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Jackknife Variance Estimation of Uniformly Minimum Variance Unbiased Estimates and the Maximum Likelihood Estimates for the Gamma Probability Density Function

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Abstract: ***The gamma distribution is an important life time distribution. However, the drawback it faces in application is that the function useful in survival analysis and life testing is that it's survivor function and hence hazard function does not exist in closed form. In this project we investigated the performance of the MLE and UMVUE of the gamma pdf. A simulation study of100 runs was carried out in R- program by fixing the shape parameter α at 0.5and 1.0 for small (n=10), moderate (n=30) and large (n=50 and n=100) sample sizes. For each fixedα, the shape parameter β was varied at β = 0.1, 0.5, 1.5, 2.0.Jackknife variance estimates for the MLE's and UMVUE's for the gamma pdfwere obtained. Confidence intervals for the pdfwere constructed and their coverage probabilities investigated at each nominal confidence coefficient. The results indicated that the Jackknife estimates of variance of the UMVUE are fairly stable compared to their MLE counterparts. In most cases the coverage probabilities of the UMVUE are generally closer to the nominal confidence coefficient than their MLE counterparts.***

Keywords: uniform minimum variance unbiased estimates, Maximum likelihoodestimates, Jackknife variance estimation, Gamma probability density function

1. Introduction

Many researchers have considered estimation of a probability density functions. They include Tate (1959) who obtained the UMVUE for the cumulative distribution function for several probability densities using transform theorem.

Barton (1961) gave the UMVUE of the normal, Poisson and binomial distribution functions while Kolmogorov (1962) studied estimation theory in particular unbiased estimation and linear statistical models and obtained the unique minimum variance unbiased estimates for a normal distribution with unknown mean and variance using Rao-Blackwell theorem. Patil (1963) pointed out elementary methods of deriving UMVUE of probability distribution functions.

Basu (1964)) estimated the tail of gamma distribution while Chikara and Folks (1974) studied UMVUE of Inverse Gaussian distribution and it's reciprocals and proposed the UMVUE of left and right limits of a certain interval which contains an Inverse Gaussian variate with an arbitrary given probability.

Schaeffer (1976), Viertl (1996), Shao (2003), Aghili(2004) all reviewed the computation of UMVUE with particular reference to exponential families.

Samanta (1988) studied and derived a unified approach to minimum variance unbiased estimation of a probability density functions belonging to an exponential family. Charturvedi and Sanjeev (2003) derived Uniformly Minimum Variance Unbiased Estimators (UMVUEs) of the powers of the parameter involved in the probabilistic model and the probability density function (pdf) at a specified point.

Krotova and Sapozhnikov (2005) proposed a method for constructing an optimal unbiased estimator of a given function of a parameter, if it exists. Walid and Rahimov(2010) used Hudson's identity to find the UMVUEs of some parameters.

Although estimation methods such as UMVUE and MLE for different distributions has been carried out but none has been done to compare the performance of UMVUE and MLE. In this article, we investigate the performance of the UMVUE and MLE of the gamma pdf based on the variances and coverage probability using Jackknife technique. We shall apply Samanta (1988) approach by using the derived formula for UMVUE for the gamma distribution and simulation experiments to compare the performance of UMVUE and MLE.

In section 2 we have reviewed the gamma distribution, its parameters and properties. In section 3 we have described the maximum likelihood procedure which has been used in estimation of parameters of a gamma distribution. UMVUE of exponential families and derived formula for UMVUE of gamma distribution using mixture of two gamma distribution with a common unknown scale parameter has been obtained in section 4.Jackknife technique has been described in section 5.Section 6 gives the simulation results and discussed in section 7.Section 8 gives the concluding remarks.

2. The Gamma Distribution

The gamma distribution is a two-parameter family of continuous probability distributions. It has a scale parameter θ and a shape parameter k. If k is an integer then the distribution represents the sum of k independent exponentially distributed random variables, each of which has a mean of θ (which is equivalent to a rate parameter of θ^{-1}) .It is sometimes called the Erlang distribution when the shape parameter is a positive integer, which is used frequently in queuing theory applications.(see Lawless ,1971).

