On Some New Results of a Subclass of Univalent Functions Defined by Ruscheweyh Derivative

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Abstract: In this paper, we introduce a new class of univalent functions defined by Ruscheweyh derivative in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1 \}$. We obtain basic properties, like, coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, closure theorems and convolution operator for functions belonging to the class $\Sigma^+(\sigma, c, \beta, \lambda)$.

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1.Introduction

Let $\Sigma$ denote the class of functions of the from:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic and univalent in the open unit disk $\mathbb{D}$.

If a function $f$ is given by (1) and $g$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (2)$$

is in the class $\Sigma$, then the convolution (or Hadamard product) of $f$ and $g$ is defined by

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}. \quad (3)$$

Let $\Sigma^+$ denote the subclass of $\Sigma$ consisting of functions of the from

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in \mathbb{N}). \quad (4)$$

We aim to study the subclass $\Sigma^+(\sigma, c, \beta, \lambda)$ consisting of function $f \in \Sigma^+$ and satisfying the condition:

$$\frac{\sigma [z(D^2 f(z))^" - ((D^2 f(z))^\prime - 1) ]}{cz(D^2 f(z))^" + ((1-\sigma)(D^2 f(z))^\prime + 1)} < \beta, \quad z \in \mathbb{D}, \quad (5)$$

where $0 \leq \sigma < 1, 0 \leq c < 1, 0 < \beta < 1$ and $D^2 f(z)$ is the Ruscheweyh derivative [6], [7] of $f$ of order $\lambda$ defined as follow:

$$D^2 f(z) = z + \sum_{n=2}^{\infty} a_n A_n(\lambda) z^n, \quad (6)$$

where

$$A_n(\lambda) = \frac{\lambda + 1)(\lambda + 2) \ldots (\lambda + n - 1)}{(n - 1)!}, \lambda > -1, \quad z \in \mathbb{D}. \quad (7)$$

Another classes studied by several authors, like, [2] and [4] consisting of functions of the from (4).

2.Coefficient Inequality

In the following theorem, we obtain necessary and sufficient condition to be the function in the class $\Sigma^+(\sigma, c, \beta, \lambda)$.

**Theorem 1:** Let the function $f$ be defined by (4). Then $f \in \Sigma^+(\sigma, c, \beta, \lambda)$ if and only if

$$\sum_{n=2}^{\infty} \left| n (\sigma(n + \beta) - \beta(c(n - 1) + 1)) A_n(\lambda) a_n \right| \leq \beta (2-\sigma). \quad (7)$$

where $0 < \beta < 1, 0 < \sigma < 1, 0 \leq c < 1, \lambda > -1$. The result (7) is sharp for the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (n \geq 2). \quad (8)$.

**Proof:** Suppose that the inequality (7) holds true and $|z| = 1$.

Then we have

$$\left| \frac{\sigma [z(D^2 f(z))^" - ((D^2 f(z))^\prime - 1) ]}{cz(D^2 f(z))^" + ((1-\sigma)(D^2 f(z))^\prime + 1)} - \beta \right| \leq 0,$$

by hypothesis, hence, by maximum modulus principle $f \in \Sigma^+(\sigma, c, \beta, \lambda)$.

Conversely, assume that $f \in \Sigma^+(\sigma, c, \beta, \lambda)$.

$$\sum_{n=2}^{\infty} \left| n (\sigma(n + \beta) - \beta(c(n - 1) + 1)) A_n(\lambda) a_n z^n \right| \leq 0,$$

by hypothesis, hence, by maximum modulus principle $f \in \Sigma^+(\sigma, c, \beta, \lambda)$.
Therefore, we get
\[
\left| \sigma \left[ D^4 f(z) \right] - \left( D^3 f(z) \right) - 1 \right| - \beta \left| \frac{c z D^4 f(z)}{D^3 f(z)} \right| + \left( 1 - \sigma \right) \left( D^3 f(z) \right) + 1 \right|.
\]

