Split Line Domination in Graphs

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Abstract: A line dominating set \( D \subseteq V(L(G)) \) is a split line dominating set, if the subgraph \( V(L(G)) - D \) is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number in \( L(G) \) and is denoted by \( \gamma_{sl}(G) \). In this paper, we introduce the new concept in domination theory. Also, we study the graph theoretic properties of \( \gamma_{sl}(G) \) and many bounds were obtained in terms of elements of \( G \) and its relationships with other domination parameters were found.

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1. Introduction

In this paper, we follow the notations of [1]. All the graphs considered here are simple and finite. As usual \( p = |V| \) and \( q = |E| \) denote the number of vertices and edges of a graph \( G \) respectively.

In general, we use \( \langle X \rangle \) to denote the subgraph induced by the set of vertices \( X \) and \( N(v) \) (\( N[v] \)) denote the open (closed) neighborhoods of a vertex \( v \).

The notation \( \alpha_0(G) (\alpha_1(G)) \) is the minimum number of vertices (edges) in a vertex (edge) cover of \( G \). The notation \( \beta_0(G) (\beta_1(G)) \) is the maximum cardinality of a vertex (edge) independent set in \( G \). Let \( \deg(v) \) is the degree of vertex \( v \) and as usual \( \delta(G) (\Delta(G)) \) is the minimum (maximum) degree. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. The degree of an edge \( e = uv \) of \( G \) is defined by \( \deg(e) = \deg(u) + \deg(v) - 2 \) and \( \delta'(G) (\Delta'(G)) \) is the minimum (maximum) degree among the edges of \( G \).

A line graph \( L(G) \) is the graph whose vertices correspond to the edges of \( G \) and two vertices in \( L(G) \) are adjacent if and only if the corresponding edges in \( G \) are adjacent. We begin by recalling some standard definitions from domination theory.

A set \( S \subseteq V(G) \) is said to be a dominating set of \( G \), if every vertex in \( V - S \) is adjacent to some vertex in \( S \). The minimum cardinality of vertices in such a set is called the domination number of \( G \) and is denoted by \( \gamma(G) \). A dominating set \( S \) is called the total dominating set, if for every vertex \( v \in V \), there exists a vertex \( u \in S \), \( u \neq v \) such that \( u \) is adjacent to \( v \). The total domination number of \( G \), denoted by \( \gamma_t(G) \) is the minimum cardinality of total dominating set of \( G \). A dominating set \( S \subseteq V(G) \) is a connected dominating set, if the induced subgraph \( \langle S \rangle \) has no isolated vertices. The connected domination number, \( \gamma_c(G) \) of \( G \) is the minimum cardinality of a connected dominating set of \( G \). A set \( D \subseteq V(L(G)) \) is said to be a line dominating set of \( G \), if every vertex not in \( D \) is adjacent to a vertex in \( D \). The line domination number of \( G \), is denoted by \( \gamma_l(G) \) is the minimum cardinality of a line dominating set. The concept of domination in graphs with its many variations is now well studied in graph theory (see [2] and [3]).

Analogously, a line dominating set \( D \subseteq V(L(G)) \) is a split line dominating set, if the subgraph \( V(L(G)) - D \) is disconnected. The minimum cardinality of vertices in such a set is called a split line domination number of \( G \) and is denoted by \( \gamma_{sl}(G) \). In this paper, we introduce the new concept in domination theory. Also we study the graph theoretic properties of \( \gamma_{sl}(G) \) and many bounds were obtained in terms of elements of \( G \) and its relationships with other domination parameters were found. Throughout this paper, we consider the graphs with \( p \geq 4 \) vertices.

2. Results

Initially, we give the split line domination number for some standard graphs, which are straight forward in the following Theorem.
Theorem 1:
a. For any cycle $C_p$ with $p \geq 4$ vertices,
$$\gamma_{sl}(C_p) = \frac{p}{3} \text{ for } p \equiv 0 \pmod{3},$$
$$= \frac{p}{3} \text{ otherwise.}$$

b. For any path $P_p$ with $p \geq 4$ vertices,
$$\gamma_{sl}(P_p) = n, \text{ for } p = 3n + 1, n = 1, 2, 3, ...$$
$$= \frac{p}{3} \text{ for } p \equiv 0 \pmod{3},$$
$$= \frac{p}{3} \text{ otherwise.}$$

Theorem 2: A split line dominating set $D \subseteq V(L(G))$ is minimal if and only if for each vertex $x \in D$, one of the following conditions holds:

a. There exists a vertex $y \in V(L(G))-D$ such that $N(y) \cap D = \{x\}$.

b. $x$ is an isolated vertex in $\{D\}$.

c. $\left(\{V(L(G))-D\} \cup \{x\}\right)$ is connected.

