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Regularization Methods to Solve Various Inverse Problems

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Abstract: ***Inverse problems arise in a wide spectrum of applications in fields ranging from engineering to scientific computation. In these problems one often has to solve operator equations of the first kind, which are usually ill-posed in the sense of Hadamard. It means that the hardest issue in the numerical computation of inverse problems is the instability of the solution with respect to the noise from the observation data; that is, small perturbations of the observation data may lead to large changes on the considered solution. Thus to ensure a feasible and stable numerical approximation solution, it is necessary to employ some kind of regularization method. The purpose of this paper is to survey some recent developments in the area of regularization methods for mathematical inverse problems. In undertaking this task it is difficult to avoid a certain amount of mathematical rigor. Nevertheless, because one of the goal of the paper is to try to explain some newer developments in mathematical regularization theory to an audience which may have the needed background in operator theory or functional analysis, we will wherever possible try to use more formal(less rigorous) but simple explanations and to supplement the mathematical concepts below with examples. We will also take a few shortcuts in some definitions.***

Keywords: Inverse problems, Ill- posedness, Regularization, Regularization parameter, Classical regularization, Iterative regularization, Local regularization, Volterra first kind integral problem.

1. Introduction

The area of mathematical inverse problems is quite broad and involves the qualitative and quantitative analysis of a wide variety of physical models. Applications include, for example, the problem of inverse heat conduction, image reconstruction, tomography, the inverse scattering problem, and the determination of unknown coefficients or boundary parameters appearing in partial differential equation models of physical phenomena.

We will survey some recent developments in the area of regularization methods for mathematical inverse problems and indicate where further contributions are needed. Finally we will discuss current work in the area of iterative solution methods, regularization schemes which have been successfully applied to a number of important non linear inverse problems.

1.1 Inverse problem and ill posedness:

Physical model:

$$y = F(x)$$

Where $x \epsilon X$, $y \epsilon Y$
and $F: X \rightarrow Y$
Direct problem: given $x \epsilon X$, calculate $y = F(x) \epsilon Y$.
Inverse problem: given $y \epsilon Y$, is there $x \epsilon X$ such that $F(x) = y$?

Well-posed problem (Hadamard):

- Existence: There exists a solution to the problem.
- Uniqueness: There is at most one solution to the problem.
- Stability: The solution depends continuously on the data.

If a problem is not well-posed, it is ill-posed.
Inverse problems are typically ill-posed.

Inverse Problem
$y = F(x)$
Where $x \epsilon X$ unknown
$y \epsilon Y$ Exact measurement
and $F : X \rightarrow Y$
operator with discontinuous inverse
Data: y^{δ} such that $\|y^{\delta} - y\| \leq \delta$ (δ is a noise level)
Ill-posedness: $F^{-1}(y\delta)$ needs not to be close to x.

1.2 Regularization

Idea of regularization: an ill-posed inverse problem is approximated by a family of nearby well-posed problems.
Regularization operators: the family $\{R_{\alpha}\}_{\alpha>0}$ such that
Stability: for any $\alpha > 0$, R_{α} is a stable operator,
Approximation:

$$\lim_{\alpha \to 0} R_{\alpha}(y) = F^{-1}(y) \; \forall y \in Y.$$

Where α is called regularization parameter.

Banach-Steinhaus theorem: it is impossible to approximate a discontinuous linear operator point wise by a uniformly bounded family of continuous linear operators.

In this paper, we consider the discrete linear inverse problem which can be expressed as
$\mathbf{Ax} = \mathbf{b}$ (1.1)
for $\mathbf{A} \in R^{m \times n}$, $\mathbf{x} \in R^{n}$, $\mathbf{b} \in R^{m}$ and $m \geq n$. The known right hand side of (1.16) consists of unknown true data $\mathbf{b}_{true} \in R^{m}$ and noise $\boldsymbol{\varepsilon} \in R^{m}$; i.e.
$\mathbf{b} = \mathbf{b}_{true} + \boldsymbol{\varepsilon}$. (1.2)

The matrix $\mathbf{A}$ in (1.16) derives from the underlying connective system and is assumed to be known. The vector $\mathbf{x}$ is the solution we want to obtain given the noisy data and the system $\mathbf{A}$. Ideally, the solution should be close to the true solution $\mathbf{x}_{true}$ that satisfies

$\mathbf{A}\mathbf{x}_{true} = \mathbf{b}_{true}$. (1.3)

In practice, all we are given is the noisy data instead of the true data. In that case, it is possible that for (1.1) any of the condition of well posedness does not satisfy. Condition of unstability is often reflected in a large condition number for **A**. In all three cases, it is necessary to use regularization to find a best (approximate) solution.

