

Common Fixed Point Theorems for Contractive Type Mappings in Complex Valued Metric Spaces

T. Senthil Kumar¹, R. Jahir Hussain²

¹P.G. Department of Mathematics, Arignar Anna Government Arts College, Musiri, 621211, TamilNadu, India

²P.G. and Research Department of Mathematics, Jamal Mohamed College (Autonomous), Tiruchirappalli-620 020, Tamil Nadu, India

Abstract: In this paper, we extend and improve the condition of contraction of results of Azam et al. for two single-valued mappings on a closed ball in complex valued metric spaces.

Keywords: Complex valued metric space, closed ball, common fixed point

1. Introduction

Azam et al. [1] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. In this paper we extend and improve the condition of contraction from the whole space to closed ball and establish the common fixed point theorems.

2. Preliminaries

Let \mathbb{C} the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. We define a partial order \leq on \mathbb{C} as follows: $z_1 \leq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$

that is $z_1 \leq z_2$ if one of the following holds

C1: $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

C2: $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

C3: $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$

C4: $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$

In particular, we will write $z_1 \approx z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 < z_2$ if only (C4) is satisfied.

Definition 2: Let X be a non empty set. A mapping $d: X \times X \rightarrow \mathbb{C}$ is called a complex valued matrix on X if the following conditions are satisfied:

(CM1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(CM3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric space.

Definition 3: Let (X, d) be a complex valued metric space.

a) A point $x \in X$ is called interior point of set $A \subseteq X$ whenever there exist $0 < r \in \mathbb{C}$ such that $B(x, r) = \{y \in X \mid d(x, y) < r\} \subseteq A$. Where $B(x, r)$ is an open Ball.

Then $\bar{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}$ is a closed ball.

b) A point $x \in X$ is called a limit of A whenever for every $0 < r \in \mathbb{C}$,

We have $B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$.

c) A subset $A \subseteq X$ is called open whenever each element A is an interior point of A .

d) A sub set $B \subseteq X$ is called closed whenever each limit point of B belongs to B .

e) A sub-basis for a Hausdorff topology τ on X is a family $\mathcal{F} = \{B(x, r) \mid x \in X \text{ and } 0 < r\}$.

Definition 4: Let (x, d) be a complex valued metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

i. If for every $c \in \mathbb{C}$, with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) < c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$, we denote this by $\lim_{n \rightarrow \infty} x_n = x$ (or) $\{x_n\} \rightarrow x$ as $n \rightarrow \infty$.

ii. If for every $c \in \mathbb{C}$, with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) < c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

iii. If for every Cauchy sequence in X is convergent, then (x, d) is said to be a complete complex valued metric space.

Lemma 5: [2] Let (x, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$, as $n \rightarrow \infty$.

Lemma 6: [2] Let (x, d) be a complex valued metric space and, let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if

$|d(x_n, x_{n+m})| \rightarrow 0$, as $n \rightarrow \infty$, where $m \in \mathbb{N}$.

Definition 7: Two families of self-mappings $\{T_i\}_{i=1}^m$ and $\{S_j\}_{j=1}^n$ are said to be pairwise commuting if:

1. $T_i T_j = T_j T_i, i, j \in \{1, 2, \dots, m\}$

2. $S_i S_j = S_j S_i, i, j \in \{1, 2, \dots, n\}$.

3. $T_i S_j = S_j T_i, i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$.

Definition 8: (i) A point $x \in X$ is said to be a fixed point of T if $Tx = x$.

(ii) A point $x \in X$ is said to be a common fixed point of T and S if $Tx = Sx = x$.

Remark 9: We obtain the following statements hold.

(i) If $z_1 \leq z_2$ and $z_2 \leq z_3$ then $z_1 \leq z_3$.

(ii) If $z \in \mathbb{C}$, $a, b \in \mathbb{R}$, and $a \leq b$, then $az \leq bz$.

(iii) If $0 \leq z_1 \leq z_2$, then $|z_1| \leq |z_2|$.

3. Main Results

In this section, we will prove some common fixed-point theorems for the generalized contractive mappings on a closed ball in complex valued metric spaces.