Proof: Suppose $D$ is a minimal split line dominating set of $G$ and there exists a vertex $x \in D$ such that $x$ does not hold any of the above conditions. Then for some vertex $v$, the set $D_1 = D - \{v\}$ forms a split line dominating set of $G$ by the conditions (a) and (b). Also by (c), $\{V(L(G))-D\}$ is disconnected. This implies that $D_1$ is a split line dominating set of $G$, a contradiction.

Conversely, suppose for every vertex $x \in D$, one of the above statements hold. Further, if $D$ is not minimal, then there exists a vertex $x \in D$ such that $D - \{x\}$ is a split line dominating set of $G$ and there exists a vertex $y \in D - \{x\}$ such that $y$ dominates $x$. That is $y \in N(x)$. Therefore, $x$ does not satisfy (a) and (b), hence it must satisfy (c). Then there exists a vertex $y \in V(L(G))-D$ such that $N(y) \cap D = \{x\}$. Since $D - \{x\}$ is a split line dominating set of $G$, then there exists a vertex $z \in D - \{x\}$ such that $z \in N(y)$. Therefore $w \in N(y) \cap D$, where $w \neq x$, a contradiction to the fact that $N(y) \cap D = \{x\}$. Clearly, $D$ is a minimal split line dominating set of $G$.

The following theorem characterizes the split line domination and line domination number of graphs.

Theorem 3: For any connected graph $G$, $\gamma_{sl}(G) = \gamma_1(G)$ if $L(G)$ contains the set of end vertices.

Proof: Let $v \in V(L(G))$ be an end vertex and there exists a support vertex $u \in N(v)$. Further, let $D$ be a split line dominating set of $G$. Suppose $u \in D$, then $D$ is a $\gamma_{sl}$-set of $G$. Suppose $u \notin D$, then $v \in D$ and hence $(D - \{v\}) \cup \{u\}$ forms a minimal $\gamma_{sl}$-set of $G$. Repeating this process for all end vertices in $L(G)$, we obtain a $\gamma_{sl}$-set of $G$ containing all the end vertices and $\gamma_{sl}(G) = \gamma_1(G)$.

The following theorem relates the split line domination and domination number in terms of vertices of $G$.

Theorem 4: For any connected $(p,q)$-graph $G$, $\gamma_{sl}(G) + \gamma(G) \leq p$.

Proof: Let $C = \{v_1,v_2,...,v_n\} \subseteq V(G)$ be the set of all non end vertices in $G$. Further, let $S \subseteq C$ be the set of vertices with $\text{diam}(u,v_i) \geq 3$, $\forall u, v_i \in S$, $1 \leq i \leq k$. Clearly, $N[S] = V(G)$ and $S$ forms a $\gamma$-set of $G$. Suppose $\text{diam}(u,v_i) < 3$. Then there exists at least one vertex $x \in V(G)-S$ such that, either $x \in N(v_i)$ or $x \in N(v_j)$, where $v \in S$ and $v_j \in S \cup \{x\}$. Then $S \cup \{x\}$ forms a minimal dominating set of $G$. Now in $L(G)$, let $F = \{u_1,u_2,...,u_n\} \subseteq V(L(G))$ the set of vertices corresponding to the edges which are incident to the vertices of $S$ in $G$. Further, let $D \subseteq F$ be the minimal set of vertices which covers all the vertices in $L(G)$, also making the subgraph $\{V(L(G))-D\}$ contains at least two components. Clearly, $D$ forms a minimal split line dominating set of $G$. Hence, it follows that $|D| + |S \cup \{x\}| \leq |V(G)|$ and gives $\gamma_{sl}(G) + \gamma(G) \leq p$.

The following theorem relates the split line domination and total domination number of $G$.

Theorem 5: For any connected graph $G$, $\gamma_{sl}(G) + \gamma_1(G) \leq \alpha_0(G) + \beta_0(G)+1$.

