On a Subclass of Multivalent Harmonic Functions Defined by a Linear Operator

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Abstract. ***In this paper, we define a subclass of p- valent harmonic functions defined by a linear operator and study some results as coefficient inequality, convolution property and convex set.***

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1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simple connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D, we call h the analytic part and g the co –analytic part of f.

A necessary and sufficient condition for f to be locally univalent and sense –preserving in D is that $|h'(z)| > |g'(z)|$ in D, see Clunie and Sheil –Small [3].

Denote by $M(p)$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense –preserving in the unit disk $U = \{z: |z| < 1\}$. For $f = h + \bar{g} \in M(p)$, we may express the analytic function h and g as:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k,$$

$$g(z) = \sum_{k=n+p-1}^{\infty} b_k z^k,$$

$$|b_k| < 1. \qquad (1.1)$$

Let $N(p)$ denote the subclass of $M(p)$ consisting of functions $= h + \bar{g}$, where h and g are given by:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} |a_k| z^k,$$

$$g(z) = \sum_{k=n+p-1}^{\infty} |b_k| z^k,$$

$$|b_k| < 1. \qquad (1.2)$$

We introduce here a class $N_\lambda(p,\alpha)$ of harmonic functions of the form (1.1) that satisfy the inequality

$$Re\left\{\frac{z^{p-1}}{[\mathcal{L}_p(h*\emptyset_1)(z)]' - \overline{[\mathcal{L}_p(g*\emptyset_1)(z)]'}}\right\} > \alpha ,$$

where $0 \le \alpha < \frac{1}{p}$, $\lambda \ge 0, p \in \mathbb{N}$ and

$$\mathcal{L}_p f(z) = \mathcal{L}_p h(z) + \overline{\mathcal{L}_p g(z)} . \qquad (1.3)$$

The operator $\mathcal{L}_p$ denotes the linear operator introduced in [6]. For h and g given by (1.1), we obtain

$$\mathcal{L}_p h(z) = z^p + \sum_{k=n+p}^{\infty} \left[\lambda\left(\frac{k}{p} - 1\right) + 1\right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}} a_k z^k ,$$

$$\mathcal{L}_p g(z) = -\sum_{k=n+p-1}^{\infty} \left[\lambda\left(\frac{k}{p} + 1\right) - 1\right] \frac{(a_2)_{k-p}}{(c_2)_{k-p}} b_k z^k ,$$

where a_1, a_2, c_1, c_2 are positive real numbers , $\lambda \ge 0, p \in \mathbb{N}$.

Now, the convolution of h, g is given by (1.2) and

$$\phi_1(z) = z^p + \sum_{k=n+p}^{\infty} |A_k| z^k , \qquad \phi_2(z) = \sum_{k=n+p-1}^{\infty} |B_k| z^k$$

is defined by

$$(h*\phi_1)(z) = z^p + \sum_{k=n+p}^{\infty} |A_k||a_k| z^k$$

$$(g*\phi_2)(z) = \sum_{k=n+p-1}^{\infty} |B_k||b_k| z^k , \quad |b_k| < 1,$$

we further denote by $N_\lambda(p,\alpha)$ the subclass of $M_\lambda(p,\alpha)$ that satisfies the relation

$$N_\lambda(p,\alpha) = N_\lambda \bigcap M_\lambda(p,\alpha).$$

Lemma (1.1)[1]: If $\alpha \ge 0$, then $Re\, w > \alpha$ if and only if $|w - (1+\alpha)| < |w + (1-\alpha)|$, where w be any complex number.

2. Main Results

Theorem 2.1: Let $f = h + \bar{g}$ (h and g are given by (1.1)). If

$$\sum_{k=n+p}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p} - 1\right) + 1\right] \frac{(a_1)_{k-p}}{(c_1)_{k-p}} |A_k||a_k| + \sum_{k=n+p-1}^{\infty} k\alpha \left[\lambda\left(\frac{k}{p} + 1\right) - 1\right] \frac{(a_2)_{k-p}}{(c_2)_{k-p}} |B_k||b_k| \le p, \qquad (2.1)$$

where $(0 \le \alpha < \frac{1}{p}, \lambda \ge 0, p \in \mathbb{N}, z \in U)$, then f is harmonic p –valent sense –preserving in U and $f \in M_\lambda(p,\alpha)$.

