Common Fixed Point Theorems for Contractive Type Mappings in Complex Valued Metric Spaces

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Abstract: In this paper, we extend and improve the condition of contraction of results of Azam et al. for two single-valued mappings on a closed ball in complex valued metric spaces.

Keywords: Complex valued metric space, closed ball, common fixed point

1. Introduction

Azam et al. [1] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. In this paper we extend and improve the condition of contraction from the whole space to closed ball and establish the common fixed point theorems.

2. Preliminaries

Let $\mathbb{C}$ the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. We define a partial order $\preceq$ on $\mathbb{C}$ as follows: $z_1 \preceq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$

that is $z_1 \preceq z_2$ if one of the following holds:

C1: $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

C2: $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

C3: $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$

C4: $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$

In particular, we will write $z_1 \preceq z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Definition 2: Let $X$ be a non empty set. A mapping $d: X \times X \rightarrow \mathbb{C}$ is called a complex valued metric on $X$ if the following conditions are satisfied:

(CM1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(CM3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then $d$ is called a complex valued metric space.

Definition 3: Let $(X, d)$ be a complex valued metric space.

a) A point $x \in X$ is called interior point of set $A \subseteq X$ whenever there exist $0 < r \in \mathbb{C}$ such that $B(x, r) = \{y \in X | d(x, y) < r\} \subseteq A$. Where $B(x, r)$ is an open ball.

b) A point $x \in X$ is called a limit point of $A$ whenever for every $0 < r \in \mathbb{C}$, we have $B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$.

c) A subset $A \subseteq X$ is called open whenever each element $A$ is an interior point of $A$.

d) A sub set $B \subseteq X$ is called closed whenever each limit point of $B$ belongs to $B$.

e) A sub-basis for a Hausdorff topology $\tau$ on $X$ is a family $F = \{B(x, r) | x \in X$ and $0 < r \}$.

Definition 4: Let $(x, d)$ be a complex valued metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$.

i. If for every $c \in \mathbb{C}$, with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) < c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to $x$ and $x$ is the limit point of $\{x_n\}$, we denote this by $\lim_{n \to \infty} x_n = x$ (or) $\{x_n\} \to x$ as $n \to \infty$.

ii. If for every $c \in \mathbb{C}$, with $0 < c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) < c$, then $\{x_n\}$ is said to be Cauchy sequence.

iii. If for every Cauchy sequence in $X$ is convergent, then $(x, d)$ is said to be a complete complex valued metric space.

Lemma 5: [2] Let $(x, d)$ be a complex valued metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x$ if and only if $d(x, x_n) \to 0$, as $n \to \infty$.

Lemma 6: [2] Let $(x, d)$ be a complex valued metric space and, let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x, x_n) \to 0$, as $n \to \infty$, where $m \in \mathbb{N}$.

Definition 7: Two families of self-mappings $\{T_i\}_{i=1}^{m}$ and $\{S_j\}_{j=1}^{n}$ are said to be pairwise commuting if:

1. $T_i T_j = T_j T_i$, $i, j \in \{1, 2, ..., m\}$
2. $S_i S_j = S_j S_i$, $i, j \in \{1, 2, ..., n\}$
3. $T_i S_j = S_j T_i$, $i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}$

Definition 8: (i) A point $x \in X$ is said to be a fixed point of $T$ if $Tx = x$.

(ii) A point $x \in X$ is said to be a common fixed point of $T$ and $S$ if $Tx = Sx = x$.

Remark 9: We obtain the following statements hold.

(i) If $z_1 \preceq z_2$ and $z_2 \preceq z_3$ then $z_1 \preceq z_3$.

(ii) If $z \in \mathbb{C}$, $a$, be $\mathbb{R}$, and $a \leq b$, then $az \preceq bz$.

(iii) If $0 \preceq z_1 \preceq z_2$ then $|z_1| \preceq |z_2|$.
3. Main Results

In this section, we will prove some common fixed-point theorems for the generalized contractive mappings on a closed ball in complex valued metric spaces.

**Theorem 10**: If Sand T are self-mapping defined on a complex valued metric space (X, d) satisfying the following condition

\[
d(Sx, Ty) \leq A\|x\| + \frac{Bd(Sx, y) + Cd(Sx, y)}{1 + d(x, y)} + \frac{Ed(y, x) + Fd(y, x) + Gd(y, x)}{1 + d(x, y)} + \frac{Hd(x, y) + Id(x, y)}{1 + d(x, y)} \quad \ldots \ldots \quad (1)
\]

For all x, y ∈ B(0, r), Where A, B, C, D, E, F are nonnegative with A + B + C + D + E + F + G < 1.

\[
|d(\beta x, \gamma y)| \leq (1 - \delta)|\beta| \frac{|\alpha|}{1 - |\alpha|} \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad (2).
\]

Where δ = max{\frac{|\alpha|}{1 - |\alpha|}, \frac{(A + E + G)}{(1 - (B + E + F + G))}}

Then there exists a unique point u ∈ B(0, r) such that u = Su.

**Proof:**

Let x₀ be an arbitrary in X and define x₂k+₁ = Sx₂k and x₂k+₂ = Tx₂k₁, for all k ≥ 0.

We will prove that xₙ ∈ B(x₀, r) for all n ∈ N, by the mathematical induction.

Using inequality (2) and the fact that

\[
\delta = \max\{\frac{|\alpha|}{1 - |\alpha|}, \frac{(A + E + G)}{(1 - (B + E + F + G))}\} < 1,
\]

we have

\[
|d(xₙ, x₀)| < |r|.
\]

It implies that xₙ ∈ B(x₀, r), Let x₂, x₃, ....... x_j ∈ B(x₀, r) for some j ∈ N.