The probability density function of the gamma distributed random variables can be expressed in terms of the gamma function parameterized in terms of a shape parameter k and scale parameter θ. Both k and θ are positive numbers. The equation defining the probability density function of a gamma-distributed random variable $X$is

$$f(x;k,\theta) = \frac{x^{k-1} e^{-x/\theta}}{\theta^k \Gamma k} \quad \text{for } x>0 \text{ and } k,\theta>0 \qquad (2.1)$$

$$\text{where } \Gamma k = \int_0^\infty y^{k-1} e^{-y}\, dy \quad \text{is the gamma function}$$

A random variable X that is gamma-distributed with scale θ and shape k is denoted

$X \sim \Gamma(k,\theta)$ or $X \sim \text{Gamma}(k,\theta)$

Alternatively, the gamma distribution can be parameterized in terms of a shape parameter $\alpha = k$ and an inverse scale parameter $\beta = 1/\theta$, called a rate parameter:

$$g(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \text{for } x>0. \qquad (2.2)$$

If α is a positive integer, the$\Gamma(\alpha) = (\alpha - 1)!$
Both parameterizations are common because either can be more convenient depending on the situation.

3. The Maximum Likelihood Estimation

The likelihood function for $n$iid observations (x_1, ..., x_n) having a pdf in (2.1) is

$$L(k,\theta) = \prod_{i=1}^{n} f(x_i;k,\theta) = \prod_{i=1}^{n} \frac{x^{k-1} e^{-x/\theta}}{\theta^k \Gamma k} = \frac{e^{-1/\theta \Sigma x_i} \prod_{i=1}^{n} x_i^{k-1}}{\theta^{nk} (\Gamma k)^n} \qquad (3.1)$$

The log-likelihood function is then given by

$$\ell(k,\theta) = (k-1)\sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} \frac{x_i}{\theta} - nk\ln(\theta) - n\ln\Gamma(k) \quad (3.2)$$

The MLE of θ is obtained by taking partial derivative of $\ell(k,\theta)$ with respect to θ and setting it equal to zero and then solving to obtain

$$\hat{\theta} = \frac{1}{kn} \sum_{i=1}^{n} x_i \qquad (3.3)$$

The MLE of k is obtained by differentiating $\ell(k, \theta)$ partially with respect to k and setting it equal to zero, then substituting (3.3) in the equation and solving to obtain

$$\ln(k) - \psi(k) = \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \frac{1}{n}\sum_{i=1}^{n} \ln(x_i) \qquad (3.4)$$

Where

$$\psi(k) = \frac{\Gamma'(k)}{\Gamma(k)}$$

is the digamma function

There is no closed-form solution for k. The equation (3.4) can be solved for k using Newton-Raphson method. An initial value of k can be found either using the method of moments, or using the approximation.

$$\ln(k) - \psi(k) \approx \frac{1}{k}\left(\frac{1}{2} + \frac{1}{12k+2}\right)$$

4. Uniformly Minimum Variance Unbiased Estimates (UMVUE) of the Gamma Distribution

A random variable y has a probability distribution belonging to an s-parameter exponential family if its pdf has the following form

$$f(y,\theta) = \exp\left[\sum_{i=1}^{s} \eta_i(\theta) T_i(y) - B(\theta)\right] \ell(y) \quad (4.1)$$

where $\eta_i$and B are real valued functions of the parameters and T_i are real valued statistics. If $Y_1, Y_2, \ldots, Y_n$ be independent random variable each having pdf in (4.1) then the joint probability density function of $Y_1, Y_2, \ldots, Y_n$ is

$$\prod_{j=1}^{n} f_j(y;\theta) = \exp\left[\sum_{i=1}^{s} \eta_i(\theta) U_i(y_1, y_2, \ldots, y_n) - \sum_{j=1}^{n} B_j(\theta)\right] \prod_{j=1}^{n} \ell(y_j) \quad (4.2)$$