Proof. Let

$$w(z) = \left\{\frac{z^{p-1}}{[\mathcal{L}_p(h*\emptyset_1)(z)]' - \overline{[\mathcal{L}_p(g*\emptyset_1)(z)]'}}\right\} = \frac{A(z)}{B(z)}.$$

By using the fact that in Lemma (1.1) $Re(w) \ge \alpha$ if and only if $|w - (1+\alpha)| < |w + (1-\alpha)|$, it is sufficient to show that

$$|A(z) - (1+\alpha)B(z)| - |A(z) + (1-\alpha)B(z)| \le 0 . \qquad (2.2)$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions (2.2) , we get

$$|A(z)-(1+\alpha)B(z)|-|A(z)+(1-\alpha)B(z)|$$

$$=\left|z^{p-1}-(1+\alpha)\left[pz^{p-1}+\sum_{k=n+p}^{\infty}k\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|z^{k-1}-\sum_{k=n+p-1}^{\infty}k\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|z^{k-1}\right]\right|$$

$$-\left|z^{p-1}+(1-\alpha)\left[pz^{p-1}+\sum_{k=n+p}^{\infty}k\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|z^{k-1}-\sum_{k=n+p-1}^{\infty}k\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|z^{k-1}\right]\right|$$

$$\leq\sum_{k=n+p}^{\infty}k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|+\sum_{k=n+p-1}^{\infty}k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|-p\leq 0,$$

by inequality (2.1), which implies that $f\in N_\lambda(p,\alpha)$. The harmonic functions

$$f(z)=z^p+\sum_{k=n+p}^{\infty}\frac{x_k}{k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}}z^k+\sum_{k=n+p-1}^{\infty}\frac{\overline{y_k}}{k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}}(\bar{z})^k, \qquad (2.3)$$

where

$$\sum_{k=n+p}^{\infty}|x_k|+\sum_{k=n+p-1}^{\infty}|\overline{y_k}|=p,$$

show that the coefficients bounds given by (2.1) is sharp. The function of the form (2.3) are in $M_\lambda(p,\alpha)$ because in view of (2.3) we infer that

$$\sum_{k=n+p}^{\infty}k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|+\sum_{k=n+p-1}^{\infty}k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|=\sum_{k=n+p}^{\infty}|x_k|+\sum_{k=n+p-1}^{\infty}|\overline{y_k}|=p.$$

Now , we need to prove that the condition (2.1) is also necessary for function of (1.2) to be in the class $N_\lambda(p,\alpha)$.

Theorem 2.2. Let $f=h+\bar{g}$ (h and g are given by (1.2)). Then $f\in N_\lambda(p,\alpha)$ if and only if

$$\sum_{k=n+p}^{\infty}k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|+\sum_{k=n+p-1}^{\infty}k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|\leq p,$$

where $(0\leq\alpha<\frac{1}{p},\lambda\geq 0,p\in\mathbb{N},z\in U)$.

Proof. By notation $N_\lambda(p,\alpha)\subset M_\lambda(p,\alpha)$, the sufficient part of Theorem (2.2) follows at once from Theorem (2.1), we get

$$Re\left\{\frac{z^{p-1}}{[\mathcal{L}_p(h*\emptyset_1)(z)]'-\overline{[\mathcal{L}_p(g*\emptyset_1)(z)]'}}\right\}$$

$$=Re\left\{\frac{z^{p-1}}{pz^{p-1}+\sum_{k=n+p}^{\infty}k\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|z^{k-1}+\sum_{k=n+p}^{\infty}k\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|z^{k-1}}\right\}>\alpha,$$

if we choose z to be real and let $z\rightarrow 1^-$, we obtain the condition (2.1).

Theorem 2.3. The class $N_\lambda(p,\alpha)$ is a convex set.

Proof. Let the function $f_j(z)(j=1,2)$ be in the class $N_\lambda(p,\alpha)$. It is sufficient to show that the function H defined by :

$$H(z)=(1-\gamma)f_1(z)+\gamma f_2(z)\,,\quad(0\leq\gamma<1)$$

is in the class $N_\lambda(p,\alpha)$, where $f_j=h_j+\overline{g_j}$ and

$$h_j(z)=z^p+\sum_{k=n+p}^{\infty}|a_{k,j}|z^k\,,$$

$$g_j(z)=\sum_{k=n+p-1}^{\infty}|b_{k,j}|(\bar{z})^k.$$

Since for $0\leq\gamma<1$

$$H(z)=z^p+\sum_{k=n+p}^{\infty}\left((1-\gamma)|a_{k,1}|-\gamma|a_{k,2}|\right)z^k - \sum_{k=n+p-1}^{\infty}\left((1-\gamma)|b_{k,1}|-\gamma|b_{k,2}|\right)(\bar{z})^k .$$

