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Common Fixed Point Theorems for Contractive Type Mappings in Complex Valued Metric Spaces

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Abstract: ***In this paper, we extend and improve the condition of contraction of results of Azam et al. for two single-valued mappings on a closed ball in complex valued metric spaces.***

Keywords: Complex valued metric space, closed ball, common fixed point

1. Introduction

Azam et al. [1] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. In this paper we extend and improve the condition of contraction from the whole space to closed ball and establish the common fixed point theorems.

2. Preliminaries

Let $\mathbb{C}$ the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. We define a partial order $\preccurlyeq$ on $\mathbb{C}$ as follows: $z_1 \preccurlyeq z_2$ if and only if Re $(z_1) \leq$ Re (z_2) and Im $(z_1) \leq$ Im (z_2)
that is $z_1 \preccurlyeq z_2$ if one of the following holds
C1: Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2)
C2: Re (z_1) < Re (z_2) and Im (z_1) = Im (z_2)
C3: Re (z_1) = Re (z_2) and Im (z_1) < Im (z_2)
C4: Re (z_1) < Re (z_2) and Im (z_1) < Im (z_2)
In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Definition 2: Let X be a non empty set .A mapping $d: X \times X \to C$ is called a complex valued matrix on X if the following conditions are satisfied:
(CM1) $0 \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(CM3) $d(x, y) \preccurlyeq d(x, z) + d(z, y)$, for all $x, y, z \in X$.
Then d is called a complex valued metric space.

Definition 3: Let (X, d) be a complex valued metric space.

a) A point $x \in X$ is called interior point of set $A \subseteq X$ whenever there exist $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X \mid d(x, y) \prec r\} \subseteq A$, Where B(x, r) is an open Ball.
Then $\overline{B(x, r)} = \{y \in X \mid d(x, y) \preccurlyeq r\}$ is a closed ball.
b) A point $x \in X$ is called a limit of A whenever for every $0 \prec r \in \mathbb{C}$,
We have $B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$.
c) A subset $A \subseteq X$ is called open whenever each element A is an interior point of A.
d) A sub set $B \subseteq X$ is called closed whenever each limit point of B belongs to B.
e) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x, r) \mid x \in X \text{ and } 0 \prec r\}$.

Definition 4: Let (x, d) be a complex valued metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

i. If for every $c \in C$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$, we denote this by $\lim_{n\to\infty} x_n = x$ (or) $\{x_n\} \to x$ as $n \to \infty$.
ii. If for every $c \in C$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.
iii. If for every Cauchy sequence in X is convergent, then (x, d) is said to be a complete complex valued metric space.

Lemma 5: [2] Let (x, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$, as $n \to \infty$.

Lemma 6: [2] Let (x, d) be a complex valued metric space and, let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if
$|d(x_n, x_{n+m})| \to 0$, as $n \to \infty$, where $m \in \mathbb{N}$.

Definition 7: Two families of self-mappings $\{T_i\}_{i=1}^{m}$ and $\{S_i\}_{i=1}^{n}$ are said to be pairwise commuting if:
1. $T_iT_j = T_jT_i, i, j \in \{1,2, \dots m\}$
2. $S_iS_j = S_jS_{i_j}, i, j \in \{1,2, \dots n\}$.
3. $T_iS_j = S_jT_i, i \in \{1,2, \dots m\}, j \in \{1,2, \dots n\}$.

Definition 8: (i) A point $x \in X$ is said to be a fixed point of T if $Tx = x$.
(ii) A point $x \in X$ is said to be a common fixed point of T and S if $Tx = Sx = x$.

Remark 9: We obtain the following statements hold.
(i) If $z_1 \preccurlyeq z_2$ and $z_2 \preccurlyeq z_3$ then $z_1 \preccurlyeq z_3$.
(ii) If $z \in \mathbb{C}$, a, $b \in \mathbb{R}$, and $a \leq b$, then $az \preccurlyeq bz$.
(iii) If $0 \preccurlyeq z_1 \preccurlyeq z_2$, then $|z_1| \preccurlyeq |z_2|$.

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3. Main Results

In this section, we will prove some common fixed-point theorems for the generalized contractive mappings on a closed ball in complex valued metric spaces.

