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q-Analogue of the Operational Methods, Fractional Operators and Special Polynomials

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Abstract: ***The aim of the present paper is to determine q-analogue of the operators, obtained by Dattoli [3] to deal with the families of q-partial differential equations of evolution type, to treat the problems involving fractional differential operators. Further, we find q-analogue of the families of special polynomials or special functions.***

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1. Introduction and Notation

In this paper, the following notations shall be used (see[9], [10]) for $|q|<1$,

A q-number or basic number is defined as

$$[a]_q=\frac{1-q^a}{1-q},\qquad q\neq 1,(1.1)$$

In 1979, Cigler [2], defined the sum of q-numbers in the following manner

$$[\alpha]_q\oplus_q[\beta]_q=[\alpha]_q+q^{\alpha}[\beta]_q=\frac{1-q^{\alpha}}{1-q}+q^{\alpha}\frac{1-q^{\beta}}{1-q}=\frac{1-q^{\alpha+\beta}}{1-q}=[\alpha+\beta]_q,(1.2)$$

The product of q-numbers is defined [8; eq.(324)] by $[a]_q=e_q^{\ln_q(a)}$, this power function allows to write

$$[a]_q[b]_q=[ab]_q.(1.3)$$

The q-number factorial of $n!$ is defined for a nonnegative integer n by

$$[n]_q!=\prod_{k=1}^{n}[k]_q,(1.4)$$

the corresponding q-number shifted factorial is defined by

$$[a]_{q;n}=\prod_{k=0}^{n-1}[a+k]_q.(1.5)$$

Clearly,

$$\lim_{q\to1}[n]_q!=n!,\qquad \lim_{q\to1}[a]_q=a,$$

and

$$[a]_{q;n}=\frac{(q^a;q)_n}{(1-q)^n},\qquad \lim_{q\to1}[a]_{q;n}=(a)_n$$

The q-shifted factorial is define as

$$(a;q)_\infty=\prod_{j=0}^{\infty}(1-aq^j),(1.6)$$

$$(a;q)_n=\frac{\Gamma_q(a+n)}{\Gamma_q(a)}.(1.7)$$

The q-derivative is defined as

$$D_qf(x)=\begin{cases}\dfrac{f(x)-f(qx)}{(1-q)x}, & x\neq0,\\ f'(0), & x=0.\end{cases}\qquad(1.8)$$

its nth iterate is written as

$$D_q^nf(x)=D_q^{n-1}(D_qf(x)),$$

for $n=1,2,\cdots,$ where D_q^0 denotes the identity operator.

The q-Gamma function is defined (see[9], [10]) as

$$\Gamma_q(x)=\int_0^{\infty}u^{x-1}e_q^{-u}d_qu=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\quad 0<q<1.(1.9)$$

The q-Exponential function is defined (see[9], [10])

$$e_q^x=\sum_{r=0}^{\infty}\frac{x^r}{[r]_q!}.(1.10)$$

The q-Taylor series is written as (see [9], [10])

$$f(x+\lambda)=\sum_{r=0}^{\infty}\frac{\lambda^r\left[\frac{\partial}{\partial x}\right]_q^r}{[r]_q!}f(x).(1.11)$$

In 2003, Dattoli [3] defined the Hermite and Laguerre polynomials through the following operational identities

$$e^{y\frac{\partial^2}{\partial x^2}} x^n = \sum_{r=0}^{\infty} \frac{(y\frac{\partial^2}{\partial x^2})^r}{r!} x^n = \sum_{r=0}^{[\frac{n}{2}]} \frac{(y)^r n! x^{n-2r}}{(n-2r)! r!} = H_n(x,y) 3.5in (1.12)$$

and

$$e^{-y\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}\left[\frac{(-1)^n}{n!}x^n\right] = n!\sum_{r=0}^{n}\frac{(-1)^r x^r y^{n-r}}{(n-r)!(r!)^2} = L_n(x,y).6.5in(1.13)$$

Let us find the q-analogue of the operators deriven by Dattoli [3]

$$e_q^{[yD_q^2]} x^n = \sum_{r=0}^{\infty} \frac{y^r D_q^{2r}}{[r]_q!} x^n = \sum_{r=0}^{\infty} \frac{y^r}{[r]_q!} D_q^{2r} x^n$$

$$= \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r}{[r]_q!}\{[n]_q [n-1]_q \cdots [n-2r+1]_q x^{n-2r}\} = \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r}{[r]_q!} \frac{[n]_q!}{[n-2r]_q!} x^{n-2r}$$

i.e.

$$e_q^{[yD_q^2]} x^n = H_n(x,y;q)(1.14)$$

q-analogue of the Hermite polynomial.