Thus, we get
\[
\sum_{n=2}^{\infty} (\sigma n^2) A_n(\lambda) a_n z^{n-2} < \beta \sum_{n=2}^{\infty} (n - \sigma) A_n(\lambda) a_n z^{n-2} + (2 - \sigma),
\]

so that
\[
\sum_{n=2}^{\infty} \frac{n(\sigma + n - \sigma + 1)}{A_n(\lambda)} A_n(\lambda) a_n \leq \beta(2 - \sigma).
\]

**Corollary 1:** Let the function \( f \in \Sigma^+(\sigma, c, \beta, \lambda) \). Then
\[
a_n \leq \frac{\beta(2 - \sigma)}{\left| \sum_{n=2}^{\infty} \frac{n(\sigma + n - \sigma + 1)}{A_n(\lambda)} A_n(\lambda) a_n \right|},
\]

**3. Distortion and Covering Theorems**

We introduce the growth and distortion theorems for the function \( f \) in the class \( \Sigma^+(\sigma, c, \beta, \lambda) \).

**Theorem 2:** Let the function \( f \in \Sigma^+(\sigma, c, \beta, \lambda) \). Then
\[
|z| - \frac{1}{\beta} \left| \sum_{n=2}^{\infty} \frac{n(\sigma + n - \sigma + 1)}{A_n(\lambda)} A_n(\lambda) a_n \right| z^2 \leq |f(z)|
\]

The result is sharp and attained.

\[
f(z) = z + \frac{\beta(2 - \sigma)}{\left| \sum_{n=2}^{\infty} \frac{n(\sigma + n - \sigma + 1)}{A_n(\lambda)} A_n(\lambda) a_n \right|} z^2.
\]

**Proof:** Notice that
\[
\sum_{n=2}^{\infty} \frac{n(\sigma + n - \sigma + 1)}{A_n(\lambda)} A_n(\lambda) a_n \leq \beta(2 - \sigma),
\]

and this completes the proof.

**4. Radii of starlikeness, convexity and close-to-convexity:**

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class \( \Sigma^+(\sigma, c, \beta, \lambda) \).

**Theorem 4:** Let \( f \in \Sigma^+(\sigma, c, \beta, \lambda) \). Then \( f \) is starlike in the disk \( |z| < R_\alpha \), where
\[
R_\alpha = \inf \left[ 1 - \frac{\alpha}{\beta} \left( \frac{n(\sigma(n - 2) + \beta(c + 1))}{A_n(\lambda)} \right) \right]^{1/3},
\]

\[
\geq 2. \tag{12}
\]

**Proof:** If \( f \) is starlike in the disk \( |z| < R_\alpha \), then
\[
R_\alpha \geq 2.
\]

Indeed, we have
\[
|zf'(z)| - 1 \leq |z| - 1 - \alpha,
\]

and this completes the proof.
Theorem 5: Let \( f \in \Sigma^+(\sigma, c, \beta, \lambda) \). Then \( f \) is convex in \(|z| < R_2\), of order \( \alpha \), \( 0 \leq \alpha < 1 \), where

\[
R_2 = \inf \left\{ \frac{n(\sigma(n + \beta) - \beta(c(n - 1) + 1))A_n(\lambda)}{(n - \alpha)\beta(2 - \sigma)} \right\}^{\frac{1}{n - 1}}.
\]

\[
\geq 2 \quad (15)
\]

Proof: \( f \) is convex of order \( \alpha \), \( 0 \leq \alpha < 1 \), if

\[
\frac{zf''(z)}{f'(z)} \leq 1 - \alpha,
\]

for \(|z| < R_2\).

