$q$-Analogue of the Operational Methods, Fractional Operators and Special Polynomials

Dr. Mohammad Asif, Dr. Anju Gupta

1Department of Mathematics, Kalindi College, New Delhi
2Director of NCWEB, University of Delhi, Department of Mathematics, Kalindi College, New Delhi

Abstract: The aim of the present paper is to determine $q$-analogue of the operators, obtained by Dattoli [3] to deal with the families of $q$-partial differential equations of evolution type, to treat the problems involving fractional differential operators. Further, we find $q$-analogue of the families of special polynomials or special functions.

2000 AMS Subject Classification: 33D05, 33C45, 47G20, 26A33.

Keywords: $q$-Calculus, Fractional calculus, Operational calculus, $q$-Hermite polynomials, $q$-Laguerre polynomials, $q$-Bessel functions.

1. Introduction and Notation

In this paper, the following notations shall be used (see[9], [10]) for $1 < |q| < 1$.

A $q$-number or basic number is defined as

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1, (1.1)$$

In 1979, Cigler [2], defined the sum of $q$-numbers in the following manner

$$[\alpha, \beta]_q = [\alpha]_q + q^\alpha[\beta]_q = \frac{1-q^{\alpha+\beta}}{1-q} = [\alpha + \beta]_q, (1.2)$$

The product of $q$-numbers is defined [8; eq.(324)] by

$$[a]_q[b]_q = [ab]_q. (1.3)$$

The $q$-number factorial of $n!$ is defined for a nonnegative integer $n$ by

$$[n]_q! = \prod_{k=1}^{n}[k]_q, (1.4)$$

the corresponding $q$-number shifted factorial is defined by

$$[a]_{q,n} = \prod_{k=0}^{n-1}[a+k]_q. (1.5)$$

Clearly,

$$\lim_{q \to 1}[n]_q! = n!, \quad \lim_{q \to 1}[a]_q = a,$$

and

$$[a]_{q,n} = \frac{(q^n:a)_{q}}{(1-q)^n}, \quad \lim_{q \to 1}[a]_{q,n} = (a)_n$$

The $q$-shifted factorial is define as

$$(a; q)_n = \prod_{j=0}^{n-1}(1-aq^j), (1.6)$$

$$(a; q)_{\infty} = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}. (1.7)$$

The $q$-derivative is defined as

$$D_q f(x) = \left\{ \begin{array}{ll}
\frac{f(x) - f(qx)}{(1-q)x}, & x \neq 0, \\
f'(0), & x = 0.
\end{array} \right. (1.8)$$

its $n$th iterate is written as

$$D_q^n f(x) = D_q^{n-1}(D_q f(x)),$$

for $n = 1, 2, \cdots$, where $D_q^0$ denotes the identity operator.

The $q$-Gamma function is defined (see[9], [10]) as

$$\Gamma_q(x) = \int_0^{q^n} e^{-u} d_q u = \frac{(q^n;x)_{\infty}}{(q^n; q)_{\infty}}(1-q)^{-x}, \quad 0 < q < 1. (1.9)$$

The $q$-Exponential function is defined (see[9], [10]) as

$$e_q^r = \sum_{r=0}^{\infty} \frac{x^r}{[r]_q!}. (1.10)$$

The $q$-Taylor series is written as (see [9], [10])

$$f(x + \lambda) = \sum_{r=0}^{\infty} \lambda^r \left[ \frac{\partial}{\partial \lambda} \right]_q f(x). (1.11)$$

In 2003, Dattoli [3] defined the Hermite and Laguerre polynomials through the following operational identities
\[ e^{\frac{\lambda^2}{2}} x^n = \sum_{r=0}^{\infty} \frac{(\frac{\lambda^2}{2})^r}{r!} x^n = \sum_{r=0}^{\infty} \frac{(y)^n}{(n-2r)!} r! = L_n(x, y) \; 3.5 \text{(12)} \]

and

\[ e^{-\lambda^2 x^2} = \frac{(-1)^n}{n!} x^n = \sum_{r=0}^{\infty} \frac{(-1)^n y^{n-r}}{(n-r)! (r!)^2} = L_n(x, y). \; 6.5 \text{ sin(13)} \]

Let us find the \( q \)-analogue of the operators derived by Dattoli [3]

\[ e_q^{2D_q^2} x^n = \sum_{r=0}^{\infty} \frac{y^r D_q x^n}{[r]_q!} = \sum_{r=0}^{\infty} \frac{y^r}{[r]_q!} D_q x^n. \]

Therefore,

\[ = \sum_{r=0}^{\infty} \frac{y^r}{[r]_q!} \left[ (n)_q [n-1]_q \ldots [n-r+1]_q x^{n-r} \right] \approx \sum_{r=0}^{\infty} \frac{y^r}{[r]_q!} [n]_q! [n-r]_q! x^{n-r} \]

i.e.

\[ e_q^{2D_q^2} x^n = H_n(x, y; q), \; 3.5 \text{(1.4)} \]

\( q \)-analogue of the Hermite polynomial.