Similarly, to find q-analogue of the Laguerre polynomial [3], we have

$$e_q^{-y[\frac{\partial}{\partial x}]_q x[\frac{\partial}{\partial x}]_q}\left[\frac{(-1)^n}{[n]_q!}x^n\right] = \sum_{r=0}^{\infty} \frac{(-1)^r\left[y[\frac{\partial}{\partial x}]_q x[\frac{\partial}{\partial x}]_q\right]^r}{[r]_q!}\left[\frac{(-1)^n}{[n]_q!}x^n\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\left[y[\frac{\partial}{\partial x}]_q x[\frac{\partial}{\partial x}]_q\right]^r}{[r]_q!}\left[\frac{(-1)^n}{[n]_q!}x^n\right] = \sum_{r=0}^{\infty} \frac{(-1)^{n+r} y^r [\frac{\partial}{\partial x}]_q^r x^n}{[r]_q! [n-r]_q!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^{n+r} y^r [n]_q! x^{n-r}}{[r]_q! ([n-r]_q!)^2} = [n]_q! \sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{[n-r]_q! ([r]_q!)^2} = L_n(x,y;q)(1.15)$$

which is the required q-analogue of the Laguerre polynomial.

Let us find q-analogue of the identity [16]

$$e_q^{b^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{(-\xi^2+2b\xi)} d_q\xi 3.5in$$

$$e_q^{yD_q^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{(-\xi^2+2\xi\sqrt{y}D_q)} d_q\xi 3.5in(1.16)$$

$$e_q^{yD_q^2} x^n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{(-\xi^2+2\xi\sqrt{y}D_q)} x^n d_q\xi 3.5in$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2} e_q^{(2\xi\sqrt{y}D_q)} x^n d_q\xi 3.5in$$

The use of the above identity and of the following fact, we have

$$e_q^{\lambda D_q} f(x) = \sum_{r=0}^{\infty} \frac{\lambda^r}{[r]_q!} D_q^r f(x) = f(x+\lambda)(1.17)$$

$$H_n(x,y;q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2} [x+2\xi\sqrt{y}]^n d_q\xi(1.18)$$

This operational identities can be used to derive integral representation.

Methods employing the combined use of generalized exponential operators and integral transforms provide a powerful tool for the solution of P.D.E. of evolution type. Let us find the q-analogue of the Black-Scholes financial model [19]

$$\left[\frac{\partial}{\partial \tau}\right]_q A = S^2\left[\frac{\partial^2}{\partial S^2}\right]_q A + \lambda S\left[\frac{\partial}{\partial S}\right]_q A - \lambda A, \qquad A(S,0) = f(S), 3.5in(1.19)$$

which can be rewritten as

$$\left[\frac{\partial}{\partial \tau}\right]_q A = \left(S\left[\frac{\partial}{\partial S}\right]_q + \frac{\lambda-1}{2}\right)^2 A - \left(\frac{\lambda+1}{2}\right)^2 A 3.5in$$

or

$$\left[\frac{\partial}{\partial \tau}\right]_q A = \left[\left(S\left[\frac{\partial}{\partial S}\right]_q + \frac{\lambda-1}{2}\right)^2 - \left(\frac{\lambda+1}{2}\right)^2\right] A = 0 3.5in(1.20)$$

which is a linear differential equation in A. whose Integrating Factor is

$$e_q^{-\left\{\left(S\left[\frac{\partial}{\partial S}\right]_q + \frac{\lambda-1}{2}\right)^2 - \left(\frac{\lambda+1}{2}\right)^2\right\}\tau} 3.5in$$

and which admits the formal solution

$$A(S,\tau) = e_q^{\left(S\left[\frac{\partial}{\partial S}\right]_q + \frac{\lambda-1}{2}\right)^2 \tau - \left(\frac{\lambda+1}{2}\right)^2 \tau} f(S) 3.5in$$

$$= e_q^{-\left(\frac{\lambda+1}{2}\right)^2 \tau} \cdot e_q^{\left(S\left[\frac{\partial}{\partial S}\right]_q + \frac{\lambda-1}{2}\right)^2 \tau} f(S) 3.5in(1.21)$$

$$= \frac{e_q^{-\left(\frac{\lambda+1}{2}\right)^2 \tau}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2+2\left(S\left[\frac{\partial}{\partial S}\right]_q + \frac{\lambda-1}{2}\right)\sqrt{\tau}\xi} f(S) d_q\xi, 3.5in$$

or

$$A(S,\tau) = \frac{e_q^{-\left(\frac{\lambda+1}{2}\right)^2 \tau}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2+(\lambda-1)\xi\sqrt{\tau}} e_q^{2\sqrt{\tau}S\left[\frac{\partial}{\partial S}\right]_q} f(S) d_q\xi, (1.22)$$

by the q-analogue of the dilatation operator[3], we find

$$e_q^{\lambda x\left[\frac{\partial}{\partial x}\right]_q} f(x)=f(xe_q^{\lambda})4.5in(1.23)$$

from the eqs. (1.17) and (1.18), we have

$$A(S,\tau)=\frac{e_q^{-\left(\frac{\lambda+1}{2}\right)^2\tau}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q^{-\xi^2+(\lambda-1)\xi\sqrt{\tau}} f(e_q^{2\xi\sqrt{\tau}}S)d_q\xi(1.24)$$

This last result shows that methods employing operational techniques can be used in fairly wide context and allow noticeable flexibility.

