$$L^b + \hat{V} = f(X^o) + \frac{\partial f(X^o)}{\partial X^a}.\hat{X} + ... \qquad (7)$$

Assuming,

$$A = \frac{\partial f(X^o)}{\partial X^a} \qquad (8)$$

and

$$L = f(X^o) - L^b \qquad (9)$$

A is the design matrix. Therefore equation 7 can be rewritten as,

$$\hat{V} = A\hat{X} + L \qquad (10)$$

Using Lagrange multiplier $\hat{K}^T$ to minimize the sum of the squares of the weighted residuals $\hat{V}^T P \hat{V}$, therefore,

$$\phi = \hat{V}^T P \hat{V} - \hat{K}^T (A\hat{X} + L - \hat{V}) \text{ is minimum} \qquad (11)$$

Differentiating equation 11 partially with respect to $\hat{V}$, $\hat{K}^T$ and $\hat{X}$,

$$\frac{\partial \phi}{\partial \hat{V}} = P\hat{V}^T + K^T = \overline{0} \qquad (12)$$

$$\frac{\partial \phi}{\partial \hat{K}^T} = -A\hat{X} - L + \hat{V} = \overline{0} \qquad (13)$$

$$\frac{\partial \phi}{\partial \hat{X}} = -K^T A = \overline{0} \qquad (14)$$

Solving equations 12-14 simultaneously yields,

$$\hat{X} = -(A^T PA)^{-1} A^T PL \qquad (15)$$

for a non-linear case and

$$\hat{X} = (A^T PA)^{-1} A^T PL^b \qquad (16)$$

for a linear case because from equation 9,

$$f(X^o) = 0 \qquad (17)$$

for the linear case, therefore

$$L^b = -L \qquad (18)$$

A detailed derivation on observation equations technique is explained in Ayeni [10].

2.3 Derivation based on condition equations

According to Ayeni [10] in the conditions equation method, the condition equations are expressed as a function of the adjusted observations,

$$f(L^a) = \overline{0} \qquad (19)$$

Recall equation 4, $L^a = L^b + \hat{V}$, therefore equation 19 becomes,

$$f(L^b + \hat{V}) = \overline{0} \qquad (20)$$

Linearizing equation 20 using Taylor's series and truncating at the first order, equation 20 becomes,

$$f(L^b) + \frac{\partial f(L^b)}{\partial L^a}\hat{V} + ... = \overline{0} \qquad (21)$$

Assuming,

$$W = f(L^b) \qquad (22)$$

and

$$B = \frac{\partial f(L^b)}{\partial L^a} \qquad (23)$$

B is the design matrix; therefore equation 21 becomes,

$$B\hat{V} + W = \overline{0} \qquad (24)$$

Using Lagrange multiplier $\hat{K}^T$ to minimise the sum of the squares of the weighted residuals $\hat{V}^T P \hat{V}$, therefore,

$$\phi = \hat{V}^T P \hat{V} - \hat{K}^T (B\hat{V} + W) \text{ is minimum} \qquad (25)$$

Differentiating equation 23 partially with respect to $\hat{V}$ and $\hat{K}^T$,

$$\frac{\partial \phi}{\partial \hat{V}} = \hat{V}^T P - K^T B = \overline{0} \qquad (26)$$

$$\frac{\partial \phi}{\partial \hat{K}^T} = -B\hat{V} - W = \overline{0} \qquad (27)$$

Solving equations 26-27 simultaneously yields

$$\hat{V} = -P^{-1} B^T (BP^{-1} B^T)^{-1} W \qquad (28)$$

Note that, the number of condition equations to be formed must equal the difference between the number of observations and the number of unknown parameters. A

detailed derivation on condition equations technique is explained in Ayeni [10].

3. Application

A short-range Electronic Distance Measurement (EDM) instrument was used to measure the distances shown in Figure 1 and Table 1 below, along a straight baseline design [1]. It was assumed that the measurements were equally weighted; therefore $P = I$.
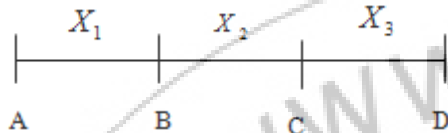

Figure 1: Baseline measurements

Table 1: Observed distances

	DISTANCES (m)
AB	11.152
BC	13.499
CD	12.052
AC	24.684
BD	25.539
AD	36.711

In order to adjust measurements AB, BC and CD, additional measurements AC, BD and AD were made. Distances AB, BC and CD were designated X_1, X_2 and X_3 respectively.