Theorem 10: If Sand T are self-mapping defined on a complex valued metric space (X, d) satisfying the following condition

$$d(Sx,Ty) \preccurlyeq Ad(x,y) + \frac{Bd(x,Sx)d(y,Ty)}{1+d(x,y)} + \frac{Cd(y,Sx)d(x,Ty)}{1+d(x,y)} + \frac{Dd(x,Sx)d(x,Ty)}{1+d(x,y)} + \frac{Ed(y,Sx)d(y,Ty)}{1+d(x,y)} + \frac{Fd(y,Sy)d(x,Tx)}{1+d(x,y)} + \frac{Gd(y,Sy)d(y,Tx)}{1+d(x,y)} \dots\dots\dots (1)$$

For all $x, y \in \overline{B(x,r)}$,

Where A, B, C, D, E, F are nonnegative with A+B+C+2D+2E+2F+2G < 1.

$$|d(x_0,Sx_0)| \preccurlyeq (1-\lambda)|r| \dots\dots (2).$$

Where $\delta = \max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\}$

Then there exists a unique point $u \in \overline{B(x,r)}$ such that $u = Su = Tu$

Proof:

Let x_0 be an arbitrary in X and define $x_{2k+1} = Sx_{2k}$ and $x_{2k+2} = Tx_{2k+1}$, for all $k \geq 0$.

We will prove that $x_n \in \overline{B(x_0,r)}$ for all $n \in N$, by the mathematical induction.

Using inequality (2) and the fact that $\delta = \max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\} < 1$, we have $|d(x_0,Sx_0)| \preccurlyeq |r|$.

It implies that $x_1 \in \overline{B(x,r)}$, Let $x_2, x_3, \dots\dots x_j \in \overline{B(x,r)}$ for some $j \in N$.

If j = 2k + 1, where $k = 0,1,2,\dots\dots\frac{j-1}{2}$ or $j = 2k+2$ where $k = 0,1,2,\dots\frac{j-2}{2}$, we obtain by using inequality (1)

$$d(x_{2k+1},x_{2k+2}) = d(Sx_{2k},Tx_{2k+1}) \preccurlyeq Ad(x_{2k},x_{2k+1}) + \frac{Bd(x_{2k},Sx_{2k})d(x_{2k+1},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} + \frac{Cd(x_{2k+1},Sx_{2k})d(x_{2k},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} + \frac{Dd(x_{2k},Sx_{2k})d(x_{2k},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} + \frac{Ed(x_{2k+1},Sx_{2k})d(x_{2k+1},Tx_{2k+1})}{1+d(x_{2k},x_{2k+1})} + \frac{Fd(x_{2k+1},Sx_{2k+1})d(x_{2k},Tx_{2k})}{1+d(x_{2k},x_{2k+1})} + \frac{Gd(x_{2k+1},x_{2k+2})d(x_{2k+1},x_{2k+1})}{1+d(x_{2k},x_{2k+1})}$$

$$d(x_{2k+1},x_{2k+2}) \preccurlyeq Ad(x_{2k},x_{2k+1}) + \frac{Bd(x_{2k},x_{2k+1})d(x_{2k+1},x_{2k+2})}{1+d(x_{2k},x_{2k+1})} + \frac{Cd(x_{2k+1},x_{2k+1})d(x_{2k},x_{2k+2})}{1+d(x_{2k},x_{2k+1})} + \frac{Dd(x_{2k},x_{2k+1})d(x_{2k},x_{2k+2})}{1+d(x_{2k},x_{2k+1})} + \frac{Ed(x_{2k+1},x_{2k+1})d(x_{2k+1},x_{2k+2})}{1+d(x_{2k},x_{2k+1})} + \frac{Fd(x_{2k+1},x_{2k+2})d(x_{2k},x_{2k+1})}{1+d(x_{2k},x_{2k+1})} + \frac{Gd(x_{2k+1},x_{2k+2})d(x_{2k+1},x_{2k+1})}{1+d(x_{2k},x_{2k+1})}$$

$$|d(x_{2k+1},x_{2k+2})| \leq A|d(x_{2k},x_{2k+1})| + \frac{B|d(x_{2k},x_{2k+1})||d(x_{2k+1},x_{2k+2})|}{|1+d(x_{2k},x_{2k+1})|} + \frac{D|d(x_{2k},x_{2k+1})||d(x_{2k},x_{2k+2})|}{|1+d(x_{2k},x_{2k+1})|} + \frac{F|d(x_{2k+1},x_{2k+2})||d(x_{2k},x_{2k+1})|}{|1+d(x_{2k},x_{2k+1})|}$$