Indeed we have

\[
\frac{zf''(z)}{f'(z)} \leq \frac{\sum_{n=2}^{\infty} n(n-1)\alpha_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} n\alpha_n|z|^{n-1}} \leq 1 - \alpha, \quad (0 \leq \alpha < 1) \quad (16)
\]

Hence by Theorem 1, (16) will be true if

\[
\frac{zf''(z)}{f'(z)} \leq \frac{\sum_{n=2}^{\infty} n(n-1)\alpha_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} n\alpha_n|z|^{n-1}} \leq 1 - \alpha, \quad (0 \leq \alpha < 1) \quad (16)
\]

Thus it is enough to show that

\[
\frac{zf''(z)}{f'(z)} \leq 1 - \alpha,
\]

5. Extreme Points:

In the following theorem, we obtain extreme points for the class \( \Sigma^+(\sigma, c, \beta, \lambda) \).

Theorem 7: Let \( f_1(z) = z \) and

\[
f_n(z) = z + \frac{\beta(2 - \sigma)}{n(\sigma(n + \beta) - \beta(c(n - 1) + 1))A_n(\lambda)}z^n, \quad \text{for } n = 2, 3, \ldots
\]

Then \( f \in \Sigma^+(\sigma, c, \beta, \lambda) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),
\]

where

\[
\begin{align*}
\mu_n &\geq 0 \\
\sum_{n=1}^{\infty} \mu_n &= 1 \\
1 &= \mu_1 + \sum_{n=2}^{\infty} \mu_n.
\end{align*}
\]

Proof: Let \( f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \).

Thus

\[
\begin{align*}
f(z) &= z + \sum_{n=2}^{\infty} \frac{\beta(2 - \sigma)}{n(\sigma(n + \beta) - \beta(c(n - 1) + 1))A_n(\lambda)}z^n \mu_n A_n(\lambda) \\
&= z + \sum_{n=2}^{\infty} \frac{\beta(2 - \sigma)}{n(\sigma(n + \beta) - \beta(c(n - 1) + 1))A_n(\lambda)}z^n \mu_n A_n(\lambda) \\
&= z + \sum_{n=2}^{\infty} \frac{\beta(2 - \sigma)}{n(\sigma(n + \beta) - \beta(c(n - 1) + 1))A_n(\lambda)}z^n \mu_n A_n(\lambda)
\end{align*}
\]

6. Hadamard Product

In the following theorem, we obtain the convolution result for function belong to the class \( \Sigma^+(\sigma, c, \beta, \lambda) \).
Theorem 8: Let \( f \) and \( g \in \Sigma^+(\sigma, c, \beta, \lambda) \). Then \( f \ast g \in \Sigma^+(\sigma, c, \delta, \lambda) \) for

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,
\]

where

\[
\delta \leq \frac{\beta^2(2-\sigma)[n(\sigma(n+\beta)) \beta(c(n-1)+1)]^2 - \beta^2(2-\sigma)[n(\sigma-c(n-1)+1)]}{A_n(\lambda)}.
\]

Proof: Since \( f, g \in \Sigma^+(\sigma, c, \beta, \lambda) \), then we have

\[
\sum_{n=2}^{\infty} \left[ n(\sigma(n+\beta) - \beta(c(n-1)+1)) \right] A_n(\lambda) \leq 1 \quad (21)
\]

and

\[
\sum_{n=2}^{\infty} \left[ n(\sigma(n+\beta) - \beta(c(n-1)+1)) \right] A_n(\lambda) b_n \leq 1 \quad (22)
\]

We must find the smallest number \( \delta \) such that

\[
\sum_{n=2}^{\infty} \left[ n(\sigma(n+\delta) - \delta(c(n-1)+1)) \right] A_n(\lambda) a_n b_n \leq 1 \quad (23)
\]

By Cauchy-Schwarz inequality, we have

\[
\sum_{n=2}^{\infty} \left[ n(\sigma(n+\delta) - \delta(c(n-1)+1)) \right] A_n(\lambda) a_n b_n \leq 1 \quad (24)
\]

Thus, it is enough to show that

\[
\delta \leq \frac{\beta^2(2-\sigma)[n(\sigma(n+\beta)) \beta(c(n-1)+1)]^2 - \beta^2(2-\sigma)[n(\sigma-c(n-1)+1)]}{A_n(\lambda)}.
\]

This complete the proof.