Similarly, to find \( q \)-analogue of the Laguerre polynomial [3], we have

\[ e_q^{D_q^2} x^n = \sum_{r=0}^{\infty} \frac{y^r D_q x^n}{[r]_q!} = \sum_{r=0}^{\infty} \frac{y^r}{[r]_q!} D_q x^n. \]

\[ = \sum_{r=0}^{\infty} \frac{(-1)^n y^{n-r}}{[r]_q!} \left[ (n)_q [n-1]_q \ldots [n-r+1]_q x^{n-r} \right] \approx \sum_{r=0}^{\infty} \frac{(-1)^n y^{n-r}}{[r]_q!} [n]_q! [n-r]_q! x^{n-r} \]

\[ = \sum_{r=0}^{\infty} \frac{(-1)^n y^{n-r} [n]_q!}{[r]_q! [n-r]_q!} = \left[ s_\xi q \left( \frac{\lambda^2}{2} \right) + \left( \frac{\lambda^2}{2} \right) \right] A \; 3.5 \text{ in} \]

and which admits the formal solution

\[ A(S, \tau) = e_q \left( s_\xi q \left( \frac{\lambda^2}{2} \right) + \left( \frac{\lambda^2}{2} \right) \right) f(S) \; 3.5 \text{ in} \]

Let us find \( q \)-analogue of the identity [16]

\[ e_q^{2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2 \xi q} d\xi \; 3.5 \text{ in} \]

\[ e_q^{2D_q^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2 \xi q} D_q d\xi \; 3.5 \text{ in} \; \text{(1.16)} \]

\[ e_q^{D_q^2} x^n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2 \xi q} D_q x^n d\xi \; 3.5 \text{ in} \]

The use of the above identity and of the following fact, we have

\[ e_q^{2D_q^2} f(x) = \sum_{r=0}^{\infty} \frac{\lambda^r}{[r]_q!} D_q^r f(x) = f(x + \lambdaq) \; \text{(1.17)} \]

\[ H_n(x, y; q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2 \xi q} [x + 2 \xi q] y^m d\xi \; \text{(1.18)} \]

This operational identities can be used to derive integral representation.

Methods employing the combined use of generalized exponential operators and integral transforms provide a powerful tool for the solution of P.D.E. of evolution type. Let us find the \( q \)-analogue of the Black-Scholes financial model [19]

\[ \left[ \frac{\partial}{\partial \tau} \right] A = S \left[ \frac{\partial}{\partial S} + A \frac{\lambda^2}{2} \right] A - \mathcal{A}, \quad A(S, 0) = f(S) \; 3.5 \text{ in} \; \text{(1.19)} \]

which can be rewritten as

\[ \left[ \frac{\partial}{\partial \tau} \right] A = \left( S \left[ \frac{\partial}{\partial S} + A \right] + \frac{\lambda^2}{2} \right) A - \left( \frac{\lambda^2}{2} \right)^2 A \; 3.5 \text{ in} \]

or

\[ \left[ \frac{\partial}{\partial \tau} \right] A = \left( S \left[ \frac{\partial}{\partial S} + A \right] + \frac{\lambda^2}{2} \right) A - \left( \frac{\lambda^2}{2} \right)^2 A = 0.35 \text{ in} \; \text{(1.20)} \]

which is a linear differential equation in \( A \) whose Integrating Factor is

\[ e_q \left( s_\xi q \left( \frac{\lambda^2}{2} \right) + \left( \frac{\lambda^2}{2} \right) \right) \; 3.5 \text{ in} \]

and which admits the formal solution

\[ A(S, \tau) = e_q \left( s_\xi q \left( \frac{\lambda^2}{2} \right) + \left( \frac{\lambda^2}{2} \right) \right) f(S) \; 3.5 \text{ in} \]