$$= A|d(x_{2k},x_{2k+1})| + B|d(x_{2k+1},x_{2k+2})|\left[\frac{|d(x_{2k},x_{2k+1})|}{|1+d(x_{2k},x_{2k+1})|}\right] + D|d(x_{2k},x_{2k+2})|\left[\frac{|d(x_{2k},x_{2k+1})|}{|1+d(x_{2k},x_{2k+1})|}\right] + F|d(x_{2k+1},x_{2k+2})|\left[\frac{|d(x_{2k},x_{2k+1})|}{|1+d(x_{2k},x_{2k+1})|}\right]$$

$$|d(x_{2k+1},x_{2k+2})| \preccurlyeq A|d(x_{2k},x_{2k+1})| + B|d(x_{2k+1},x_{2k+2})| + D|d(x_{2k},x_{2k+1})| + D|d(x_{2k+1},x_{2k+2})| + F|d(x_{2k+1},x_{2k+2})|$$

$$|d(x_{2k+1},x_{2k+2})[1-(B+D+F)] \preccurlyeq [A+D]d(x_{2k},x_{2k+1})|$$

It follows that

$$|d(x_{2k+1},x_{2k+2})| \preccurlyeq \frac{[A+D]}{[1-(B+D+F)]}|d(x_{2k},x_{2k+1})| \dots\dots\dots (3)$$

Similarly, we get

$$|d(x_{2k+2},x_{2k+3})| \preccurlyeq \frac{(A+E+G)}{(1-(B+E+F+G))}|d(x_{2k+2},x_{2k+1})| \dots\dots (4).$$

Putting $\delta = max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\}$

We obtain that $|d(x_j, x_{j+1})| \leq \delta^j|d(x_0, x_1)|$ $for\ all\ j \in \mathbb{N} \dots\dots\dots (5)$

Now

$$|d(x_0, x_{j+1})| \leq |d(x_0, x_1)| + \dots + |d(x_j,x_{j+1})|$$
$$\leq |d(x_0,x_1)| + \dots + \delta^j|d(x_0, x_1)|$$
$$\leq |d(x_0,x_1)|[1 + \dots + \delta^{j-1} + \delta^j]$$
$$\leq |d(x_0,x_1)|\frac{(1-\delta^{j+1})}{(1-\delta)}$$
$$\leq (1-\delta)|r|\frac{(1-\delta^{j+1})}{(1-\delta)} \leq |r|$$

gives $x_{j+1} \in \overline{B(x_0,r)}$. Hence $x_n \in \overline{B(x_0,r)}$ $for\ all\ n \in N$ and

$$|d(x_n,x_m)| \leq \delta^n|d(x_0, x_1)|$$

$for\ all\ n \in N$. Without loss of generality, we take $m > n$, then

$$d(x_n, x_m)| \leq |d(x_n, x_{n+1})| \leq d(x_{n+1}, x_{n+2})| \dots$$
$$\leq |d(x_{m-1}, x_m)|$$
$$\leq (\delta^n + \delta^{n+1} + \delta^{n+2} + \dots + \delta^{m-1})|d(x_0, x_1)|$$
$$\leq (\frac{\delta^n}{1-\delta})|d(x_0, x_1)|$$

$$|d(x_n, x_m)| \leq \left(\frac{\delta^n}{1-\delta}\right)|d(x_0, x_1)| \to 0\ as\ m,n \to \infty.$$

This implies that the sequence $\{x_n\}$ is a Cauchy in $\overline{B(x_0,r)}$. Therefore there exists a point $z \in \overline{B(x_0,r)}$ with $\lim_{n\to\infty} x_n = z$.