1.3 Regularization Parameter

α controls the compromise between the approximation and the stability. Thus to ensure a feasible and stable numerical approximation solution, it is necessary to develop appropriate strategies for choosing the regularization parameters. In practice, the effectiveness of a regularization method depends strongly on the choice of a good regularization parameter.

Three types of parameter choices:

- A-priori parameter choice rules where α = α (δ).
- A-posteriori parameter choice rules where α = α (y^{δ}, δ).
- noise level free parameter choice rules where α = α (y^{δ}).

A regularization method is always defined as a family of regularization operators together with a parameter choice rule.

Regularized solution:

$$x_{\alpha}^{\delta} = R_{\alpha(y\delta,\delta)}(y\delta)$$

Convergence:

$$x_{\alpha}^{\delta} \to x \; as \; \delta \to 0$$

- A-priori parameter choice rule: α depends only on the noise level possibly including information about the a-priori smoothness of the solution.
- A-posteriori parameter choice rule: α depends on both the noise level and the data, e.g., Morozov's discrepancy principle:

$$\propto = sup\{\propto > 0 : \|F(X_{\alpha}^{\delta}) - Y^{\delta}\|\} \leq \mathcal{T}\delta\}$$
$$where \; \mathcal{T} > 1$$

- Noise level free parameter choice rule: α depends only on the data, e.g. L-curve and Quasi-optimality principle. Bakushinskii veto: a noise level free parameter choice rule cannot yield a convergent regularization method for ill-posed linear inverse problems.

2. Regularization Methods

In this section we review some of the most commonly used methods when ill - posed inverse problems are treated. These methods are called regularization methods. Although the emphasis in this paper is not on regularization techniques, it is important to understand the philosophy behind them and how the methods work.

Basically, Inverse problems are used to solve the integral equation of the first kind. On the basis of the nature of these integral equations, regularization methods can be divided into two categories broadly-: Classical regularization methods and Local regularization methods

- **Classical regularization methods** are designed to overcome the obstacles in The *Fredholm integral equations of the first kind*

To explain the basic ideas of regularization, we consider a simple linear inverse problem. Let H_1 and H_2 be separable Hilbert spaces of finite or infinite dimensions and $A: H_1 \to H_2$ a compact operator or mapping from H_1 to H_2. Consider first the problem of finding $f \in D(A) \in H_1$ satisfying the equation

Af=g (2.1)

Where $y \in R(A) \in H_2$ (Range of A) is given.

The Fredholm integral equation of the first kind takes the generic form

$$\int_0^1 K(s,t)f(t)dt = g(s), 0 \leq s \leq 1$$

Where the function K, is given by

$$K(s,t) = \frac{d}{(d^2 + (s-t)^2)^{3/2}}$$

and

$$Af(t) = \int_0^1 K(s,t)f(t)dt = g(s), 0 \leq s \leq 1$$

Here, both the kernel K and the right hand side g are known functions, while f is the unknown function. This equation establishes a linear relationship between the two functions f and g and the kernel K describes the precise relationship between the two quantities. Thus the function K describes the underlying model. *If f and K are known, then we can compute g by evaluating the integral; this is called the forward computation. The inverse problem consists of computing f given the right hand side and the kernel.* Now from the above examples it is very clear to understand the concept of inverse problems

Here, we discuss three families of classical methods. These methods are *(1) regularization by singular value truncation, (2) the Tikhonov regularization and (3) regularization by truncated iterative methods.*

- **Local regularization methods** are designed to overcome the obstacles in The *Volterra integral equations of the first kind*

