_1 = D - \{v\}$ forms a split line dominating set of G by the conditions (a) and (b). Also by (c), $\langle V(L(G)) - D \rangle$ is disconnected. This implies that D_1 is a split line dominating set of G, a contradiction.

Conversely, suppose for every vertex $x \in D$, one of the above statements hold. Further, if D is not minimal, then there exists a vertex $x \in D$ such that $D - \{x\}$ is a split line dominating set of G and there exists a vertex $y \in D - \{x\}$ such that y dominates x. That is $y \in N(x)$. Therefore, x does not satisfy (a) and (b), hence it must satisfy (c). Then there exists a vertex $y \in V(L(G)) - D$ such that $N(y) \cap D = \{x\}$. Since $D - \{x\}$ is a split line dominating set of G, then there exists a vertex $z \in D - \{x\}$ such that $z \in N(y)$. Therefore $w \in N(y) \cap D$, where $w \neq x$, a contradiction to the fact that $N(y) \cap D = \{x\}$. Clearly, D is a minimal split line dominating set of G.

The following Theorem characterizes the split line domination and line domination number of graphs.

Theorem 3: For any connected graph G, $\gamma_{sl}(G) = \gamma_l(G)$ if $L(G)$ contains the set of end vertices.

Proof: Let $v \in V(L(G))$ be an end vertex and there exists a support vertex $u \in N(v)$. Further, let D be a split line dominating set of G. Suppose $u \in D$, then D is a γ_{sl} - set of G. Suppose $u \notin D$, then $v \in D$ and hence $(D - \{v\}) \cup \{u\}$ forms a minimal γ_{sl} - set of G. Repeating this process for all end vertices in $L(G)$, we obtain a γ_{sl} - set of G containing all the end vertices and $\gamma_{sl}(G) = \gamma_l(G)$.

The following Theorem relates the split line domination and domination number in terms of vertices of G.

Theorem 4: For any connected (p,q) - graph G, $\gamma_{sl}(G) + \gamma(G) \leq p$.

Proof: Let $C = \{v_1, v_2, ..., v_n\} \subseteq V(G)$ be the set of all non end vertices in G. Further, let $S \subseteq C$ be the set of vertices with $diam(u_i, v_i) \geq 3$, $\forall u_i, v_i \in S$, $1 \leq i \leq k$. Clearly, $N[S] = V(G)$ and S forms a γ - set of G. Suppose $diam(u_i, v_i) < 3$. Then there exists at least one vertex $x \in V(G) - S$ such that, either $x \in N(v)$ or $x' \in N(v')$, where $v \in S$ and $v' \in S \cup \{x\}$. Then $S \cup \{x\}$ forms a minimal dominating set of G. Now in $L(G)$, let $F = \{u_1, u_2, ..., u_n\} \subseteq V(L(G))$ be the set of vertices corresponding to the edges which are incident to the vertices of S in G. Further, let $D \subseteq F$ be the minimal set of vertices which covers all the vertices in $L(G)$, also making the subgraph $\langle V(L(G)) - D \rangle$ contains at least two components. Clearly, D forms a minimal split line dominating set of G. Hence, it follows that $|D| \cup |S \cup \{x\}| \leq |V(G)|$ and gives $\gamma_{sl}(G) + \gamma(G) \leq p$.

The following Theorem relates the split line domination and total domination number of G.

Theorem 5: For any connected graph G, $\gamma_{sl}(G) + \gamma_t(G) \leq \alpha_0(G) + \beta_0(G) + 1$.