Next we will show that $Sz = z$. By the notion complete complex valued metric d, we have $d(z,Sz) \preccurlyeq d(z,x_{2k+2}) + d(x_{2k+2},Sz)$

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$$= d(z, x_{2k+2}) + d(Sz, Tx_{2k+1})$$
$$\precsim d(z, x_{2k+2}) + Ad(z, x_{2k+1})$$
$$+ \frac{Bd(z,Sz)d(x_{2k+1},Tx_{2k+1})}{1+d(z,x_{2k+1})} + \frac{Cd(x_{2k+1},Sz)d(z,Tx_{2k+1})}{1+d(z,x_{2k+1})}$$
$$+ \frac{Dd(z,Sz)d(z,Tx_{2k+1})}{1+d(z,x_{2k+1})} + \frac{Ed(x_{2k+1},Sz)d(x_{2k+1},Tx_{2k+1})}{1+d(z,x_{2k+1})}$$
$$+ \frac{Fd(x_{2k+1},Sx_{2k+1})d(z,Tz)}{1+d(z,x_{2k+1})} + \frac{Gd(x_{2k+1},Sx_{2k+1})d(x_{2k+1},Tz)}{1+d(z,x_{2k+1})}$$

Taking $k \to \infty$, we have $|d(z, Sz)| = 0$, it is obtained that $d(z, Sz) = 0$.
Thus $Sz = z$.
It follows that similarly $Tz = z$.
Therefore, z is common fixed point of S and T.
Finally, to prove the uniqueness of common fixed point.
Let $z^* \in \overline{B(x_0, r)}$ be another common fixed point of S and T such that $Sz^* = Tz^* = z^*$.
Consider

$$d(z, z^*) = d(Sz, Tz^*) \precsim Ad(z, z^*) + \frac{Bd(z,Sz)d(z^*,Tz^*)}{1+d(z,z^*)} + \frac{Cd(z^*,Sz)d(z,Tz^*)}{1+d(z,z^*)} + \frac{Dd(z,Sz)d(z,Tz^*)}{1+d(z,z^*)} + \frac{Ed(z^*,Sz)d(z^*,Tz^*)}{1+d(z,z^*)} + \frac{Fd(z^*,Sz^*)d(z,Tz)}{1+d(z,z^*)} + \frac{Gd(z^*,Sz^*)d(z^*,Tz)}{1+d(z,z^*)}$$

So that $d(z, z^*)| \precsim A|d(z, z^*)| + \frac{C|d(z^*,Sz)||d(z,Tz^*)|}{|1+d(z,z^*)|}$

$$|d(z, z^*)| = A|d(z, z^*)| + C|d(z^*, Sz)| \frac{|d(z,z^*)|}{|1+d(z,z^*)|},$$

Since $|1 + d(z, z^*)| > d(z, z^*)|$,
Therefore $|d(z, z^*)| <$ A $|d(z, z^*)| + C|d(z^*, z)| = (A + C)|d(z, z^*)|$.
This is contraction to $A + C < 1$.
Hence, $z = z^*$.

There z is a unique common fixed point of S and T.

Corollary 11: If T is a self-mapping defined on a complete complex-valued metric space (X, d) satisfying the condition

$$d(Tx, Ty) \precsim Ad(x, y) + \frac{Bd(x,Tx)d(y,Ty)}{1+d(x,y)} + \frac{Cd(y,Tx)d(x,Ty)}{1+d(x,y)} + \frac{Dd(x,Tx)d(x,Ty)}{1+d(x,y)} + \frac{Ed(y,Tx)d(y,Ty)}{1+d(x,y)} + \frac{Fd(y,Ty)d(x,Tx)}{1+d(x,y)} + \frac{Gd(y,Ty)d(y,Tx)}{1+d(x,y)}$$

for all $x, y \in \overline{B(x_0, r)}$, where A, B, C, D, E, F are nonnegative with A+B+C+2D+2E+2F+2G< 1. $|d(x_0, Sx_0)| \precsim (1 - \lambda)|r|$, where $\delta = \max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\}$ then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.

Proof: We can prove this result by applying Theorem 10 by putting T = S.

Corollary 12: If S and T are self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition

$$d(Sx, Ty) \precsim Ad(x, y) + \frac{Bd(x,Sx)d(y,Ty)}{1+d(x,y)} + \frac{Cd(y,Sx)d(x,Ty)}{1+d(x,y)} + \frac{Dd(x,Sx)d(x,Ty)}{1+d(x,y)} + \frac{Ed(y,Sx)d(y,Ty)}{1+d(x,y)}$$

For all $x, y \in \overline{B(x_0, r)}$,
Where A, B, C, D, E, F are nonnegative with A + B + C + 2D + 2E + 2F + 2G < 1.
$|d(x_0, Sx_0)| \precsim (1 - \lambda)|r|$, where $\delta = \max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\}$
Then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Su = Tu$.