In view of Theorem (2.2), we have

$$\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k|\left((1-\gamma)|a_{k,1}|-\gamma|a_{k,2}|\right)+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k|\left((1-\gamma)|b_{k,1}|-\gamma|b_{k,2}|\right)$$

$$=(1-\gamma)\left(\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_{k,1}|+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_{k,1}|\right)$$

$$+\gamma\left(\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_{k,2}|+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_{k,2}|\right)$$

$$\le (1-\gamma)p+\gamma p=p ,$$

hence $H(z)\in N_\lambda(p,\alpha)$. For harmonic functions

$$f(z)=z^p+\sum_{k=n+p}^{\infty}|a_k|z^k+\sum_{k=n+p-1}^{\infty}|b_k|(\bar{z})^k \qquad (2.4)$$

and

$$F(z)=z^p+\sum_{k=n+p}^{\infty}|r_k|z^k+\sum_{k=n+p-1}^{\infty}|s_k|(\bar{z})^k , \qquad (2.5)$$

we define the convolution of f and F as

$$(f*F)(z)=z^p+\sum_{k=n+p}^{\infty}|a_k r_k|z^k+\sum_{k=n+p-1}^{\infty}|b_k s_k|(\bar{z})^k . \quad (2.6)$$

In the following theorem , we examine the convolution property of the class $N_\lambda(p,\alpha)$.

Theorem 2.4. If f and F are in $N_\lambda(p,\alpha)$, then $(f*F)\in N_\lambda(p,\alpha)$.

Proof. Let f and F of the forms (2.4) and (2.5) belongs to $N_\lambda(p,\alpha)$. Then the convolution of f and F is given by (2.6). Note that $|r_k|\le 1$ and $|s_k|\le 1$, since $F\in N_\lambda(p,\alpha)$. Then we can write

$$\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k r_k|+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k s_k|$$

$$\le \sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|A_k||a_k|+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|B_k||b_k|.$$

The right hand side of the above inequality is bounded by p because $f\in N_\lambda(p,\alpha)$. Therefore $(f*F)\in N_\lambda(p,\alpha)$.

Now, we will examine the closure property of the class $N_\lambda(p,\alpha)$ under the generalized Bernardi –Libera –Livingston integral operator (see [2],[4] and [5]) $D_{c,p}(f)$ which is defined by

$$D_{c,p}(f)(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt , \quad (c>-p). \qquad (2.7)$$

Theorem 2.5. Let $f\in N_\lambda(p,\alpha)$. Then $D_{c,p}(f)$ belong to the class $N_\lambda(p,\alpha)$.

Proof. From the representation of $D_{c,p}(f)$, it follows that

$$D_{c,p}(f)=\frac{c+p}{z^c}\int_0^z t^{c-1}\{h(t)+\overline{g(t)}\}dt$$

$$=\frac{c+p}{z^c}\left\{\int_0^z t^{c-1}\left(t^p+\sum_{k=n+p}^{\infty}|a_k|t^k\right)dt+\overline{\int_0^z t^{c-1}\left(\sum_{k=n+p-1}^{\infty}|b_k|t^k\right)dt}\right\}$$

$$=z^p+\sum_{k=n+p}^{\infty}v_k z^k+\sum_{k=n+p-1}^{\infty}w_k(\bar{z})^k ,$$

where $v_k=\frac{c+p}{c+k}|a_k|$and $w_k=\frac{c+p}{c+k}|b_k|$. Therefore

$$\sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|\,A_k|\left(\frac{c+p}{c+k}\right)|a_k|$$
$$+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)\right.$$
$$\left.-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|\,B_k|\left(\frac{c+p}{c+k}\right)|a_k|$$
$$\leq \sum_{k=n+p}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}-1\right)+1\right]\frac{(a_1)_{k-p}}{(c_1)_{k-p}}|\,A_k|\,|a_k|$$
$$+\sum_{k=n+p-1}^{\infty} k\alpha\left[\lambda\left(\frac{k}{p}+1\right)\right.$$
$$\left.-1\right]\frac{(a_2)_{k-p}}{(c_2)_{k-p}}|\,B_k||b_k| \leq p\,.$$

Since $\in N_\lambda(p,\alpha)$, by Theorem (2.2) , we have $D_{c,p}(f) \in N_\lambda(p,\alpha)$.

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