We consider the following scalar Volterra first-kind integral problem. Given a suitable function f(·) defined on [0, 1], find $\bar{u}(\cdot)$ satisfying, for a.e.

t ϵ [0, 1],

Au (t) = f (t), (2.2)

Where A is the bounded linear operator on $L^2(0, 1)$ given by

$$Au(t) = \int_0^t K(t,s)u(s)ds, a.e. \; t \in [0,1]$$

In the typical case that the range of A is not closed, it is well-known that problem (2.2) is ill -posed, lacking continuous dependence on data f ϵ $L^2(0, 1)$

Note: The type with integration over a fixed interval is called a Fredholm equation, while if the upperlimit is a variable it is a Volterra equation.

The basic idea of regularization methods is that, instead of trying to solve equation (2.1) and (2.2) exactly, one seeks to find a nearby problem that is uniquely solvable and that is

robust in the sense that small errors in the data do not corrupt excessively this approximate solution.

2.1 Classical regularization methods:

This category includes three families of methods. These methods are *(1) regularization by singular value truncation, (2) the Tikhonov regularization and (3) regularization by truncated iterative methods.*

2.1.1 Regularization by singular value decomposition:

Let H_1 and H_2 be separable Hilbert spaces with inner products $\langle\cdot, \cdot\rangle H_i$, and norms $\|\cdot\|$ H_i, i= 1, 2. The set of bounded operators from H_1 to H_2 will be denoted by B (H_1, H_2).

Definition 2.1.1.1. For an operator A ∈ B (H_1, H_2), the adjoint A* of *A* is the element of B(H_2,H_1) that satisfies ⟨Ag, f⟩H_2 = ⟨g,A*f⟩H_1 for any g ∈ H_1 and f ∈ H_2.

Definition 2.1.1.2. An operator A ∈ B (H, H) is self-adjoint if A = A*.

Definition 2.1.1.3. An operator A ∈ B (H, H) is unitary if A*A = AA* = I.

Definition 2.1.1.4. A: $H_1 \rightarrow H_2$ is compact if, for any bounded sequence $\{g_n\} \in H_1$, the sequence $\{Ag_n\} \in H_2$ contains a convergent subsequence.

Our interest will be directed toward the case where *A* is compact. In that event, A^*A is compact, nonnegative definite and self-adjoint. As such, it has a pure point spectrum with nonzero eigen values λ^2_j that provide eigenvalue – eigen function pairs (λ^2_j, g_j), $j = 1, 2\ldots..$ where
$\lambda^2_1 \geq \lambda^2_2 \geq \cdots > 0$ and the g_j's are orthonormal [11].

Now, AA^* is also compact, nonnegative definite and self adjoint with

$$A\,(A^*Ag_j) = (AA^*)\,Ag_j = \lambda^2_j Ag_j\,,$$

Where j = 1, 2, ...

Thus, by letting $f_j = Ag_j/\lambda_j$, we obtain the pairs (λ^2_j, f_j), $j = 1, 2, \ldots$, that form the eigen value –eigen function system for AA^* This follows from observing that

$$\langle f_i, f_j\rangle_{H2} = \langle Ag_i/\lambda_i, Ag_j/\lambda_j\rangle_{H2} = (1/(\lambda_i\lambda_j)\langle g_i, A*Ag_j\rangle_{H1} = (\lambda_j/\lambda_i)\langle g_i, g_j\rangle_{H1} = \delta_{ij}\,,$$

Where δ_{ij}=0 for i≠ j and δ_{ij} = 1 for i = j.

Corollary 2.1.1.1. *If* $\mathbf{A} \in C^{m\times n}$ *has rank* $k \leq \min(m, n) < \infty$, *there exist unitary matrices*

$$\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_m] \in C^{m\times m}$$

and

$$\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n] \in C^{n\times n}$$

such that

$$\mathbf{A} = \mathbf{U\Lambda V}^* = \sum_{j=1}^{k} \lambda_j u_j v_j^* \quad (2.1.1)$$

Where $\mathbf{\Lambda} = \text{diag}(\lambda_1, \ldots\ldots\ldots \lambda_k) \in R^{m\times n}$ is a diagonal matrix for which $\lambda_1, \ldots, \lambda_k$ satisfy $\lambda_1 \geq \lambda_2 \geq \ldots\ldots\ldots\ldots\ldots\ldots \geq \lambda_k > 0$.