Theorem 10: If S and T are self-mapping defined on a complex valued metric space (X, d) satisfying the following condition

$$d(Sx, Ty) \leq Ad(x, y) + \frac{Bd(x, Sx)d(y, Ty)}{1+d(x, y)} + \frac{Cd(y, Sx)d(x, Ty)}{1+d(x, y)} + \frac{Dd(x, Sx)d(x, Ty)}{1+d(x, y)} + \frac{Ed(y, Sx)d(y, Ty)}{1+d(x, y)} + \frac{Fd(y, Sy)d(x, Tx)}{1+d(x, y)} + \frac{Gd(y, Sy)d(y, Tx)}{1+d(x, y)} \dots \dots \dots (1)$$

For all $x, y \in \overline{B(x, r)}$, Where A, B, C, D, E, F are nonnegative with $A+B+C+2D+2E+2F+2G < 1$.

$$|d(x_0, Sx_0)| \leq (1 - \lambda)|r| \dots \dots \dots (2)$$

Where $\delta = \max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\}$

Then there exists a unique point $u \in \overline{B(x, r)}$ such that $u = Su = Tu$

Proof:

Let x_0 be an arbitrary in X and define $x_{2k+1} = Sx_{2k}$ and $x_{2k+2} = Tx_{2k+1}$, for all $k \geq 0$.

We will prove that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$, by the mathematical induction.

Using inequality (2) and the fact that

$$\delta = \max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\} < 1, \text{ we have}$$

$$|d(x_0, Sx_0)| \leq |r|.$$

It implies that $x_1 \in \overline{B(x, r)}$, Let $x_2, x_3, \dots \dots x_j \in \overline{B(x, r)}$ for some $j \in N$.

If $j = 2k + 1$, where $k = 0, 1, 2, \dots \dots \frac{j-1}{2}$ or $j = 2k + 2$ where

$k = 0, 1, 2, \dots \dots \frac{j-2}{2}$, we obtain by using inequality (1)

$$d(x_{2k+1}, x_{2k+2}) \leq d(Sx_{2k}, Tx_{2k+1}) \leq Ad(x_{2k}, x_{2k+1}) + \frac{Bd(x_{2k}, Sx_{2k})d(x_{2k+1}, Tx_{2k+1})}{1+d(x_{2k}, x_{2k+1})} + \frac{Cd(x_{2k+1}, Sx_{2k})d(x_{2k}, Tx_{2k+1})}{1+d(x_{2k}, x_{2k+1})} + \frac{Dd(x_{2k}, Sx_{2k})d(x_{2k}, Tx_{2k+1})}{1+d(x_{2k}, x_{2k+1})} + \frac{Ed(x_{2k+1}, Sx_{2k})d(x_{2k+1}, Tx_{2k+1})}{1+d(x_{2k}, x_{2k+1})} + \frac{Fd(x_{2k+1}, Sx_{2k+1})d(x_{2k}, Tx_{2k})}{1+d(x_{2k}, x_{2k+1})} + \frac{Gd(x_{2k+1}, x_{2k+2})d(x_{2k+1}, x_{2k+1})}{1+d(x_{2k}, x_{2k+1})} \leq Ad(x_{2k}, x_{2k+1}) + \frac{Bd(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1+d(x_{2k}, x_{2k+1})} + \frac{Cd(x_{2k+1}, x_{2k+1})d(x_{2k}, x_{2k+1})}{1+d(x_{2k}, x_{2k+1})} + \frac{Dd(x_{2k}, x_{2k+1})d(x_{2k}, x_{2k+1})}{1+d(x_{2k}, x_{2k+1})} + \frac{Ed(x_{2k+1}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1+d(x_{2k}, x_{2k+1})}$$