3.1 Solution using method of first principles

First $P_i V_i^2$ was stated for the six observations,

$$P_1 V_1^2 = 1(X_1^a - 11.152)^2 \tag{29}$$

$$P_2 V_2^2 = 1(X_2^a - 13.449)^2 \tag{30}$$

$$P_3 V_3^2 = 1(X_3^a - 12.052)^2 \tag{31}$$

$$P_4 V_4^2 = 1(X_1^a + X_2^a - 24.684)^2 \tag{32}$$

$$P_5 V_5^2 = 1(X_2^a + X_3^a - 25.539)^2 \tag{33}$$

$$P_6 V_6^2 = 1(X_1^a + X_2^a + X_3^a - 36.711)^2 \tag{34}$$

Recall from equation 1 that $\sum_{i=1}^{n} P_i V_i^2$ is minimum; therefore,

$$\phi = (X_1^a - 11.152)^2 + (X_2^a - 13.499)^2 + (X_3^a - 12.052)^2 +$$
$$(X_1^a + X_2^a - 24.684)^2 + (X_2^a + X_3^a - 25.539)^2 +$$
$$(X_1^a + X_2^a + X_3^a - 36.711)^2 \text{ is minimum} \tag{35}$$

$$\frac{\partial \phi}{\partial X_1^a} = 2(X_1^a - 11.152) + 2(X_1^a + X_2^a - 24.684) +$$
$$2(X_1^a + X_2^a + X_3^a - 36.711) = 0 \tag{36}$$

$$\frac{\partial \phi}{\partial X_2^a} = 2(X_2^a - 13.499) + 2(X_1^a + X_2^a - 24.684) +$$
$$2(X_2^a + X_3^a - 25.539) +$$
$$2(X_1^a + X_2^a + X_3^a - 36.711) = 0 \tag{37}$$

$$\frac{\partial \phi}{\partial X_3^a} = 2(X_3^a - 12.052) + 2(X_2^a + X_3^a - 25.539) +$$
$$2(X_1^a + X_2^a + X_3^a - 36.711) = 0 \tag{38}$$

Equations 36-38 can be simplified as,

$$6X_1^a + 4X_2^a + 2X_3^a = 145.094 \tag{39}$$

$$4X_1^a + 8X_2^a + 4X_3^a = 200.866 \tag{40}$$

$$2X_1^a + 4X_2^a + 6X_3^a = 148.604 \tag{41}$$

Solving equations 39-41 simultaneously yielded,

$$\begin{bmatrix} X_1^a \\ X_2^a \\ X_3^a \end{bmatrix} = \begin{bmatrix} 11.165m \\ 13.504m \\ 12.043m \end{bmatrix} \tag{42}$$

3.2 Solution using method of observation equations

Recall from section 2 that the number of observation equations formed must be equal to the number of field observations. Therefore six observations will be formed, since six field observations were made. The observation equations were given in equations 43-48.

$$L_1^a = X_1^a \tag{43}$$

$$L_2^a = X_2^a \tag{44}$$

$$L_3^a = X_3^a \tag{45}$$

$$L_4^a = X_1^a + X_2^a \tag{46}$$

$$L_5^a = X_2^a + X_3^a \tag{47}$$

$$L_6^a = X_1^a + X_2^a + X_3^a \tag{48}$$

Recall that $\hat{X} = (A^T P A)^{-1} A^T P L^b$

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$$\hat{X} = \begin{bmatrix} \hat{X}_1^a \\ \hat{X}_2^a \\ \hat{X}_3^a \end{bmatrix}, \tag{49}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{50}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \tag{51}$$

$$L^b = \begin{bmatrix} 11.152 \\ 13.499 \\ 12.052 \\ 24.684 \\ 25.539 \\ 36.711 \end{bmatrix} \tag{52}$$