$$Av_i = \lambda_i u_i\,,\ A^*u_i = \lambda_i v_i$$

$$(u_i,u_j) = \delta_{ij},\ (v_i,v_j) = \delta_{ij},\ u^*u=I,\ v^*v = I$$

Writing the SVD (from 2.1.1)) of **A** in (1.1) as $\sum_{j=1}^{k} \lambda_j u_j v_j^T$

then $X = \sum_{j=1}^{k} \frac{u_j^T b}{\lambda_j} v_j = \sum_{j=1}^{k} \frac{u_j^T b_{true}}{\lambda_j} v_j + \sum_{j=1}^{k} \frac{u_j^T \varepsilon}{\lambda_j} v_j$ (2.1.2)

From the above eqn it becomes obvious that the solution is contaminated by the second term of the last expression, especially when λj is much less than $u_j^T \varepsilon$.

2.1.2 Tikhonov Regularization

As we have seen, the truncated SVD method relies on computing the singular values and singular vectors of the matrix **A**. The resulting computational task can be heavy or not feasible for large-scale problems. In contrast, the Tikhonov regularization method [39] does not require the calculation of the SVD. Instead, we solve the problem

$$\min_X\{\|AX - b\|_2^2 + \alpha^2\|X\|_2^2\} \quad (2.1.3)$$

for **x**, where $\alpha > 0$ is a parameter that governs the weight of the regularizationor penalty term in (2.1.3). That is,

$X_\alpha = \arg\min_X\{\|AX - b\|_2^2 + \alpha^2\|X\|_2^2\}$ (2.1.4) The first term of the right hand side of (2.1.3), i.e., the fidelity term, measures the fit of the solution to the noisy data and the second term controls the norm of the solution as a means of governing the noise distortion. There is a trade off between these two aspects of the criterion and we want to attain a suitable balance through adjusting the parameter *α*.

To obtain a more explicit form for (2.1.4), we can write it as

$$X_\propto = arg\min_X \left\| \begin{pmatrix} A \\ \propto I \end{pmatrix} - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2^2$$

Which is now just an ordinary least-squares problem with the consequence that

$$X_\propto = \left(\begin{pmatrix} A \\ \propto I \end{pmatrix}^T \begin{pmatrix} A \\ \propto I \end{pmatrix}^{-1} \begin{pmatrix} A \\ \propto I \end{pmatrix}^T \begin{pmatrix} b \\ 0 \end{pmatrix} \right)$$

$$X_\propto = (A^TA + \propto^2 I)^{-1}A^Tb.$$

2.1.3 Iterative Regularization

Iterative regularization methods are efficient regularization tools for image restoration problems. Certain iterative methods, e.g., steepest descent, conjugate gradients, and Richardson Lucy (EM), have regularizing effects with the regularization parameter equal to the number of iterations. These are useful in applications, like 3D imaging, with many unknowns. An example is Landweber iteration, a variant of steepest descent.

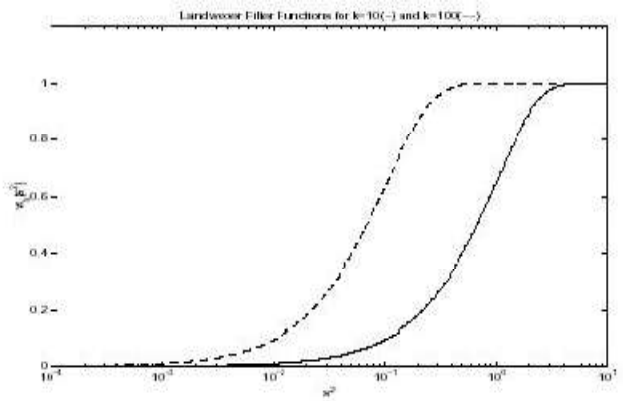

Image deblurring is one of the most classic linear inverse problems. Blurring in images can arise from many sources, such as limitations of the optical system, camera and object motion, astigmatism, and environmental effects. The blurring process of an image can be formulated as a Fredholm integral equation of the first kind which has the