Proof: Let $C = \{v_1,v_2,...,v_n\} \subseteq V(G)$ be the minimal set of vertices with $\text{dist}(u,v) \geq 2$ for all $u,v \in C$, covers all the edges in $G$. Clearly, $|C| = \alpha_0(G)$. Further, if for any vertex $x \in C$, $N(x) \subseteq V(G)-C$, then $C$ itself is an independent vertex set. Otherwise, $C_1 \cup C_2$ where $C_1 \subseteq C$ and $C_2 \subseteq V(G)-C$, forms a maximum independent set of vertices $|C_1 \cup C_2| = \beta_0(G)$. Now, let $S = C \cup C'$, where $C' \subseteq C$ and $C' \subseteq V(G)-C$, be the minimal set of vertices with $N[S] = V(G)$ and $\deg(x) \geq 1$, $\forall x \in S$ in the sub graph $\{S\}$. Clearly, $S$ forms a minimal total dominating set in $G$. Now by the definition of line graph, let $F = \{u_1,u_2,...,u_n\} \subseteq V(L(G))$ be the set of vertices corresponding to the edges which are incident with the vertices of $S$ in $G$. Let there exists a set $D \subseteq F$ of vertices
which are minimally independent and covers all the vertices in line graph. Clearly, $D$ itself is a $\gamma_d$- set of $G$. Therefore, it follows that $|D|\cup|S|\leq |C|\cup|C_2|+1$ and hence $\gamma_d(G)+\gamma_c(G)\leq \alpha_0(G)+\beta_0(G)+1$.

The following Theorem relates the split line domination, connected domination and domination number of $G$.

**Theorem 6:** For any connected graph $G$, $\gamma_d(G)+\gamma_c(G)\leq \text{diam}(G)+\gamma(G)+\alpha_0(G)$. 

**Proof:** Let $C\subseteq V(G)$ be the minimal set of vertices which covers all the edges in $G$ with $|C| = \alpha_0(G)$. Further, there exists an edge set $J \subseteq E$, where $J$ is the set of edges which are incident with the vertices of $C$, constituting the longest path in $G$ such that $|J|$ = $\text{diam}(G)$. Let $S = \{v_1, v_2, \ldots, v_n\} \subseteq C$ be the minimal set of vertices which covers all the vertices in $G$. Clearly, $S$ forms a minimal dominating set of $G$. Suppose the subgraph $\langle S \rangle$ is connected, then $S$ itself is a $\gamma_c$-set. Otherwise, there exists at least one vertex $v \in V(G)-S$ such that $S_v = S \cup \{x\}$ forms a minimal dominating set of $G$. Now, in $L(G)$, let $F = \{u_1, u_2, \ldots, u_k\} \subseteq V(L(G))$ be the set of vertices such that $u_j = \{v_j\} \in E(G)$, $1 \leq j \leq k$, where $\{v_j\}$ are incident with the vertices of $S$. Further, let $D \subseteq F$ be the set of vertices with $|V(D)| = V(L(G))$ and if the subgraph $\langle V(L(G)) - D \rangle$ contains more than one component. Then $D$ forms a split line dominating set of $G$. Otherwise, there exists at least one vertex $u \in V(L(G)) - D$ such that $\langle V(L(G)) - D - \{u\} \rangle$ yields more than one component. Clearly, $D \cup \{u\}$ forms a minimal $\gamma_d$- set of $G$. Therefore, it follows that $|D| \cup |u| \leq |S| \cup |S| \cup |C|$ and hence $\gamma_d(G)+\gamma_c(G)\leq \text{diam}(G)+\gamma(G)+\alpha_0(G)$.

In the following Theorems we give lower bounds to split line domination number of graphs.

**Theorem 7:** If every non end vertex of a tree $T$ is adjacent to at least one end vertex with $T$, containing at least two cut vertices, then $\gamma_d(T) \leq c-1$, where $c$ is the number of cut vertices in $T$.

**Proof:** Let $F = \{v_1, v_2, \ldots, v_m\} \subseteq V(T)$ be the set of all cut vertices in $T$ with $|F| = c$. Further, let $A = \{e_1, e_2, \ldots, e_h\}$ be the set of edges which are incident with the vertices of $F$. Now by the definition of line graph, suppose $D = \{u_1, u_2,\ldots, u_m\} \subseteq A$ be the set of vertices which covers all the vertices in $L(T)$. Clearly, $D$ forms a minimal split line dominating set of $L(T)$. Therefore, it follows that $|D| \leq |F|-1$ and hence $\gamma_d(T) \leq c-1$.