\[ = e_q \left( \frac{\lambda^2}{2} \right)^2 \left( s_\xi q \left( \frac{\lambda^2}{2} \right) + \left( \frac{\lambda^2}{2} \right) \right) f(S) \; 3.5 \text{ in} \; \text{(1.21)} \]

\[ = e_q \left( \frac{\lambda^2}{2} \right)^2 \left( s_\xi q \left( \frac{\lambda^2}{2} \right) + \left( \frac{\lambda^2}{2} \right) \right) f(S) \; 3.5 \text{ in} \]

or

\[ A(S, \tau) = e_q \left( \frac{\lambda^2}{2} \right)^2 \left( s_\xi q \left( \frac{\lambda^2}{2} \right) + \left( \frac{\lambda^2}{2} \right) \right) f(S) \; 3.5 \text{ in} \; \text{(1.22)} \]

by the \( q \)-analogue of the dilatation operator[3], we find
\[ A(S, \tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{2\xi^2} f(e^{2\xi^2} S) d\xi. \]

This last result shows that methods employing operational techniques can be used in fairly wide context and allow noticeable flexibility.

In this paper we introduce \( q \)-analogue of the new families of special polynomials. It shall be shown that the concept we develop is useful in different variety including the theory of \( q \)-fractional derivatives.

2. \( q \)-Analogue of the Fractional Operators, Integral Transforms and a New Class of Special Polynomials

From the theory of fractional operators [3], let us write \( q \)-analogue of the operators [15, p.218] raised to a fractional power, is the identity

\[ \lambda^{-\nu} = \frac{1}{\Gamma_q(\nu)} \int_{0}^{\infty} e^{-\nu t} t^{-\nu-1} d_q t3.5 \sin t. \]

It is, therefore, evident that

\[ f(x) = e_q^{-x^2} \]

and using the eq. (1.16), produces

\[ \left( \frac{d^2}{dx^2} \right)_q e_q^{-x^2} = \frac{1}{\Gamma_q(\nu)} \int_{0}^{\infty} e^{-\nu t} t^{-\nu-1} d_q t3.5 \sin t. \]

Let us consider the under-brace function of eq. (2.3)

\[ e_q^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{x^2/\xi^2} d\xi. \]

since under-brace is the \( q \)-shift operator, therefore, from eqs. (1.17) and (2.4), we have

\[ e_q^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{x^2/(\xi^2+1)} d\xi. \]

Let us now consider the simpler case \( f(x) = x^n \).

According to eq. (2.2) and eq. (1.14), we have

\[ \left( \frac{d^2}{dx^2} \right)_q x^n = \frac{1}{\Gamma_q(\nu)} \int_{0}^{\infty} e^{-\nu t} t^{-\nu-1} d_q t4.5 \sin (2.5) \]

substituting the value of the under-brace from the eq. (1.6) into the eq. (2.6), we get the desired result

\[ e_q^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{x^2/\xi^2} d\xi. \]
\[
\left[1 - \frac{z^2}{\cos z}\right]^{-1} = \frac{1}{\Gamma(q)} \int_0^\infty e^{-t} t^{q-1} H_n(x, y; q) dt 3\sin(2.7)
\]

\[
= \frac{1}{\Gamma(q)} \int_0^\infty e^{-t} t^{q-1} [n]_q \sum_{r=0}^{\infty} [r]_q [n-r]_q! d_q t 3\sin
\]

\[
= \frac{1}{\Gamma(q)} \int_0^\infty e^{-t} t^{q-1} [n]_q \sum_{r=0}^{\infty} [r]_q [n-r]_q! d_q t 3\sin
\]

\[
= \frac{1}{\Gamma(q)} \sum_{r=0}^{\infty} [r]_q [n-r]_q! 4\sin
\]

where \((q^n)_r\) is a -analogue of the pockham symbol.