n

where $U_i = U_i(y_1, y_2, \ldots, y_n) = \Sigma T_{ij}(y_j)$ i= 1,2,…,s . It follows that $U_1, U2, \ldots, Us$
j=1
have a joint- distribution belonging to an s- parameter exponential family with a pdf

$$g_n(u_1, u_2, \ldots, u_s) = \exp\left[\sum_{i=1}^{s} \eta_i(\theta) u_i - \sum_{j=1}^{n} B_j(\theta)\right] k_n(u_1, u_2, \ldots, u_s) \quad (4.3)$$

where $U = (U_1, U_2, \ldots, Us)$ is a complete minimal sufficient statistic for $\eta_1(\theta), \eta_2(\theta), \ldots, \eta_s(\theta)$ Lehmann (1983). It is usually possible to take u to be either Lebesgue measure or counting measure. If u is a Lebesgue measure then we get from (4.3)

$$\int\int\ldots\int \exp\left[\sum^{s} \eta_i(\theta) u_i - \sum^{n} B_j(\theta)\right] k_n(u_1, u_2, \ldots, u_s) du_1, du_2, \ldots, du_s = 1 \quad (4.4)$$

where the integration extends over all values in the range of U. If u is a counting measure then the integral operators in (4.4) are replaced by summation operators. The conditional pdf of y_j given $U = u = (u_1, u_2, \ldots, u_s)$ is

$$h_j(y;u) = \frac{\ell(y)\, k_{n-1}(u_1 - T_{1j}(y), u_2 - T_{2j}(y), \ldots, u_s - T_{sj}(y))}{k_n(u_1, u_2, \ldots, u_s)} \quad (4.5)$$

Since U is a complete sufficient statistic, it follows by the Rao- Blackwell- Lehmann – Scheffe theorem that $h_j(y;u)$ is the MVUE of $fj(y;\theta)$. The evaluation of the function $k_n(u_1, u_2, \ldots, u_s)$ as defined in (4.4) is not always an easy task. There are many probability distributions belonging to an exponential family for which it is impossible to compute the function $k_n(u_1, u_2, \ldots, u_s)$.

Known results of Laplace and Mellin transform was used to evaluate $k_n(u_1, u_2, \ldots, u_s)$ as described by Erdelyi (1954) and Abramowitz and Stegun (1970).

The UMVUE of the gamma probability density function was derived for a mixture of two gamma distributions with a common unknown scale parameter (Samanta ,1988).

The pdf of the mixture of two gamma distributions having the same scale parameter is given by

$$f(y;\theta) = C(\theta)\,[a_1 y^{p_1-1} + a_2 y^{p_2-1}] \exp[-\theta y] \quad 0 < y < \infty \quad (4.6)$$

where $a_1 \geq 0$, $a_2 \geq 0$, $a_1 + a_2 = 1$ $p_1 > 0$, $p_2 > 0$ are known constants,

$0 < \theta < \infty$ is unknown and $C(\theta) = \left(\frac{a_1 \Gamma(p_1)}{\theta^{p_1}} + \frac{a_2 \Gamma(p_2)}{\theta^{p_2}}\right)^{-1}$

Defining $T(y) = y$ and $U = \sum_{i=1}^{n} T(y_i) = \sum_{i=1}^{n} y_i$

from equation (4.4)

$$\int_0^{\infty} \exp[-\theta u]\, k_n(u)\, du = \left(\frac{a_1 \Gamma(p_1)}{\theta^{p_1}} + \frac{a_2 \Gamma(p_2)}{\theta^{p_2}}\right)^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{a_1^i a_2^{n-i} [\Gamma(p_1)]^i [\Gamma(p_2)]^{n-i}}{\theta^{ip_1 + (n-i)p_2}} \quad (4.7)$$

Using tables of Laplace transforms we have

$$k_n(u) = \sum_{i=0}^{n} \binom{n}{i} \frac{a_1^i a_2^{n-i} [\Gamma(p_1)]^i [\Gamma(p_2)]^{n-i} u^{ip_1 + (n-i)p_2 - 1}}{\Gamma[ip_1 + (n-i)p_2]} \quad (4.8)$$

and the UMVUE of $f(y;\theta)$ is

$$h(y;u) = \frac{(a_1 y^{p_1-1} + a_2 y^{p_2-1})\, k_{n-1}(u-y)}{k_n(u)},\ 0 < y < u \quad (4.9)$$