Therefore, the solution using equation 16 yielded,

$$\begin{bmatrix} \hat{X}_1^a \\ \hat{X}_2^a \\ \hat{X}_3^a \end{bmatrix} = \begin{bmatrix} 11.165m \\ 13.504m \\ 12.043m \end{bmatrix} \tag{53}$$

3.3 Solution using method of condition equations

Recall from section 2 that the number of condition equations must equal the difference between the number of observations and the number of unknown parameters. Therefore three condition equations will be formed. The condition equations were given in equations 54-56. Equations 54-56 were formed using the information in Table 1 and Figure 1,

$$L_1^a + L_2^a - L_4^a = 0 \tag{54}$$

$$L_2^a + L_3^a - L_5^a = 0 \tag{55}$$

$$L_1^a + L_2^a + L_3^a - L_6^a = 0 \tag{56}$$

Recall from equation 28 that $\hat{V} = -P^{-1}B^T(BP^{-1}B^T)^{-1}W$. Also recall from equation 4 that $L^a = L^b + \hat{V}$, therefore,

$$B = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 \end{bmatrix} \text{ and } \tag{57}$$

$$W = \begin{bmatrix} 11.152+13.499-24.684 \\ 13.499+12.052-25.539 \\ 11.152+13.499+12.052-36.711 \end{bmatrix} = \begin{bmatrix} -0.033 \\ 0.012 \\ -0.008 \end{bmatrix} \tag{58}$$

Recall that

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore the solution of $\hat{V}$ yielded,

$$\hat{V} = \begin{bmatrix} 0.013 \\ 0.005 \\ -0.009 \\ -0.015 \\ 0.008 \\ 0.001 \end{bmatrix} \tag{59}$$

Recalling L^b from equation 52 and $\hat{V}$ from equation 59, therefore,

$$L^a = L^b + \hat{V} = \begin{bmatrix} 11.165m \\ 13.504m \\ 12.043m \\ 24.670m \\ 25.547m \\ 36.712m \end{bmatrix} \tag{60}$$

Recall from equation 53 that rows 1-3 of equation 60 are the same as X_1^a, X_2^a and X_3^a in equation 53.

3.4 Solution using other methods of adjustment

For LSC, the basic of LSC is such that,

$$Y = A\hat{X} + Z \tag{61}$$

Where $Z = R' + S'$ \tag{62}

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The assumption is that the following covariances exist, $\sum_Y$, $\sum_S$, $\sum_R$, $\sum_Z$ and also that R' and S' are random. A is the design matrix, X is the estimated parameters and Y is the observation. The solution of $\hat{X}$ is given as,

$$\hat{X} = (A^T \sum_Y^{-1} A)^{-1} A^T \sum_Y^{-1} Y \text{ [5, 10]} \tag{63}$$

For KF, KF predicts or estimates the state of a dynamic system from a series of incomplete and /or noisy measurements. Suppose we have a noisy linear system that is defined by the following equations:

$$X_k = A\hat{X}_{k-1} + w_{k-1} \tag{64}$$

$$Z_k = HX_k + v_k \tag{65}$$

Where X_k is estimated state at time k, A is the state transition matrix, X_{k-1} is estimated state for preceding time $k-1$, w is process noise at time $k-1$, Z_k is the measurement, H is the measurement design matrix and v_k is the measurement noise [3].

For TLS, it assumes that all the elements of the data are erroneous. This situation can be stated mathematically as,

$$b + \Delta b = (A + \Delta A)x \text{ , } rank(A) = m < n \tag{66}$$

where, Δb is error vector of observations and ΔA is error matrix of data matrix A. Both errors are assumed independently and identically distributed with zero mean and with same variance [9].

Table 2: Adjusted distances

	LS	LSC	KF	TLS
X_1^a	11.1652	11.1652	11.1650	11.1653
X_2^a	13.5042	13.5042	13.5042	13.5043
X_3^a	12.0428	12.0428	12.0425	12.0428