Proof: Let $C = \{v_1, v_2, ..., v_n\} \subseteq V(G)$ be the minimal set of vertices with $dist(u,v) \geq 2$ for all $u, v \in C$, covers all the edges in G. Clearly, $|C| = \alpha_0(G)$. Further, if for any vertex $x \in C$, $N(x) \in V(G) - C$. Then C itself is an independent vertex set. Otherwise, $C_1 \cup C_2$ where $C_1 \subseteq C$ and $C_2 \subseteq V(G) - C$, forms a maximum independent set of vertices $|C_1 \cup C_2| = \beta_0(G)$. Now, let $S = C' \cup C''$, where $C' \subseteq C$ and $C'' \subseteq V(G) - C$, be the minimal set of vertices with $N[S] = V(G)$ and $\deg(x) \geq 1$, $\forall x \in S$ in the sub graph $\langle S \rangle$. Clearly, S forms a minimal total dominating set in G. Now by the definition of line graph, let $F = \{u_1, u_2, ..., u_n\} \subseteq V(L(G))$ be the set of vertices corresponding to the edges which are incident with the vertices of S in G. Let there exists a set $D \subseteq F$ of vertices which are minimally independent and covers all the vertices in line graph. Clearly, D itself is a γ_{sl}- set of G. Therefore, it follows that $|D|\cup|S|\le|C|\cup|C_1\cup C_2|\cup 1$ and hence $\gamma_{sl}(G)+\gamma_t(G)\le\alpha_0(G)+\beta_0(G)+1$.

The following Theorem relates the split line domination, connected domination and domination number of G.

Theorem 6: For any connected graph G, $\gamma_{sl}(G)+\gamma_c(G)\le diam(G)+\gamma(G)+\alpha_0(G)$.

Proof: Let $C'\subseteq V(G)$ be the minimal set of vertices which covers all the edges in G with $|C'|=\alpha_0(G)$. Further, there exists an edge set $J'\subseteq J$, where J is the set of edges which are incident with the vertices of C', constituting the longest path in G such that $|J'|=diam(G)$. Let $S=\{v_1,v_2,...,v_k\}\subseteq C$ be the minimal set of vertices which covers all the vertices in G. Clearly, S forms a minimal dominating set of G. Suppose the subgraph $\langle S\rangle$ is connected, then S itself is a γ_c- set. Otherwise, there exists at least one vertex $x\in V(G)-S$ such that $S_1=S\cup\{x\}$ forms a minimal connected dominating set of G. Now, in $L(G)$, let $F=\{u_1,u_2,...,u_k\}\subseteq V(L(G))$ be the set of vertices such that $\{u_j\}=\{e_j\}\in E(G)$, $1\le j\le k$, where $\{e_j\}$ are incident with the vertices of S. Further, let $D\subseteq F$ be the set of vertices with $N[D]=V(L(G))$ and if the subgraph $\langle V(L(G))-D\rangle$ contains more than one component. Then D forms a split line dominating set of G. Otherwise, there exists at least one vertex $\{u\}\in V(L(G))-D$ such that $\langle V(L(G))-D-\{u\}\rangle$ yields more than one component. Clearly, $D\cup\{u\}$ forms a minimal γ_{sl}- set of G. Therefore, it follows that $|D\cup\{u\}|\cup|S_1|\le|J'|\cup|S|\cup|C'|$ and hence $\gamma_{sl}(G)+\gamma_c(G)\le diam(G)+\gamma(G)+\alpha_0(G)$.

In the following Theorems we give lower bounds to split line domination number of graphs.

Theorem 7: If every non end vertex of a tree T is adjacent to at least one end vertex with T containing at least two cut vertices, then $\gamma_{sl}(T)\le c-1$, where c is the number of cut vertices in T.

Proof: Let $F=\{v_1,v_2,...,v_m\}\subseteq V(T)$ be the set of all cut vertices in T with $|F|=c$. Further, let $A=\{e_1,e_2,...,e_k\}$ be the set of edges which are incident with the vertices of F. Now by the definition of line graph, suppose $D=\{u_1,u_2,...,u_i\}\subseteq A$ be the set of vertices which covers all the vertices in $L(T)$. Clearly, D forms a minimal split line dominating set of $L(T)$. Therefore, it follows that $|D|\le|F|-1$ and hence $\gamma_{sl}(T)\le c-1$.

Theorem 8: For any connected (p,q)- graph G, $\gamma_{sl}(G)\le\left\lceil\frac{p}{2}\right\rceil$.