Theorem 9: Let \( h \in \Sigma^+(\sigma, c, \beta, \lambda) \). Then

\[
h(z) = z + \sum_{a_2}^{\infty} (a_2^2 + b_2^2) z^n
\]

belong to the class \( \Sigma^+(\sigma, c, \delta, \lambda) \), where

\[
\delta \geq \frac{2\beta^2(2-\sigma)n\sigma}{A_n(\lambda)}.
\]

Proof: Since \( f, g \in \Sigma^+(\sigma, c, \beta, \lambda) \) so by Theorem 1, yields

\[
\sum_{n=2}^{\infty} \left[ n(\sigma(n+\beta) - \beta(c(n-1)+1)) \right] A_n(\lambda) \leq 1 \quad (25)
\]

and

\[
\sum_{n=2}^{\infty} \left[ n(\sigma(n+\beta) - \beta(c(n-1)+1)) \right] A_n(\lambda) b_n \leq 1 \quad (26)
\]

we obtain from the last two inequalities

\[
\sum_{n=2}^{\infty} \left[ n(\sigma(n+\delta) - \delta(c(n-1)+1)) \right] A_n(\lambda) (a_n^2 + b_n^2) \leq 1 \quad (27)
\]

but \( h(z) \in \Sigma^+(\sigma, c, \delta, \lambda) \) if and only if

\[
\sum_{n=2}^{\infty} \left[ n(\sigma(n+\delta) - \delta(c(n-1)+1)) \right] A_n(\lambda) (a_n^2 + b_n^2) \leq 1 \quad (28)
\]

where \( 0 < \delta < 1 \), however (27) implies (28) if

\[
\frac{\delta(2-\sigma)}{A_n(\lambda)} \leq \frac{\beta^2(2-\sigma)[n(\sigma(n+\beta)) \beta(c(n-1)+1)]^2 - \beta^2(2-\sigma)[n(\sigma-c(n-1)+1)]}{A_n(\lambda)}.
\]

Simplifying, we get

\[
\delta \geq \frac{2\beta^2(2-\sigma)n\sigma}{A_n(\lambda)}.
\]

7. Closure theorems

We shall prove the following closure theorems for the class \( \Sigma^+(\sigma, c, \beta, \lambda) \), let the function \( f_i(z) \) be defined by

\[
f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n, \quad (a_{n,i} \geq 0, n \in N, n \geq 2) \quad (29).
\]

Theorem 10: Let the functions \( f_i(z) \) defined by (29) be in the class \( \Sigma^+(\sigma, c, \beta, \lambda) \) for every \( i = 1, 2, \ldots, m \). Then the function \( h(z) \) defined by

\[
h(z) = z + \sum_{n=2}^{\infty} c_n z^n, \quad (c_n \geq 0, n \in N, n \geq 2)
\]

also belongs to the class \( \Sigma^+(\sigma, c, \beta, \lambda) \), where

\[
c_n = \frac{1}{m} \sum_{i=1}^{m} a_{n,i}.
\]

Proof: Since \( f_i(z) \in \Sigma^+(\sigma, c, \beta, \lambda) \), therefore from Theorem 1, we obtain

\[
\sum_{n=2}^{\infty} n(\sigma(n+\beta) - \beta(c(n-1)+1)) A_n(\lambda) a_{n,i} \leq \beta(2-\sigma), \quad (29)
\]
For every \( \beta \leq 1 \).