By introducing the notation

\[
(q^n)_r = \left(\frac{(q^n)_r}{(1-q)^r}\right)
\]

and the operators, \(q\) -analogue to the operators [3], are

\[
[Y_r]_q \quad \text{and} \quad [D_{y,r}]_q
\]

we have

\[
[Y_r]_q (q^n)_r \frac{y^r}{(1-q)^r} = \left[\frac{(q^n)_r y^r}{(1-q)^r}\right] = (q^n)_r y^r (1-q)^r^{(r+1)}
\]

\[
[D_{y,r}]_q (q^n)_r \frac{y^r}{(1-q)^r} = \left[\frac{(q^n)_r y^r}{(1-q)^r}\right] = (q^n)_r y^r (1-q)^r
\]

It is not difficult to realize that the polynomial (2.8) is \(q\) -analogue of an umbral image [6] of the ordinary Hermite polynomials and that satisfy the recurrences

\[
\frac{\partial}{\partial x} H_n(x, y; q) = \left[\frac{\partial}{\partial x} \right]^n [n]_q \sum_{r=0}^{\infty} \frac{(q^n)_r y^r}{(1-q)^r [r]_q [n-r]_q!} 4\sin
\]

or

\[
\frac{\partial}{\partial x} \left(\frac{v_n(x, y; q)}{H_n(x, y; q)}\right) = [n]_q \frac{v_n(x, y; q).3\sin(2.12)}{H_n(x, y; q)}
\]

and

\[
2[Y_r]_q \left[\frac{\partial}{\partial x} + x\right] \frac{v_n(x, y; q)}{4.5\sin}
\]

or

\[
\frac{\partial}{\partial x} [n]_q \frac{v_n(x, y; q)}{4.5\sin}
\]

and consider the following \(q\) -analogue of the differential equations, we have

\[
2[Y_r]_q \left[\frac{\partial^2}{\partial x^2} \right] + x\left[\frac{\partial}{\partial x} \right] [n]_q \frac{v_n(x, y; q)}{4.5\sin}
\]

or

\[
2[Y_r]_q \left[\frac{\partial^2}{\partial x^2} \right] + x\left[\frac{\partial}{\partial x} \right] [n]_q \frac{v_n(x, y; q)}{4.5\sin}
\]

Volume 3 Issue 7, July 2014
The operator \([\mathcal{D}_{\nu}]\) is defined as

\[
\mathcal{D}_{\nu} f(x) = \left( x^n \right)_{\nu} f(x)
\]

where \([\mathcal{D}_{\nu}]\) is a differentiation operator. The \([\mathcal{D}_{\nu}]\) operator can be written as

\[
\mathcal{D}_{\nu} f(x) = \left( x^n \right)_{\nu} f(x) = \left( \frac{\partial}{\partial x} \right)^{\nu} f(x)
\]

for a function \(f(x)\). The \([\mathcal{D}_{\nu}]\) operator can be used to obtain \(\nu\)-analogous results for a new family of polynomials, which can be viewed as an umbral image of the \(\nu\)-Laguerre family.

\[
[D_{\nu}, q] \cdot H_n(x, y; q) = \left[ \frac{\partial^2}{\partial x^2} \right] q \cdot H_n(x, y; q) 3\sin(2.15)
\]

It is worth noting that \(q\)-analogues of the Hermite polynomial \(H_n(x, y; q)\) can be derived from the operational rule.

\[
\left[ D_{\nu}, q \right] \cdot \left( x^n \right)_{\nu} = \sum_{r=0}^{n} \left( \frac{\partial^2}{\partial x^2} \right)^r q \cdot \left( [n]_q ! \right)^{3\sin(2.16)}
\]

\(\nu\)-analogous results can be obtained for a new family of polynomials, which can be viewed as an umbral image of the \(\nu\)-Laguerre family.

Now consider the operational definition in the following manner:

\[
\left( 1 + y \left( \frac{\partial}{\partial x} \right)_q \right) \cdot H_n(x, y; q) = 0.45\sin(2.14)
\]

and \(\nu\)-analogue of the operator \([\mathcal{D}_{\nu}]\) is written as \([\mathcal{D}_{\nu}, q]\)

\[
[D_{\nu}, q] \cdot H_n(x, y; q) = \sum_{r=0}^{n} \left( \frac{\partial^2}{\partial x^2} \right)^r q \cdot \left( [n]_q ! \right)^{3\sin(2.15)}
\]

and the under-brace has been determined in eq. (1.10), therefore:

\[
\Gamma(q) \cdot \int_0^{\infty} \frac{e^{-t} t^{q-n-1}}{\Gamma(q)} \cdot \left( (\nu)_{\nu} \right)_{\nu} \cdot \left( (-1)^n x^n \right)_{\nu} \cdot \left( [n]_q ! \right)^{3\sin(2.15)}
\]