$$+ \frac{Fd(x_{2k+1}, x_{2k+2})d(x_{2k}, x_{2k+1})}{1+d(x_{2k}, x_{2k+1})} + \frac{Gd(x_{2k+1}, x_{2k+2})d(x_{2k+1}, x_{2k+1})}{1+d(x_{2k}, x_{2k+1})} |d(x_{2k+1}, x_{2k+2})| \leq A|d(x_{2k}, x_{2k+1})| + \frac{B|d(x_{2k}, x_{2k+1})||d(x_{2k+1}, x_{2k+2})|}{|1+d(x_{2k}, x_{2k+1})|} + \frac{D|d(x_{2k}, x_{2k+1})||d(x_{2k}, x_{2k+1})|}{|1+d(x_{2k}, x_{2k+1})|} + \frac{F|d(x_{2k+1}, x_{2k+2})||d(x_{2k}, x_{2k+1})|}{|1+d(x_{2k}, x_{2k+1})|} = A|d(x_{2k}, x_{2k+1})| + B|d(x_{2k+1}, x_{2k+2})| \left[\frac{|d(x_{2k}, x_{2k+1})|}{|1+d(x_{2k}, x_{2k+1})|} \right] + D|d(x_{2k}, x_{2k+1})| \left[\frac{|d(x_{2k}, x_{2k+1})|}{|1+d(x_{2k}, x_{2k+1})|} \right] + F|d(x_{2k+1}, x_{2k+2})| \left[\frac{|d(x_{2k}, x_{2k+1})|}{|1+d(x_{2k}, x_{2k+1})|} \right] |d(x_{2k+1}, x_{2k+2})| \leq A|d(x_{2k}, x_{2k+1})| + B|d(x_{2k+1}, x_{2k+2})| + D|d(x_{2k}, x_{2k+1})| + D|d(x_{2k+1}, x_{2k+2})| + F|d(x_{2k+1}, x_{2k+2})|$$

$$|d(x_{2k+1}, x_{2k+2})[1 - (B + D + F)]| \leq [A + D]|d(x_{2k}, x_{2k+1})|$$

It follows that

$$|d(x_{2k+1}, x_{2k+2})| \leq \frac{[A+D]}{[1-(B+D+F)]} |d(x_{2k}, x_{2k+1})| \dots \dots \dots (3)$$

Similarly, we get

$$|d(x_{2k+2}, x_{2k+3})| \leq \frac{(A+E+G)}{(1-(B+E+F+G))} |d(x_{2k+2}, x_{2k+1})| \dots \dots (4)$$

$$\text{Putting } \delta = \max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\}$$

We obtain that $|d(x_j, x_{j+1})| \leq \delta^j |d(x_0, x_1)|$ for all $j \in N \dots \dots \dots (5)$

Now $|d(x_0, x_{j+1})| \leq |d(x_0, x_1)| + \dots +$

$$|d(x_j, x_{j+1})| \leq |d(x_0, x_1)| + \dots +$$

$$\delta^j |d(x_0, x_1)| \leq |d(x_0, x_1)| [1 + \dots + \delta^{j-1} +$$

$$\delta^j] \leq |d(x_0, x_1)| \frac{(1-\delta^{j+1})}{(1-\delta)}$$

$$\leq (1 - \delta)|r| \frac{(1-\delta^{j+1})}{(1-\delta)} \leq |r|$$

$$\leq (1 - \delta)|r| \frac{(1-\delta^{j+1})}{(1-\delta)} \leq |r|$$

$$\leq (1 - \delta)|r| \frac{(1-\delta^{j+1})}{(1-\delta)} \leq |r|$$

gives $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$ and

$$|d(x_n, x_m)| \leq \delta^n |d(x_0, x_1)|$$

for all $n \in N$. Without loss of generality, we take $m > n$, then

$$|d(x_n, x_m)| \leq |d(x_n, x_{n+1})| \leq |d(x_{n+1}, x_{n+2})| \dots \leq |d(x_{m-1}, x_m)|$$

$$\leq (\delta^n + \delta^{n+1} + \delta^{n+2} + \dots + \delta^{m-1}) |d(x_0, x_1)|$$

$$\leq \left(\frac{\delta^n}{1-\delta}\right) |d(x_0, x_1)|$$

$$|d(x_n, x_m)| \leq \left(\frac{\delta^n}{1-\delta}\right) |d(x_0, x_1)| \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

This implies that the sequence $\{x_n\}$ is a Cauchy in $\overline{B(x_0, r)}$.

Therefore there exists a point $z \in \overline{B(x_0, r)}$ with $\lim_{n \rightarrow \infty} x_n = z$.