The Gaussian hypergeometric function is defined by

\[
\sum_{n=0}^{\infty} \frac{[n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda)c_n}
\]

\[
= \sum_{n=0}^{\infty} [n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda) \left( \frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right)
\]

\[\leq \beta(2 - \sigma).\]

Hence \( h(z) \in \Sigma^+(\sigma, c, \beta, \lambda). \)

**Theorem 11:** Let the functions \( f_i(z) \) defined by (29) be in the class \( \Sigma^+(\sigma, c, \beta, \lambda) \), for every \( i = 1, 2, \ldots, m \). Then the function \( h(z) \) defined by

\[ h(z) = \sum_{i=1}^{m} d_i f_i(z) \text{ and } d_i = 1, d_i \geq 0 \]

in the class \( \Sigma^+(\sigma, c, \beta, \lambda). \)

**Proof:** By definition of \( h(z) \), we have

\[ h(z) = \sum_{i=1}^{m} d_i z^i + \sum_{n=1}^{\infty} \sum_{i=1}^{m} d_i a_{n,i} z^n, \]

since \( f_i(z) \) are in the class \( \Sigma^+(\sigma, c, \beta, \lambda) \), for every \( i = 1, 2, \ldots, m \), we obtain

\[ \sum_{n=0}^{\infty} [n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda)a_{n,i} \]

\[ \leq \beta(2 - \sigma) \]

for every \( i = 1, 2, \ldots, m \), hence we can see that

\[ \sum_{n=0}^{\infty} [n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda) \left( \sum_{i=1}^{m} d_i a_{n,i} \right)
\]

\[ = \sum_{i=1}^{m} d_i \left( \sum_{n=0}^{\infty} [n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda)a_{n,i} \right)
\]

\[ \leq \beta(2 - \sigma) \sum_{i=1}^{m} d_i = \beta(2 - \sigma). \]

Thus \( h(z) \in \Sigma^+(\sigma, c, \beta, \lambda). \)

8. Convolution Operator

**Definition 1 [2,5]:** The Gaussian hypergeometric function denoted by

\[
\sum_{|z| < 1}
\]

\[
\frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} [z] \leq 1,
\]

where \( c > b > 0, c > a + b \) and

\[
(x)_n = \begin{cases} (x(x + 1)(x + 2) \ldots (x + n - 1)) & \text{for } n = 1, 2, 3, \ldots \\ 1 & \text{otherwise} \end{cases}
\]

**Definition 2 [3]:** For every \( f \in \Sigma^+ \), we defined the convolution operator \( W_{a,b,c}(f)(z) \) as below:

\[
W_{a,b,c}(f)(z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} [z] \leq 1,
\]

where \( z^2 f_1(a, b; c; z) \) is Gaussian hypergeometric function (see[2] and [5]) introduced in Definition 1.

**Theorem 12:** Let \( f \) be given by (4) be in the class \( \Sigma^+(\sigma, c, \beta, \lambda) \). Then the convolution operator \( W_{a,b,c}(f) \) is in the class \( \Sigma^+(\sigma, c, \beta, \lambda) \) for \( |z| \leq r(\beta, \delta) \), where

\[
r(\beta, \delta) = \inf \left[ \frac{\delta \left[ n\sigma(n + \beta) - \beta(c(n - 1) + 1) \right]}{\beta \left[ n\sigma(n + \beta) - \beta(c(n - 1) + 1) \right]} \right]^\frac{1}{n+1}.
\]

The result is sharp for the function

\[
f_n(z) = z + \sum_{n=0}^{\infty} \frac{n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda)}{\beta(2 - \sigma)} \]

\[ \geq z \]

\[ \sum_{n=0}^{\infty} \frac{n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda)}{\beta(2 - \sigma)} \]

**Proof:** Since \( f \in \Sigma^+(\sigma, c, \beta, \lambda) \), we have

\[ \sum_{n=0}^{\infty} \frac{n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda)}{\beta(2 - \sigma)} \]

\[ \leq \beta(2 - \sigma) \]

\[ n \leq 1. \]

Note that (30) is satisfied if

\[ \sum_{n=0}^{\infty} \frac{n\sigma(n + \beta) - \beta(c(n - 1) + 1)]]A_n(\lambda)}{\beta(2 - \sigma)} \]

\[ = \beta(2 - \sigma) \sum_{n=0}^{\infty} a_n \]

\[ \leq \beta(2 - \sigma) \sum_{n=0}^{\infty} a_n \]

solving for \( |z| \) we get the result.

**Reference**


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