following classic form: $\int_{R_2} K(s-s',t-t')f(s',t')ds'dt' = g(s,t), (s,t) \in R^2$ (2.1.5)
where f and g are the original image and the blurred image respectively. K is a given point spread function (PSF). PSF describes the blurring and the resulting image of the point source.
The degree of blurring of the point object is a measure for the quality of an imaging system. More information about different PSFs can be found in [17]. Equation (2.1.5) can be discretized to form a linear system

$$Ax_{true} = b_{true}, \quad A \in R^{m^2 \times m^2}$$
$$x, b \in R^{m^2} \quad (2.1.6)$$

where the matrix A is ill-conditioned since it has many singular values close to zero [17]. For simplicity, we assume that A in this paper is nonsingular. However, the right-hand side b is not available in many practical applications of image restoration because of the contamination of noise, so the linear system (2.1.6) can be reformulated as
$Ax = b, and\ b = b_{true} + e, e \in R^{m^2}$ (2.1.7)
where e represents the noise.

Our goal is to obtain a good approximation of the original image x_{true} by solving the system (2.1.7) instead of the system (2.1.6) since b_{true} is not known. However, the solution of (2.1.7) is not a good approximation of the solution of (2.1.6) because of ill conditioned A. In fact, there is a quite remarkable disparity among the corresponding solutions of (2.1.7) and (2.1.6) even if the norm of e is small.

However, the previously mentioned techniques (SVD and Tikhonov regularization) are usually computationally expensive for large-scale problems like image deblurring. Iterative methods, especially Krylov subspace iterative methods, are used to solve these problems due to their inexpensive computational cost and easy implementation (see [29] for more details about Krylov subspace methods).

When we apply a Krylov subspace method to solve (2.1.7), the semi convergence property is observed. Iterative methods can produce a sequence of iterates $\{x_1, x_2, \ldots, x_k, \ldots\}$ that initially tends to get closer to the exact solution but diverges again from the exact solution in later stages because the influence of the noise starts to dominate the solution. In this situation, the iteration number k can be considered as a regularization parameter. We also have different methods for choosing an effective k.

Two robust methods appropriate in different situations are: the discrepancy principle (DP) and the discrete L-curve criterion. DP is the most famous and frequently used method. It has distinct advantages and disadvantages: it is simple and easy to be utilized, but it needs to know the norm of the noise in advance. The discrete L-curve criterion is used to choose the regularization parameter by using the curve of $(\log \|r_k\|, \log \|x_k\|)$. For more information about methods for choosing the regularization parameter, we suggest [16, Chapter 5] and references therein.

The conjugate gradient (CG) The conjugate gradient (CG) method is a classical Krylov subspace iterative method for solving the linear systems with symmetric positive definite (SPD) matrix, and its regularizing effects are well known (see [27] and [18, Chapter 6] for more details).When A is not SPD, the method can be applied to the normal equations $A^TAx = A^Tb$.. CGLS is the most stable way to implementation of CG algorithm for the normal equations ([18, Chapter 6], [16, Chapter 6]). Other Krylov subspace methods also have been used to solve image restoration problems.

2.2 Local regularization methods:

Inverse problems based on first-kind Volterra integral equations appear naturally in the study of many applications, from geophysical problems to the inverse heat conduction problem. The ill-posedness of such problems means that a regularization technique is required, but classical regularization schemes like Tikhonov regularization destroy the causal nature of the underlying Volterra problem and, in general, can produce over smoothed results.

For these kinds of problems there is a class of local regularization methods in which the original (unstable) problem is approximated by a parameterized family of well-posed, second–kind Volterra equations. Being Volterra, these approximating second–kind equations retain the causality of the original problem and allow for quick sequential solution techniques. In addition, the regularizing method we develop is based on the use of a regularization parameter which is a function (rather than a single constant), allowing for more or less smoothing at localized points in the domain
We consider the following scalar Volterra first-kind integral problem. Given a suitable function f(·) defined on [0, 1], find $\bar{u}(\cdot)$ satisfying, for a.e. $t \in [0,1]$,.
$Au(t) = f(t)$ (2.2.1) Where A is the bounded linear operator on $L^2(0,1)$ given by
$Au(t) = \int_0^t K(t,s)u(s)ds, a.e. t \in [0,1]$

Ill-posedness of Problem

In the typical case that the range of A is not closed, it is well-known that problem (2.2.1) is ill-posed, lacking continuous dependence on data $f \in L^2(0,1)$. Thus, when using measured or numerically-approximated data, one must resort to a regularization method to ensure stability.