Next we will show that $Sz = z$. By the notion complete complex valued metric d , we have $d(z, Sz) \leq$

$$d(z, x_{2k+2}) + d(x_{2k+2}, Sz)$$

$$\begin{aligned}
 &= d(z, x_{2k+2}) + d(Sz, Tx_{2k+1}) \\
 &\leq d(z, x_{2k+2}) + Ad(z, x_{2k+1}) \\
 &+ \frac{Bd(z, Sz)d(x_{2k+1}, Tx_{2k+1})}{1+d(z, x_{2k+1})} + \frac{Cd(x_{2k+1}, Sz)d(z, Tx_{2k+1})}{1+d(z, x_{2k+1})} \\
 &+ \frac{Dd(z, Sz)d(z, Tx_{2k+1})}{1+d(z, x_{2k+1})} + \frac{Ed(x_{2k+1}, Sz)d(x_{2k+1}, Tx_{2k+1})}{1+d(z, x_{2k+1})} \\
 &+ \frac{Fd(x_{2k+1}, Sx_{2k+1})d(z, Tz)}{1+d(z, x_{2k+1})} + \frac{Gd(x_{2k+1}, Sx_{2k+1})d(x_{2k+1}, Tz)}{1+d(z, x_{2k+1})}
 \end{aligned}$$

Taking $k \rightarrow \infty$, we have $|d(z, Sz)| = 0$, it is obtained that $d(z, Sz) = 0$.

Thus $Sz = z$.

It follows that similarly $Tz = z$.

Therefore, z is common fixed point of S and T .

Finally, to prove the uniqueness of common fixed point.

Let $z^* \in \overline{B(x_0, r)}$ be another common fixed point of S and T such that $Sz^* = Tz^* = z^*$.

Consider

$$\begin{aligned}
 d(z, z^*) &= d(Sz, Tz^*) \leq Ad(z, z^*) + \frac{Bd(z, Sz)d(z^*, Tz^*)}{1+d(z, z^*)} + \\
 &\frac{Cd(z^*, Sz)d(z, Tz^*)}{1+d(z, z^*)} + \frac{Dd(z, Sz)d(z, Tz^*)}{1+d(z, z^*)} + \\
 &\frac{Ed(z^*, Sz)d(z^*, Tz^*)}{1+d(z, z^*)} + \frac{Fd(z^*, Sz^*)d(z, Tz)}{1+d(z, z^*)} + \\
 &\frac{Gd(z^*, Sz^*)d(z^*, Tz)}{1+d(z, z^*)}
 \end{aligned}$$

$$\begin{aligned}
 \text{So that } d(z, z^*) &\leq A|d(z, z^*)| + \frac{C|d(z^*, Sz)||d(z, Tz^*)|}{|1+d(z, z^*)|} \\
 |d(z, z^*)| &= A|d(z, z^*)| + C|d(z^*, Sz)| \frac{|d(z, z^*)|}{|1+d(z, z^*)|},
 \end{aligned}$$

Since $|1 + d(z, z^*)| > |d(z, z^*)|$,

Therefore $|d(z, z^*)| < A|d(z, z^*)| + C|d(z^*, z)| = (A + C)|d(z, z^*)|$.

This is contraction to $A + C < 1$.

Hence, $z = z^*$.

There z is a unique common fixed point of S and T .

Corollary 11: If T is a self-mapping defined on a complete complex-valued metric space (X, d) satisfying the condition

$$\begin{aligned}
 d(Tx, Ty) &\leq Ad(x, y) + \frac{Bd(x, Tx)d(y, Ty)}{1+d(x, y)} + \frac{Cd(y, Tx)d(x, Ty)}{1+d(x, y)} + \\
 &\frac{Dd(x, Tx)d(x, Ty)}{1+d(x, y)} + \frac{Ed(y, Tx)d(y, Ty)}{1+d(x, y)} + \frac{Fd(y, Ty)d(x, Tx)}{1+d(x, y)} + \\
 &\frac{Gd(y, Ty)d(y, Tx)}{1+d(x, y)}
 \end{aligned}$$

for all $x, y \in \overline{B(x_0, r)}$, where A, B, C, D, E, F are nonnegative with $A+B+C+2D+2E+2F+2G < 1$.

$|d(x_0, Sx_0)| \leq (1 - \lambda)|r|$,

where $\delta = \max\left\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{[1-(B+E+F+G)]}\right\}$ then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.

Proof: We can prove this result by applying Theorem 10 by putting $T = S$.

Corollary 12: If S and T are self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition

$$\begin{aligned}
 d(Sx, Ty) &\leq Ad(x, y) + \frac{Bd(x, Sx)d(y, Ty)}{1+d(x, y)} + \frac{Cd(y, Sx)d(x, Ty)}{1+d(x, y)} + \\
 &\frac{Dd(x, Sx)d(x, Ty)}{1+d(x, y)} + \frac{Ed(y, Sx)d(y, Ty)}{1+d(x, y)}
 \end{aligned}$$

For all $x, y \in \overline{B(x_0, r)}$,

Where A, B, C, D, E, F are nonnegative with $A + B + C + 2D + 2E + 2F + 2G < 1$.