\[
= \Gamma(q) \cdot \int_0^{\infty} \frac{e^{-t} t^{q-n-1}}{\Gamma(q)} \cdot \left( (\nu)_{\nu} \right)_{\nu} \cdot \left( (-1)^n x^n \right)_{\nu} \cdot \left( [n]_q ! \right)^{3\sin(2.15)}
\]

\[
= \Gamma(q) \cdot \int_0^{\infty} \frac{e^{-t} t^{q-n-1}}{\Gamma(q)} \cdot \left( (\nu)_{\nu} \right)_{\nu} \cdot \left( (-1)^n x^n \right)_{\nu} \cdot \left( [n]_q ! \right)^{3\sin(2.15)}
\]

\[
= \Gamma(q) \cdot \int_0^{\infty} \frac{e^{-t} t^{q-n-1}}{\Gamma(q)} \cdot \left( (\nu)_{\nu} \right)_{\nu} \cdot \left( (-1)^n x^n \right)_{\nu} \cdot \left( [n]_q ! \right)^{3\sin(2.15)}
\]

\[
= \Gamma(q) \cdot \int_0^{\infty} \frac{e^{-t} t^{q-n-1}}{\Gamma(q)} \cdot \left( (\nu)_{\nu} \right)_{\nu} \cdot \left( (-1)^n x^n \right)_{\nu} \cdot \left( [n]_q ! \right)^{3\sin(2.15)}
\]
\[
= [n_q]^n \sum_{r=0}^{n} \frac{(-1)^r (q^r; q)_{n-r} y^{n-r} x^r}{[n-r]_q! [(r)_q]!} = x^n L_n (x, y; q).3in
\]

or

\[
\left(1 + y \left[ \frac{\partial}{\partial x} \right]_q \frac{x}{[n]_q!} \right)^{-n} \left[ \frac{\partial}{\partial x} \right]_q \left[ \frac{\partial}{\partial x} \right]_q \left(-1\right)^n x^n \right. \left[ \frac{\partial}{\partial x} \right]_q = L_n (x, y; q).3in (2.17)
\]

The use of the same operators as before allows the derivation of the following as well

\[
[D_{r+r}]_q \cdot L_n (x, y; q) = [n_q]^n \sum_{r=0}^{n} \frac{(-1)^r \left( \frac{\partial}{\partial x} \right)_q (q^r; q)_{n-r} y^{n-r} \left[ \frac{\partial}{\partial x} \right]_q x^n}{(1-q)^{n-r}(r)_q!} x^n 4in
\]

\[
= [n_q]^n \sum_{r=0}^{n} \frac{(-1)^r (q^r; q)_{n-r} y^{n-r} [r]_q^2}{(1-q)^{n-r}(r)_q!} [n-r]_q! x^n 4in
\]

\[
= [n_q]^n \sum_{r=0}^{n} \frac{(-1)^r (q^r; q)_{n-r} y^{n-r} (r)_q^2}{(1-q)^{n-r}(r)_q!} [n-r-1]_q! x^n 4in
\]

\[
= [n_q]^n \sum_{r=0}^{n} \frac{(-1)^r (q^r; q)_{n-r} y^{n-r} (r)_q^2}{(1-q)^{n-r}(r)_q!} [n-r-1]_q! x^n 4in
\]

\[
= \left[ \frac{\partial}{\partial x} \right]_q [n_q]^n \sum_{r=0}^{n} \frac{(-1)^r (q^r; q)_{n-r} y^{n-r} (r)_q^2}{(1-q)^{n-r}(r)_q!} [n-r-1]_q! x^n 4in
\]

\[
= \left[ \frac{\partial}{\partial x} \right]_q x[n_q]^n \sum_{r=0}^{n} \frac{(-1)^r (q^r; q)_{n-r} y^{n-r} (r)_q^2}{(1-q)^{n-r}(r)_q!} [n-r-1]_q! x^n 3in
\]

\[
= \left[ \frac{\partial}{\partial x} \right]_q \left[ \frac{x}{[n]_q!} \left[ \frac{\partial}{\partial x} \right]_q \right. \left. \left[ \frac{x}{[n]_q!} \right] \right] \cdot L_n (x, y; q) 4in (2.18)
\]

with the use of the following operators, we find

\[
[D_{r+r}]_q \cdot L_n (x, y; q) = \left[ \frac{\partial}{\partial x} \right]_q \left[ \frac{x}{[n]_q!} \right] \cdot L_n (x, y; q) .3in 4in
\]

\[
[D_{r+r}]_q \cdot L_n (x, y; q) = \left[ \frac{\partial}{\partial x} \right]_q \left[ \frac{x}{[n]_q!} \right] \cdot L_n (x, y; q) .3in 4in
\]
The advantage offered by the use of the \( q \)-integral representation in dealing with \( q \)-differential operators is the possibility of giving a meaning to apparently meaningless operations. This is indeed the case of the \( q \)-analogue of the Euler \([14] \); see also \([15, Chapter 5] \).