Proof: Let $D=\{v_1,v_2,...,v_n\}\subseteq V(L(G))$ be the minimal split line dominating set of G. Suppose $|V(L(G))-D|=0$. Then the result follows immediately. Further, if $|V(L(G))-D|\ge 2$, then $V(L(G))-D$ contains at least two vertices such that $2n<p$. Clearly, it follows that $\gamma_{sl}(G)=n<\lceil p/2\rceil$.

Theorem 9: For any connected (p,q)- tree T, $\gamma_{sl}(T)\le q-\Delta'(T)$.

Proof: Let $A=\{v_1,v_2,...,v_n\}\subseteq V(L(T))$ be the set of all support vertices. Suppose there exists a set of vertices $A_1=\{u_1,u_2,...,u_m\}\subseteq V(L(T))-A$ such that $dist(u_i,v_j)\ge 2$, $\forall u_i\in A_1$, $v_j\in A$, $1\le i\le m$, $1\le j\le n$. Then, clearly $S=A\cup A_1$ forms a split line dominating set of T. Otherwise, if $A\not\subset V(L(T))$, then select the set of vertices $S=A_1$ such that $N[S]=V(L(T))$ and the subgraph $\langle V(L(T))-S\rangle$ is disconnected. Clearly, in any case S forms a minimal split line dominating set of T. Since for any tree T, there exists at least one edge $e\in E(T)$ with $\deg(e)=\Delta'(T)$, we obtain $|S|\le|E(T)|-\Delta'(T)$. Therefore, $\gamma_{sl}(T)\le q-\Delta'(T)$.

Theorem 10: For any connected unicyclic graph $G=(V,E)$, $\gamma_{sl}(G)\le q-\Delta'(G)+1$, if one of the following conditions hold:

a. $G=C_4$.
b. $G=C_3(u_1,u_2,...,u_n)$, $\deg(u_1)\ge 3$, $\deg(u_2)=\deg(u_3)=2$, $diam(u_1,w)\le 2$ for all vertices w not on C_3 and $\deg(w)\ge 3$ for at most one vertex w not on C_3.
c. $G=C_3$, $\deg(u_1)\ge 3$, $\deg(u_2)\ge 3$, $\deg(u_3)=2$, all vertices not on C_3 adjacent to u_1 have degree at most 2 and all vertices whose distance from u_1 is 2 are end vertices.
d. $G=C_3$, $\deg(u_1)=3$, $\deg(u_2)\ge 3$, $\deg(u_3)\ge 3$ and all vertices not on C_3 are end vertices.
e. $G=C_4$, either exactly one vertex of C_4 or two vertices of C_4 have degree at least 3 and all vertices not on C_3 are end vertices.

Proof: Assume $\gamma_{sl}(G)=q-\Delta'(G)+1$. Let A denote the set of all end vertices of $L(G)$ with $|A|=m$. Since

$V(L(G))-(A\cup\{v_1\})$ is a split line dominating set for any vertex v_1 of C, $\gamma_{sl}(G)\le q-m$ so that $m\le\Delta'(G)$. Let e be an edge of maximum degree $\Delta'(G)$. Analogously in $L(G)$, $e=u\in V(L(G))$ such that $|u|=\Delta(L(G))$. If u is not on C, then $m=\Delta'(G)$ and there exists vertices v_1 and v_2 on C such that $V(L(G))-(A\cup\{v_1,v_2\})$ is a split line dominating set of cardinality $q-\Delta'(G)$, which is a contradiction. Hence u lies on C and $m\ge\Delta'(G)-1$, we now consider the following cases.

Case 1: $m=\Delta'(G)-1$. In this case, all vertices other than u and v have degree either one or two. Hence $C=C_3$ or C_4 and G is isomorphic to one of the graphs described in (a) to (e).

Case 2: $m=\Delta'(G)$. In this case, there exists a unique vertex u on C such that $V(L(G))-(A\cup\{u\})$ is a minimum split line dominating set of G. It follows that $C=C_3$ and G is isomorphic to the graph described in (d).

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