Proof: We can prove this result by applying Theorem 10 by putting F = G = 0.

Theorem 13: If $\{T_i\}_{i=1}^{m}$ and $\{S_i\}_{i=1}^{n}$ are two finite pairwise commuting finite families of self-mapping defined on complete complex-valued metric space (X, d) such that the mappings and S and T with $T = T_1, T_2, T_3, \ldots T_m$ and $S = S_1, S_2, S_3, \ldots S_n$ satisfy condition (1) and 2 , then the component maps of the two families $\{T_i\}_{i=1}^{m}$ and $\{S_i\}_{i=1}^{n}$ have unique common fixed point.

Proof: By theorem (10), one can infer that T and S have a unique common fixed point Z (ie., Tz = Sz = z). Now we will show that z is common fixed point of all the component maps of both families. In view of pairwise commutativity of the families $\{T_i\}_{i=1}^{m}$ and $\{S_i\}_{i=1}^{n}$, for every $1 \le k \le m$, we can write $T_k z = T_k Sz = ST_k z$, $T_k z = T_k Tz = TT_k z$. It implies that $T_k z$ for all k is also a common fixed point of T and S. By using the uniqueness of common fixed point, we have $T_k z = z$ for all k. Hence, z is a common fixed point of the family $\{T_i\}_{i=1}^{m}$. Similarly, we can show that z is a common fixed point of the family $\{S_i\}_{i=1}^{n}$. This completes the proof of the theorem.

Corollary 14: If F and G are self-mappings defined on a complex valued metric space (X, d) satisfying the condition

$$d(F^m x, G^n y) \precsim Ad(x, y) + \frac{Bd(x,F^m x)d(y,G^n y)}{1+d(x,y)} + \frac{Cd(y,F^m x)d(x,G^n y)}{1+d(x,y)} + \frac{Dd(x,F^m x)d(x,G^n y)}{1+d(x,y)} + \frac{Ed(y,F^m x)d(y,G^n y)}{1+d(x,y)} + \frac{Fd(y,F^m y)d(x,G^n x)}{1+d(x,y)} + \frac{Gd(y,F^m y)d(y,G^n x)}{1+d(x,y)}$$

For all $x, y \in \overline{B(x_0, r)}$.
Where A, B, C, D, E, F are nonnegative with A+B+C+2D+2E+2F+2G < 1.
$|d(x_0, Sx_0)| \precsim (1 - \lambda)|r|$,
where $\delta = \max\{\frac{[A+D]}{[1-(B+D+F)]}, \frac{(A+E+G)}{(1-(B+E+F+G))}\}$
then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.

Proof: We can prove this result by applying Theorem 13 by setting

$$T_1 = T_2 = \cdots = T_m = F \text{ and } \quad S_1 = S_2 = \cdots S_n = G$$

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References

[1] A. Azam, B. Fisher, and M. Khan, “common fixed point theorems in complex valued metric spaces” Numerical Functional Analysis and Optimization, vol.32, no.3, pp.243-253, 2011.
[2] Chakkrid Klim-eam and Cholatis Suanoom, “Some Common Fixed Point Theorems for Generalized-Contractive-Type Mappings on Complex-Valued Metric Spaces,”Hindawi- Abstract and Applied Anlysis, vol 2013, Article ID 604215.
[3] Jamshaid Ahmad, Akbar Azam and Satit Saejung, “Common Fixed Point Results for contractive mappings in complex valued metric spaces, Fixed Point theory and Applicatons”2014, 2014:67
[4] F. Rouzkard and Imdad, “Some Common Fixed Theorems in Complex Valued metric spaces, “Computers & Mathematics with Applications, vol. 64, no.6,pp. 1866-1874,2012.
[5] W. Sintunavarat and P. Kumam, “Generalized common fixed point in complex valued metric spaces and applications”. Journal of inequalities and Applications, vol. 2012, article 84, 2012.
[6] R.K Verma and H.K Pathak, “Common fixed point theorems using property (E.A) in complex valued metric spaces,” Thai Journal of Mathematics, 2012.

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