If j = 2k + 1, where k = 0, 1, 2, ....... i-1 or j = 2k + 2 where

\[
k = 0, 1, 2, \ldots \ldots i-2, \text{ we obtain by using inequality (1)}
\]

\[
d(x₂k+1, x₂k+2) = d(Sx₂k, Tx₂k₁+₁) \leq A\|x₂k\| + B\|x₂k₁, x₂k+1\| + C\|x₂k₁, x₂k+1\| + D\|x₂k₁, x₂k+1\| + E\|x₂k₁, x₂k+1\| + F\|x₂k₁, x₂k+1\| + G\|x₂k₁, x₂k+1\| + H\|x₂k₁, x₂k+1\| + I\|x₂k₁, x₂k+1\| + J\|x₂k₁, x₂k+1\| + K\|x₂k₁, x₂k+1\| + L\|x₂k₁, x₂k+1\| + M\|x₂k₁, x₂k+1\| + N\|x₂k₁, x₂k+1\| + O\|x₂k₁, x₂k+1\| + P\|x₂k₁, x₂k+1\| + Q\|x₂k₁, x₂k+1\| + R\|x₂k₁, x₂k+1\| + S\|x₂k₁, x₂k+1\| + T\|x₂k₁, x₂k+1\| + U\|x₂k₁, x₂k+1\| + V\|x₂k₁, x₂k+1\| + W\|x₂k₁, x₂k+1\| + X\|x₂k₁, x₂k+1\| + Y\|x₂k₁, x₂k+1\| + Z\|x₂k₁, x₂k+1\| + a\|x₂k₁, x₂k+1\| + b\|x₂k₁, x₂k+1\| + c\|x₂k₁, x₂k+1\| + d\|x₂k₁, x₂k+1\| + e\|x₂k₁, x₂k+1\| + f\|x₂k₁, x₂k+1\| + g\|x₂k₁, x₂k+1\| + h\|x₂k₁, x₂k+1\| + i\|x₂k₁, x₂k+1\| + j\|x₂k₁, x₂k+1\| + k\|x₂k₁, x₂k+1\| + l\|x₂k₁, x₂k+1\| + m\|x₂k₁, x₂k+1\| + n\|x₂k₁, x₂k+1\| + o\|x₂k₁, x₂k+1\| + p\|x₂k₁, x₂k+1\| + q\|x₂k₁, x₂k+1\| + r\|x₂k₁, x₂k+1\| + s\|x₂k₁, x₂k+1\| + t\|x₂k₁, x₂k+1\| + u\|x₂k₁, x₂k+1\| + v\|x₂k₁, x₂k+1\| + w\|x₂k₁, x₂k+1\| + x\|x₂k₁, x₂k+1\| + y\|x₂k₁, x₂k+1\| + z\|x₂k₁, x₂k+1\| + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ ld
Let $z \in \mathbb{B}(x_0, r)$ be another common fixed point of $S$ and $T$ such that $Sz = Tz = z$. Consider

$$d(z, z') = d(Sz, Tz') \leq A d(z, z') + \frac{Bd(z, z')d(z', z')}{{1+d(z, z')}} + \frac{Cd(z, z')d(z', z')}{{1+d(z, z')}}$$

Then there exists a unique point $u \in \mathbb{B}(x_0, r)$ such that $u = Tu$.

**Proof:** We can prove this result by applying Theorem 10 by putting $F = G = 0$.

**Theorem 13:** If $(T_i)_{i=1}^n$ and $(S_i)_{i=1}^n$ are two finite pairwise commuting finite families of self-mapping defined on complete complex-valued metric space $(X, d)$ such that the mappings and $S$ and $T$ with $T = T_1, T_2, T_3, ..., T_m$ and $S = S_1, S_2, S_3, ..., S_n$ satisfy condition (1) and (2), then the component maps of the two families $(T_i)_{i=1}^m$ and $(S_i)_{i=1}^n$ have unique common fixed point.

**Proof:** By theorem (10), one can infer that $T$ and $S$ have a unique common fixed point $z$ (i.e., $Tz = Sz = z$). Now we will show that $z$ is common fixed point of all the component maps of both families. In view of pairwise commutativity of the families $(T_i)_{i=1}^m$ and $(S_i)_{i=1}^n$, for every $1 \leq k \leq m$, we can write $T_kz = T_kSz = ST_kz$, $T_kz = T_kT_kz = TT_kz$. It implies $TT_kz$ for all $k$ is also a common fixed point of $T$ and $S$. By using the uniqueness of common fixed point, we have $T_kz = z$ for all $k$. Hence, $z$ is a common fixed point of the family $(T_i)_{i=1}^m$. Similarly, we can show that $z$ is a common fixed point of the family $(S_i)_{i=1}^n$. This completes the proof of the theorem.

**Corollary 14:** If $F$ and $G$ are self-mappings defined on a complex valued metric space $(X, d)$ satisfying the condition

$$d(F(x, y), G(x, y)) \leq A d(x, y) + \frac{Bd(x, x')d(y, y')}{{1+d(x,y)}} + \frac{Cd(x, x')d(x', y')}{{1+d(x,y)}}$$

Then there exists a unique point $u \in \mathbb{B}(x_0, r)$ such that $u = Tu$.

**Proof:** We can prove this result by applying Theorem 13 by setting $T_1 = T_2 = ... = T_m = F$ and $S_1 = S_2 = ... = S_n = G$. 


References