$|d(x_0, Sx_0)| \leq (1 - \lambda)|r|$,

where

$$\delta = \max\left\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{[1-(B+E+F+G)]}\right\}$$

Then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Su = Tu$.

Proof: We can prove this result by applying Theorem 10 by putting $F = G = 0$.

Theorem 13: If $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ are two finite pairwise commuting finite families of self-mapping defined on complete complex-valued metric space (X, d) such that the mappings and S and T with $T = T_1, T_2, T_3, \dots, T_m$ and $S = S_1, S_2, S_3, \dots, S_n$ satisfy condition (1) and 2, then the component maps of the two families $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$ have unique common fixed point.

Proof: By theorem (10), one can infer that T and S have a unique common fixed point Z (ie., $Tz = Sz = z$). Now we will show that z is common fixed point of all the component maps of both families. In view of pairwise commutativity of the families $\{T_i\}_{i=1}^m$ and $\{S_i\}_{i=1}^n$, for every $1 \leq k \leq m$, we can write $T_k z = T_k S z = S T_k z$, $T_k z = T_k T z = T T_k z$. It implies that $T_k z$ for all k is also a common fixed point of T and S . By using the uniqueness of common fixed point, we have $T_k z = z$ for all k . Hence, z is a common fixed point of the family $\{T_i\}_{i=1}^m$. Similarly, we can show that z is a common fixed point of the family $\{S_i\}_{i=1}^n$. This completes the proof of the theorem.

Corollary 14: If F and G are self-mappings defined on a complex valued metric space (X, d) satisfying the condition

$$\begin{aligned}
 d(F^m x, G^n y) &\leq Ad(x, y) + \frac{Bd(x, F^m x)d(y, G^n y)}{1+d(x, y)} + \\
 &\frac{Cd(y, F^m x)d(x, G^n y)}{1+d(x, y)} + \frac{Dd(x, F^m x)d(x, G^n y)}{1+d(x, y)} + \frac{Ed(y, F^m x)d(y, G^n y)}{1+d(x, y)} + \frac{Fd(y, F^m y)d(x, G^n x)}{1+d(x, y)} + \\
 &\frac{Gd(y, F^m y)d(y, G^n x)}{1+d(x, y)}
 \end{aligned}$$

For all $x, y \in \overline{B(x_0, r)}$.

Where A, B, C, D, E, F are nonnegative with $A+B+C+2D+2E+2F+2G < 1$.

$|d(x_0, Sx_0)| \leq (1 - \lambda)|r|$,

where $\delta = \max\left\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{[1-(B+E+F+G)]}\right\}$

then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.

Proof: We can prove this result by applying Theorem 13 by setting

$$T_1 = T_2 = \dots = T_m = F \quad \text{and} \quad S_1 = S_2 = \dots = S_n = G$$

References

- [1] A. Azam, B. Fisher, and M. Khan, "common fixed point theorems in complex valued metric spaces" Numerical Functional Analysis and Optimization, vol.32, no.3, pp.243-253, 2011.
- [2] Chakkrid Klim-eam and Cholatis Suanoom, "Some Common Fixed Point Theorems for Generalized-Contractive-Type Mappings on Complex-Valued Metric Spaces,"Hindawi- Abstract and Applied Analysis, vol 2013, Article ID 604215.
- [3] Jamshaid Ahmad, Akbar Azam and Satit Saejung, "Common Fixed Point Results for contractive mappings in complex valued metric spaces, Fixed Point theory and Applicatons"2014, 2014:67
- [4] F. Rouzkard and Imdad, "Some Common Fixed Theorems in Complex Valued metric spaces, "Computers & Mathematics with Applications, vol. 64, no.6,pp. 1866-1874,2012.
- [5] W. Sintunavarat and P. Kumam, "Generalized common fixed point in complex valued metric spaces and applications". Journal of inequalities and Applications, vol. 2012, article 84, 2012.
- [6] R.K Verma and H.K Pathak, "Common fixed point theorems using property (E.A) in complex valued metric spaces," Thai Journal of Mathematics, 2012.