One of the advantages offered by the use of the operational rules and integral representations may provide unsuspected links between apparently disconnected fields.

3. The Remarks

One of the advantages offered by the use of the \( q \)-integral representation in dealing with \( q \)-differential operators is the possibility of giving a meaning to apparently meaningless operations. This is indeed the case of the Riemann–Liouville definition of fractional derivative (see \([14] \); see also \([15, Chapter 5] \)).

By taking advantage from the definition of the Euler \( \Gamma \) function (see9, 10) and use of the identity (see3, p. 153, eq. (12)) allows to determine the \( q \)-analogue of the results obtained by Dattoli et. al [3], we conclude that

\[
\Theta[n_q][n_q]\frac{1}{t^{n_q}}\sum_{r=0}^{n_q} \frac{(-1)^r (q^r ; q)_{n_q-r}}{(n_q - r)!} \bigg[ \frac{d^{\nu}}{d^n} \bigg] \bigg[ f(x) \bigg] d_q t^{4in}
\]

\[
= \frac{1}{\Gamma_q(\nu)} \int_0^1 f(x) t^{\nu-1} d_q t^{4in}
\]

or

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

If we set the \( q \)-analogue of \( \frac{1}{1-x} \) in place of \( f(x) \) in eq. (3.1) we find that

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]

\[
\left[ \frac{d}{d_q}\right] f(x) = \frac{1}{\Gamma_q(\nu)} \int_0^1 t^{\nu-1} f(x t^{\nu}) d_q t^{4in}(3.1)
\]
In eq. (3.3), we find that

\[
\text{L}_n(x, y; \alpha, \nu; q) = [n]_q \sum_{\alpha + r \in \mathbb{N}} \frac{(-1)^r x^{\alpha + r}}{[r]_q!} [a - 1 - r]_q! 3in \quad (3.4)
\]

or

\[
\sum_{\alpha + r \in \mathbb{N}} \frac{(-1)^r x^{\alpha + r}}{[r]_q!} [a - 1 - r]_q! 3in
\]

Denoting the polynomial \( \text{L}_n(x, y; \alpha, \nu; q) \) on the right hand side of the above eq. (3.5) we find the following recurrences

\[
\frac{\partial}{\partial y} \text{L}_n(x, y; \alpha, \nu; q) = [n]_q \sum_{r \in \mathbb{N}} \frac{(-1)^r x^{\alpha + r - 1}}{[r]_q!} [a - 1 - r]_q! 4in
\]

\[
\text{L}_n(x, y; \alpha, \nu; q) = [n]_q \text{L}_{n-1}(x, y; \alpha, \nu; q) 4in
\]

\[
\frac{\partial}{\partial y} \text{L}_n(x, y; \alpha, \nu; q) = [n]_q \text{L}_{n-1}(x, y; \alpha, \nu; q) 4in
\]
The analogue of the Gauss analogue of the integral is convergent and can be viewed as a kind of convolution of $S(x)$ on the $0^\text{th}$-order cylindrical Bessel function.

As a final example, we will consider the solution of the fractional diffusive equation

$$
\left[ \frac{\partial}{\partial y} \right]^\alpha f(x, y) = - \left[ \frac{\partial^{1/2}}{\partial x^{1/2}} \right]^\alpha f(x, y),
$$

and by proceeding as before, we find the solution of eq. (3.17) as

$$
e^{-\sqrt{\frac{y^2}{2}}}g(x)= \frac{y}{2\sqrt{\pi}} \int_0^y e^{-\frac{y^2}{4t}} g(x) d_t t. 
$$

This result can be viewed as the $q$-analogue of the Gauss transform for the solution of the heat diffusion equation. In this paper it has been shown that $q$ - analogue of operational method can provide a fairly useful tool to solve a large number of problems including fractional propagation equations.

**References**


[3] Dattoli G., Operational methods, fractional operators

---

**Volume 3 Issue 7, July 2014**

www.ijsr.net

Licensed Under Creative Commons Attribution CC BY