Letting $a_1 = 1$ and $a_2 = 0$ in (2.9) gives the MVUE of the gamma pdf as

$$h(y;u) = \frac{y^{P-1} k_{n-1}(u-y)}{k_n(u)} \quad (4.10)$$

where

$$k_n(u) = \frac{a^n \Gamma(p)^n u^{np}}{\Gamma np}$$

But a = 1
Therefore

$$k_n(u) = \frac{\Gamma(p)^n u^{np}}{\Gamma np} \quad (4.11)$$

5. Jackknife Estimates of Variances^

Jackknife technique is an approach to estimation where $\theta_j(p)$ is a linear function of P which matches $\theta(p)$ at n points corresponding to the deletion of a single x_i from observed data set $x_1, x_2, \ldots .x_n$ as given by Efron and Tibshirani (1983) .

Hence Jackknife mean of MLE / UMVUE will be obtained as

$$\frac{\hat{\theta}_{\backslash 1} + \hat{\theta}_{\backslash 2} + \hat{\theta}_{\backslash 3} + \ldots + \hat{\theta}_{\backslash R-1} + \hat{\theta}_{\backslash R}}{R} = \hat{\theta}_{MLE/UMVUE} \quad (5.1)$$

where R is the number of runs for each sample size.

The Jackknife Variances of MLE and UMVUE are given as follows.

$$\delta^2_{J(MLE)} = \left[\frac{R-1}{R} \sum_{i=1}^{R} \left\{\hat{\theta}_{\backslash i} - \hat{\theta}_{MLE/UMVUE}\right\}^2\right] \quad (5.2)$$

Confidence intervals for the Jackknife variances were constructed and were used to obtain coverage probabilities for MLE and UMVUE of $f(x; k, \theta)$ using R program.

6. Simulation Results

Simulation experiments to compare the performance of MLE and UMVUE in terms of their Jackknife estimates of variances were carried out. For purposes of illustration a simulation study for different sample sizes and for different parameter values was performed. Sample sizes were taken as n = 10, 30, 50,100; the scale parameter was fixed at β = 0.1, 0.5, 1.0, 1.5 and 2.0 while shape parameter was fixed at α = 0.5, 1.0. All the results were based on 100 runs for each sample size and the MLE and UMVUE of the estimates of Jackknife variances were obtained and presented in Table 6.1 and the comparison of MLE and UMVUE has been given in Figures 6.1 to 6.10. The coverage probabilities are given in Table 6.2.

Table 6.1: MLE and UMVUE estimates of Jackknife variances of gamma pdf atα = 1.0 and α = 0.5 for β = 0.1, 0.5, 1.0, 1.5 and 2.0 for different sample sizes at n = 10, 30, 50 and 100.

Sample Sizes		Jacknife variances ESTIMATES AT α		Jacknife variances ESTMATES AT α	
	B	MLE	UMVUE	MLE	UMVUE
n = 10	0.1	2.200754	0.183193	2.736723	0.070502
n =30	0.1	1.452181	0.13017	2.092641	0.018173
n =50	0.1	0.649787	0.163284	1.059083	0.023029
n = 100	0.1	1.653837	0.101417	1.30003	0.009737
n = 10	0.5	2.852338	0.286356	0.75916	0.102121
n =30	0.5	0.393473	0.187964	0.068433	0.024958
n =50	0.5	1.005423	0.123434	0.146584	0.018377
n = 100	0.5	1.071594	0.125861	0.116203	0.018504
n = 10	1	2.599742	0.351251	0.595053	0.142364
n =30	1	0.786979	0.182424	0.170797	0.03321
n =50	1	0.147389	0.214914	0.092839	0.03354
n = 100	1	0.056954	0.136707	0.108504	0.012869
n = 10	1.5	2.372357	0.349805	0.142654	0.081316
n =30	1.5	0.337826	0.127208	0.086309	0.028666
n =50	1.5	0.088219	0.164591	0.073699	0.032082
n = 100	1.5	0.051351	0.169832	0.095585	0.016683
n = 10	2	1.362984	0.467653	0.09135	0.064721
n =30	2	0.123934	0.200438	0.150726	0.023202
n =50	2	0.067671	0.059083	0.051917	0.042096
n = 100	2	0.075367	0.185733	0.065103	0.029146