3. Why classical methods can't be used for solving these equations?

There are many classical regularization techniques are available for solving ill posed problems but these methods are less than optimal for Volterra problems of the form (2.2.1). For example, Tikhonov regularization replaces the original "causal" problem with a full-domain one. By the causal nature of the original problem we mean that problem (2.2.1) has the property that, for any t∈(0, 1], the solution u on the interval [0, t] is determined only from values of f on that same interval; for this reason, sequential solution techniques are optimal for causal problems. In contrast, to determine a solution via Tikhonov regularization one must use data values from the interval [t, 1] (i.e., future data

values), thus destroying the causal nature of the original problem and leading to non-sequential solution techniques.

Another difficulty arising in classical regularization techniques involves the use of a single regularization parameter when a priori information indicates that a solution is rough in some .areas of the domain and smooth in others. In recent years a number of approaches have been developed to handle this difficulty, among them the technique of bounded variation regularization [1, 3, 10, 11, 12, 26], as well as the method of "regularization for curve representations" [2]. Although effective, these approaches do not preserve the causal nature of the original Volterra problem and, in addition, can require a reformulation of the linear problem (or linear least-squares problem) into either a non- differentiable or non- quadratic optimization problem. In [14], a unified approach to regularization with non differentiable functional (including functional of bounded variation type) is considered, with theoretical treatment based on the concept of distributional approximation. The approach in [14] may be adapted so that a localized type of regularization is possible, however the application of this approach to Volterra equations has evidently not been studied.

4. What is local regularization technique?

Local regularization techniques form a different class of methods which have been the focus of study in recent years. These methods retain both the linear and causal structure of the original Volterra problem, allowing for solution via fast sequential methods, and rely on differentiable optimization techniques for solution. And, because regularization occurs in local regions only, sharp/fine structures of true solutions can often be recovered. The development in [19, 20, 21, 23, 24] of such methods grew out of a desire to construct a theoretical framework for understanding a popular numerical method developed by J. V. Beck in the 1960's for the IHCP. In this sequential method, Beck held solutions rigid for a short time into the future (forming a locally regularized "prediction"), and then truncated the prediction in order to improve accuracy ("correction") before moving to the next step in the sequence. Generalizations of Beck's ideas also retain this "predictor-corrector" characteristic when discretized. In general, the methods are easy to implement numerically and provide fast results in almost real-time. (We note that mollifier methods for regularization can also be considered local regularization methods; however such methods do not easily apply to general equations of the form (1). See [22] and the references therein for a more complete discussion.