The data given in Table 1 and Figure 1 were applied to LSC, KF and TLS. From Table 2 all the results for X_1^a, X_2^a and X_3^a from the four models yielded basically the same values apart from very slight difference in the results from KF and TLS.
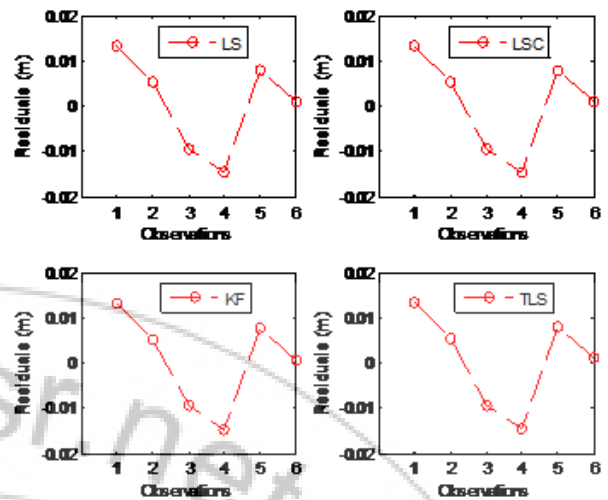

Figure 2: Residuals

The computed residuals for each observation were given in Figure 2. From Figure 2, similar residual values were yielded all the four models. When the sum of the absolute values of the residuals was computed for the four models (Figure 3), it was found that KF yield the least value, while the remaining three models yielded relatively the same values.
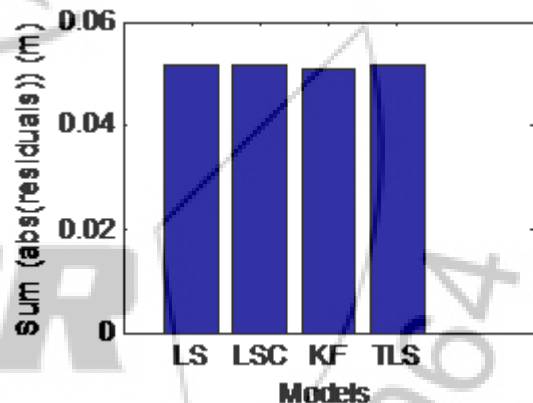

Figure 3: Sum of absolute values of the residuals

4. Conclusion

The solutions from the first principles, observation equations and condition equations for X_1^a, X_2^a and X_3^a must be the same. From Figure 3, KF yielded the least value of the sum of the absolute values of the residuals, and therefore yielded relatively the most accurate result. In summary, recalling from section 2, some of the primary conditions for LS adjustment among others are that: (i) the number of field observations must exceed the number of parameters to be determined (ii) the number of observation equations formed must be equal to the number of field observations (iii) the number of condition equations formed must equal the difference between the number of observations and the number of unknown parameters to be determined.

References

[1] O. Okwuashi, "Adjustment computation and statistical methods in surveying," A Manual, in the Department of Geoinformatics & Surveying, Faculty of Environmental Studies, University of Uyo, Nigeria, 2014.

[2] C. F. Gauss, "Theoria combinationis obsevationum erroribus minimis obnoxiae," Werke, Vol. 4. Göttingen, Germany, 1823.

[3] J. Bezrucka, "The use of a Kalman filter in geodesy and navigation," Slovak Journal of Civil Engineering, Vol. 2, pp. 8-15, 2011.

[4] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," Transactions of the ASME-Journal of Basic Engineering, (Series D), Vol. 82, pp. 35-45, 1960.

[5] H. Moritz, "Least squares collocation," Reviews of Geophysics and Space Physics, vol. 16, No. 3, pp 421-430, 1978.

[6] G. H. Golub, C. F. V. Loan, "An analysis of the total least squares problem," SIAM Journal of Numerical Analysis, Vol. 17, p. 883, 1980.

[7] S. V. Huffel, J. Vandewalle, "The Total Least Squares Problem: Computational Aspects and Analysis," 1st Ed., SIAM, Philadelphia, 1991.

[8] O. Akyilmaz, "Total least squares solution of coordinate transformation," Survey Review, Vol. 39, p. 68, 2007.

[9] S. Dogan, M. O. Altan, "Total least squares registration of images for change detection," *ISPRS Archive* Vol. XXXVIII, Part 4-8-2-W9, Core Spatial Databases – Updating, Maintenance and Services – from Theory to Practice, Haifa, Israel, 2010.

[10] O. O. Ayeni, "Statistical adjustment and analysis of data," A Manual, in the Department of Surveying & Geoinformatics, Faculty of Engineering, University of Lagos, Nigeria, 2001.