Table 6.2: Coverage probabilities of MLE and UMVUE variance estimates of gamma pdf

	β = 0.1		β = 0.5		β = 1.0		β = 1.5		β =2.0	
	MLE	UMVUE	MLE	UMVUE	MLE	UMVUE	MLE	UMVUE	MLE	UMVUE
α = 0.5 , n=10										
90%	86	94	94	90	95	91	96	96	97	92
95%	90	99	96	96	96	95	98	99	99	99
99%	94	100	97	100	97	100	100	100	99	99
α = 0.5 , n=30										
90%	94	93	90	91	94	95	92	93	95	93
95%	95	96	93	96	96	97	96	98	97	98
99%	95	99	96	100	98	100	98	100	99	100
α = 0.5 , n=50										
90%	92	97	95	93	93	95	96	89	90	91
95%	93	98	98	95	95	97	97	93	96	93
99%	96	99	98	98	98	99	97	99	98	97
α = 0.5 ,n=100										
90%	87	92	95	91	92	91	91	90	95	90
95%	90	96	97	95	97	93	95	95	97	97
99%	100	100	98	98	99	98	98	97	98	99
α = 1 , n=10										
90%	93	93	99	96	93	93	95	91	98	91
95%	94	96	99	98	94	94	97	94	98	94
99%	96	97	99	100	97	97	98	98	99	97
90%	91	92	92	92	95	93	99	91	96	90
95%	94	95	94	95	99	95	100	94	98	96
99%	100	98	98	99	99	97	100	98	100	98
α = 1 , n=50										
90%	91	90	97	93	98	92	92	91	95	95
95%	93	96	97	95	98	95	95	96	96	97
99%	98	98	97	97	98	97	98	98	96	98
α = 1 , n=100										
90%	97	92	98	89	89	90	80	92	90	92
95%	98	95	99	93	93	94	98	95	92	96
99%	99	98	99	98	98	97	100	98	95	98