References

[1] A.Chambolle and P.-L. Lions, Image recovery via total variation minimization and related problems, Numer. Math., 72 (1997), pp. 167–188.
[2] D. C. Dobson and C. R. Vogel, Convergence of an iterative method for total variation denoising, SIAM J. Numerical Analysis, 34 (1997), pp. 1779-1971.
[3] [4] D. Calvetti, B. Lewis, and L. Reichel, "On the regularizing properties of the GMRES method," *Numerische Mathematik*, vol. 91, no. 4, pp. 605–625, 2002.
[4] D. Calvetti, B. Lewis, and L. Reichel, "Krylov subspace iterative methods for nonsymmetric discrete ill-posed problems in image restoration," in *Advanced Signal Processing: Algorithms, Architectures and Implementations XI*, vol. 4474 of *Proceedings of SPIE*, pp. 224–233, The International Society for Optical Engineering, Bellingham,Wash, USA, August 2001.
[5] H, K. Pikkarainen, Regularization of Ill-posed Problems: What and Why?, Johann Radon Institute for Computational and Applied Mathematics Austrian Academy of Sciences, Helsinki, June 14, 2010
[6] H. W. Engl, M. Hanke,A. Neubauer: "Regularization of inverse problems", Kluwer Academic Publishers Group, Dordrecht, 1996.
[7] H. W. Engl, H Gfrerer, A posteriori parameter choice for general regularization methods for solving linear ill-posed problems, Appl. Numer. Math. 4 (1988) no. 5,395–417.
[8] J. Kaipio and E. Somersalo, Statistical and Computational Inverse Problems, Springer, ISBN: 978-0-387-22073-4, XVI, 339p., 102 illus., Hardcover, 2005.
[9] K. Ito and K. Kunisch, An active set strategy based on the augmented Lagrangian formulation for image restoration, R.A.I.R.O. Math. Mod. and Numer. Anal., to appear.
[10] K. Ito and K. Kunisch, Augmented Lagrangian formulation of nonsmooth convex optimization in Hilbert spaces, in Control of Partial Differential Equations and Applications, Proceedings of the IFIP TC7/WG- 7.2 International Conference, Laredo, Spain, 1994, Lect. Notes Pur Appl. Math. 174, E. Casas, ed., Dekker, New York (1996), pp. 107–117.
[11] L. I. Rudin, S. Osher and E. Fatemi, Nonlinear total variation based noise
[12] removal algorithms, Physica D, 60 (1992), pp. 259–268.
[13] M. M. Lavrent0ev, M. M. Savel0ev, Linear operators and ill-posed problems, Consultants Bureau, New York, 1995.
[14] M. Z. Nashed and O. Scherzer, Least squares and bounded variation regularization with nondifferentiable functionals, Numer. Funct. Anal. and Optimz., 19 (1998), pp. 873–901.
[15] P. Brianzi, P. Favati, O.Menchi, and F. Romani, "A framework for studying the regularizing properties of Krylov subspace methods," *Inverse Problems*, vol. 22, no. 3, pp. 1007–1021, 2006.
[16] P. C. Hansen, *Discrete Inverse Problems: Insight and Algorithms* ,Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 2010.
[17] P. C.Hansen, J. G. Nagy, and D. P. O'Leary, *Deblurring Images:Matries, Spectra and Filtering*, Society for Industrial andAppliedMathematics, Philadelphia, Pa, USA, 2006.
[18] P. C. Hansen, *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion*, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 1998.

[19] P. K. Lamm, Approximation of ill-posed Volterra problems via predictor–corrector regularization methods, SIAM J. Appl. Math., 56 (1996), pp. 524–541.
[20] P. K. Lamm, Future-sequential regularization methods for ill-posed Volterra equations: applications to the inverse heat conduction problem, J. Math. Anal. Appl., 195 (1995), pp. 469–494.
[21] P. K. Lamm, Regularized inversion of finitely smoothing Volterra operators: predictor–corrector regularization methods, Inverse Problems, 13 (1997), pp. 375–402.
[22] P. K. Lamm, A survey of regularization methods for first-kind Volterra equations, to appear.
[23] P. K. Lamm, On the local regularization of inverse problems of Volterra type, in Proc. 1995 ASME Design Engineering Technical Conferences – Parameter Identification Symposium, Boston, MA, 1995.
[24] P. K. Lamm, P. K and L. Eld´en, Numerical solution of first-kind Volterra equations by sequential Tikhonov regularization, SIAM J. Numer. Anal., 34 (1997), pp. 1432–1450.
[25] Q. Huang, "Some Topics Concerning the Singular Value Decomposition and Generalized Singular Value Decomposition", A Dissertation Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy, ARIZONA STATE UNIVERSITY, August, 2012.
[26] R. Acar and C. Vogel, Analysis of bounded variation penalty methods for ill-posed problems, Inverse Problems, 10 (1994), pp. 1217–1229.
[27] R. Plato, "Optimal algorithms for linear ill-posed problems yield regularization methods," *Numerical Functional Analysis and Optimization*, vol. 11, no. 1-2, pp. 111–118, 1990.
[28] W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*. Kluwer Academic Publishers, Netherlands, 2000.
[29] Y. Saad, *Iterative Methods for Sparse Linear Systems*, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 2nd edition, 2003.

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