**Theorem 8:** For any connected $(p,q)$ - graph $G$, $\gamma_d(G) \leq \left\lceil \frac{p}{2} \right\rceil$.

**Proof:** Let $D = \{v_1,v_2,\ldots,v_n\} \subseteq V(L(G))$ be the minimal split line dominating set of $G$. Suppose $|V(L(G)) - D| = 0$. Then the result follows immediately. Further, if $|V(L(G)) - D| \geq 2$, then $V(L(G)) - D$ contains at least two vertices such that $2n < p$. Clearly, it follows that $\gamma_d(G) = n < \left\lceil \frac{p}{2} \right\rceil$.

**Theorem 9:** For any connected $(p,q)$-tree $T$, $\gamma_d(T) \leq q - \Delta(T)$.

**Proof:** Let $A = \{v_1,v_2,\ldots,v_q\} \subseteq V(L(T))$ be the set of all support vertices. Suppose there exists a set of vertices $A_i = \{u_1, u_2,\ldots, u_m\} \subseteq V(L(T)) - A$ such that $\text{dist}(u_i,v_j) \geq 2$, $\forall u_i \in A_i$, $v_j \in A$, $1 \leq i \leq m$, $1 \leq j \leq n$. Then, clearly $S = A \cup A_i$ forms a split line dominating set of $T$. Otherwise, if $A \subseteq V(L(T))$, then select the set of vertices $S = A_i$ such that $N[S] = V(L(T))$ and the subgraph $\langle V(L(T)) - S \rangle$ is disconnected. Clearly, in any case $S$ forms a minimal split line dominating set of $T$. Since for any tree $T$, there exists at least one edge $e \in E(T)$ with $\text{deg}(e) = \Delta(T)$, we obtain $|S| \leq |E(T)| - \Delta(T)$. Therefore, $\gamma_d(T) \leq q - \Delta(T)$.

**Theorem 10:** For any connected unicyclic graph $G = (V,E)$, $\gamma_d(G) \leq q - \Delta(G) + 1$, if one of the following conditions hold:

a. $G = C_4$

b. $G = C_3(u_1, u_2, \ldots, u_n)$, $\text{deg}(u_i) \geq 3$

deg$(u_2) = \text{deg}(u_3) = 2$, $\text{diam}(u_1, w) \leq 2$ for all vertices $w$ not on $C_3$ and $\text{deg}(w) \geq 3$ for at most one vertex $w$ not on $C_3$.

c. $G = C_3$, $\text{deg}(u_i) \geq 3$, $\text{deg}(u_1) \geq 3$, $\text{deg}(u_2) = 2$, all vertices not on $C_3$ adjacent to $u_i$ have degree at most 2 and all vertices whose distance from $u_1$ is 2 are end vertices.

d. $G = C_3$, $\text{deg}(u_1) = 3$, $\text{deg}(u_2) \geq 3$, $\text{deg}(u_3) \geq 3$ and all vertices not on $C_3$ are end vertices.

e. $G = C_4$, either exactly one vertex of $C_4$ or two vertices of $C_4$ have degree at least 3 and all vertices not on $C_3$ are end vertices.

**Proof:** Assume $\gamma_d(G) = q - \Delta(G) + 1$. Let $A$ denote the set of all end vertices of $L(G)$ with $|A| = m$. Since
$V(L(G)) \setminus (A \cup \{v_1\})$ is a split line dominating set for any vertex $v_1$ of $C$, $\gamma_d(G) \leq q - m$ so that $m \leq \Delta'(G)$. Let $e$ be an edge of maximum degree $\Delta(G)$. Analogously in $L(G)$, $e = u \in V(L(G))$ such that $|u| = \Delta(L(G))$. If $u$ is not on $C$, then $m = \Delta(G)$ and there exists vertices $v_1$ and $v_2$ on $C$ such that $V(L(G)) \setminus (A \cup \{v_1, v_2\})$ is a split line dominating set of cardinality $q - \Delta'(G)$, which is a contradiction. Hence $u$ lies on $C$ and $m \geq \Delta'(G) - 1$, we now consider the following cases.

Case 1: $m = \Delta'(G) - 1$. In this case, all vertices other than $u$ and $v$ have degree either one or two. Hence $C = C_4$ or $C_3$ and $G$ is isomorphic to one of the graphs described in (a) to (e).

Case 2: $m = \Delta'(G)$. In this case, there exists a unique vertex $u$ on $C$ such that $V(L(G)) \setminus (A \cup \{u\})$ is a minimum split line dominating set of $G$. It follows that $C = C_3$ and $G$ is isomorphic to the graph described in (d).

References