7. MLE and UMVUE estimates of Jackknife variances of gamma pdf at:

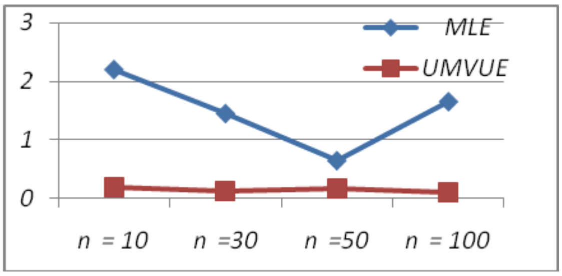

Figure 6.1: α=0.5 and β =0.1

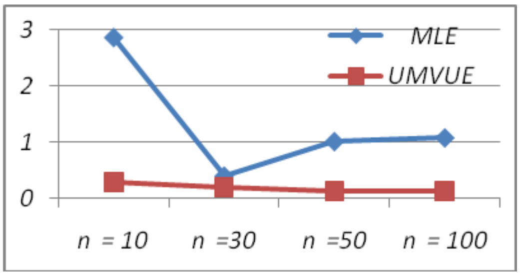

Figure 6.2: α=0.5 and β =0.5

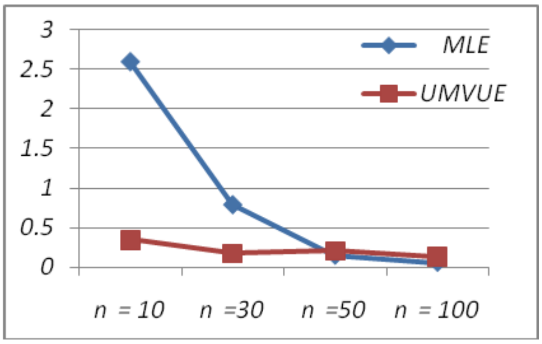

Figure 6.3: α=0.5and β =1.0

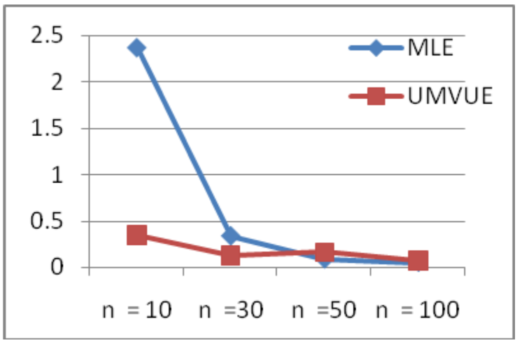

Figure 6.4:α=0.5 and β =1.5

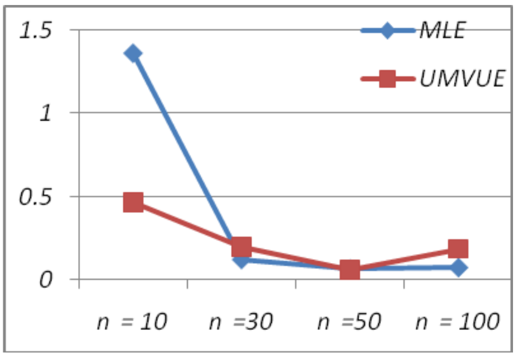

Figure 6.5: α=0.5 and β =2.0

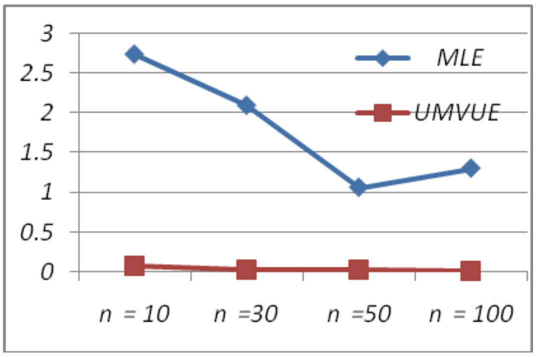

Figure 6.6: α=1.0 and β =0.1

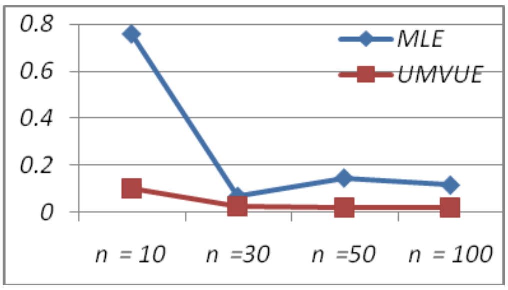

Figure 6.7: α= 1.0 and β =0.5

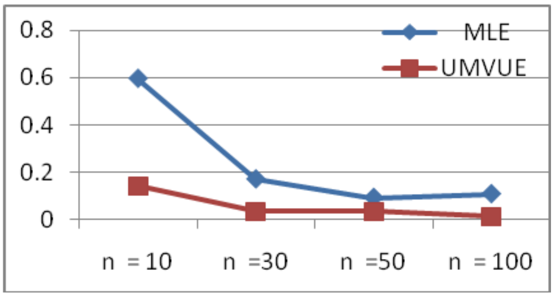

Figure 6.8: α= 1.0 and β =1.0

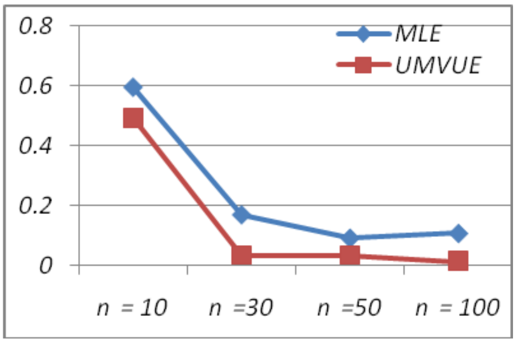

Figure 6.9: α= 1.0 and β =1.5

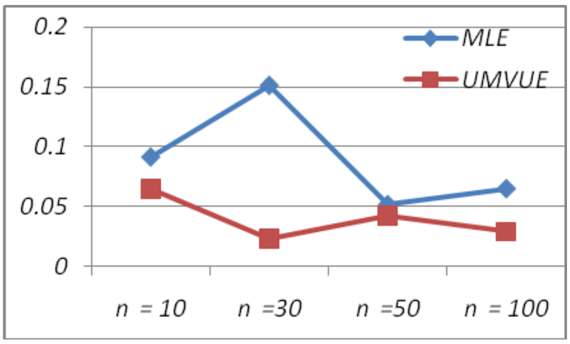

Figure 6.10: α= 1.0 and β =2.0

8. Numerical illustration from life time data

Jackknife technique on life time data to compare the performance of MLE and UMVUE, life time data fromBirnbaum & Saunders (1958) and Balakrishnan et al. (2009)were taken and analysed.

Lifetimes of aluminum specimens exposed to 31,000 psi. from Birnbaum & Saunders(1958)

70 90 96 97 99 100 103 104 104 105
107 108 108 108 109 109 112 112 113 114
114 114 116 119 120 120 120 121 121 123
124 124 124 124 124 128 128 129 129 130
130 130 131 131 131 131 131 132 132 132
133 134 134 134 134 134 136 136 137 138
138 138 139 139 141 141 142 142 142 142
142 142 144 144 145 146 148 148 149 151
151 152 155 156 157 157 157 157 158 159
162 163 163 164 166 166 168 170 174 196212

The Jackknife estimate of variance for UMVUE for the data was **0.181176**while that of MLE was **0.196551.** Another set of data used as an example was from Balakrishnan et al. (2009) that used mixture inverse Gaussian distributions to describe several datasets.

Data sets obtained from Balakrishnan et al. (2009).

22 2425(2) 272829(4) 30 31(6) 32(7)
33(3) 34(6) 35(4) 36(11) 37(5) 38(3) 39(6) 40(14) 41(12)
42(6) 43(5) 44(7) 45(10) 46(6) 47(5) 48(11) 49(8) 50(8)
51(8)52(14) 53(10) 54(13) 55(11) 56(10) 57(15) 58(11) 59(9)
60(7) 61(2) 62 63 64(4) 65(2)66(3) 7174 7579 86

The Jackknife estimate of variance for UMVUE for the data was **0.048815**while that of MLE was **0.053137**.

9. Discussion

From Table 6.1,whenα is fixed at α = 1.0 and α = 0.5 and varying values of β i.e. β = 0.1, 0.5, 1.0, 1.5 and 2.0 for small, medium and large sample sizes, we observe that the Jackknife estimate of variances for UMVUE are smaller for small and medium sample sizes as compared to the Jackknife estimates of variances for MLE. Table 6.2 shows that in most cases the coverage probabilities of MLE shows over coverage data given nominal confidence coefficients, while coverage probabilities of UMVUE are generally closer to the nominal confidence coefficients.

Figure 6.1 – Figure 6.10displays graphs from which it can be observed that for varying sample sizes, Jackknife estimates of variances for MLE experiences large fluctuations as compared to those of UMVUE. From numerical illustrations of life time data from Birnbaum & Saunders (1958) and Balakrishnan et al. (2009), it can be seen that the Jackknife estimate of variances for UMVUE are generally smaller compared to variances for MLE.

10. Summary and Conclusion

In sectio2, we have reviewed the derivation of UMVUE and MLE of the gamma distribution. The Jackknife variance estimates of MLE and UMVUE of the gamma pdf have been obtained using Samanta (1988).From the simulation results it can be observed that the Jackknife estimates of the variances for UMVUE are generally smaller and experiences smaller fluctuations as compared to the MLE counterparts. From coverage probabilities UMVUE shows generally a better coverage than those of MLE in terms of closeness to nominal confidence coefficients hence we can generally conclude that UMVUE gives more efficient estimates than those given by MLE particularly for small and medium sample sizes.

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