In this paper we introduce q-analogue of the new families of special polynomials. It shall be shown that the concept we develop is useful in different variety including the theory of q-fractional derivatives.

2. q-Analogue of the Fractional Operators, Integral Transforms and a New Class of Special Polynomials

From the theory of fractional operators [3], let us write q-analogue of the operators [15, p.218] raised to a fractional power, is the identity

$$\lambda^{-\nu}=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\lambda t}t^{\nu-1}d_qt3.5in(2.1)$$

It is, therefore, evident that

$$\left[\alpha-\left[\frac{\partial^2}{\partial x^2}\right]_q\right]^{-\nu} f(x)=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\left(\alpha-\left[\frac{\partial^2}{\partial x^2}\right]_q\right)t} t^{\nu-1}f(x)d_qt3.5in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}e_q^{t\left[\frac{\partial^2}{\partial x^2}\right]_q} f(x)d_qt3.5in$$

or

$$\left[\alpha-\left[\frac{\partial^2}{\partial x^2}\right]_q\right]^{-\nu} f(x)=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}e_q^{t\left[\frac{\partial^2}{\partial x^2}\right]_q} f(x)d_qt.3.5in(2.2)$$

Substituting $f(x)=e_q^{-x^2}$ and using the eq. (1.16), produces

$$\left[\alpha-\left[\frac{\partial^2}{\partial x^2}\right]_q\right]^{-\nu} e_q^{-x^2}=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}\underbrace{e_q^{t\left[\frac{\partial^2}{\partial x^2}\right]_q} e_q^{-x^2}}d_qt,3.5in(2.3)$$

let us consider the under-brace function of eq. (2.3)

$$e_q^{t\left[\frac{\partial^2}{\partial x^2}\right]_q} e_q^{-x^2}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q^{-\xi^2+2\sqrt{t}\xi\left[\frac{\partial}{\partial x}\right]_q} e_q^{-x^2}d_q\xi4.5in$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp_q\left[-\xi^2\right]\underbrace{e_q^{2\sqrt{t}\xi\left[\frac{\partial}{\partial x}\right]_q} e_q^{-x^2}}d_q\xi4.5in(2.4)$$

since under-brace is the q-shift operator, therefore, from eqs. (1.17) and (2.4), we have

$$e_q^{t\left[\frac{\partial^2}{\partial x^2}\right]_q} e_q^{-x^2}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q^{-\xi^2}e_q^{-(2\sqrt{t}\xi+x)^2}d_q\xi4.5in$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q^{-\xi^2}e_q^{-(4t\xi^2+x^2+2\cdot2\sqrt{t}\xi x)}d_q\xi4.5in$$

$$=\frac{e_q^{-x^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q^{-\{(1+4t)\xi^2+2(2\sqrt{t}\xi)x\}}d_q\xi4.5in$$

$$=\frac{e_q^{-x^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q^{-(1+4t)\left(\xi+\frac{(2\sqrt{t})x}{1+4t}\right)^2} e_q^{\frac{4tx^2}{1+4t}}d_q\xi4.5in$$

$$=\frac{e_q^{-\frac{x^2}{1+4t}}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q^{-\left((\sqrt{1+4t})\xi+\frac{2\sqrt{t}x}{\sqrt{1+4t}}\right)^2}d_q\xi,4.5in$$

let us suppose that

$$(\sqrt{1+4t})\xi+\frac{2\sqrt{tx}}{\sqrt{1+4t}}=z,$$

therefore,

$$e_q^{t\left[\frac{\partial^2}{\partial x^2}\right]_q} e_q^{-x^2}=\frac{e_q^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e_q^{-z^2}d_qz4.5in$$

$$=\frac{e_q^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}\frac{2}{\sqrt{\pi}}\int_0^{\infty} e_q^{-z^2}d_qz=\frac{e_q^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}4.5in$$

finally, substituting this value into the eq. (2.3), we have

$$\left[\alpha-\left[\frac{\partial^2}{\partial x^2}\right]_q\right]^{-\nu} e_q^{-x^2}=\frac{1}{\Gamma_q(\nu)}\int_0^\infty \frac{e_q^{-\alpha t}t^{\nu-1}}{\sqrt{1+4t}}e_q^{-\frac{x^2}{1+4t}}d_qt.4.5in(2.5)$$

Let us now consider the simpler case $f(x)=x^n$. According to eq. (2.2) and eq. (1.14), we have

$$\left[1-y\left[\frac{\partial^2}{\partial x^2}\right]_q\right]^{-\nu} x^n=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu-1}\underbrace{e_q^{yt\left[\frac{\partial^2}{\partial x^2}\right]_q} x^n}d_qt,4.5in(2.6)$$

substituting the value of the under-brace from the eq. (1.6) into the eq. (2.6), we get the desired result

$$\left[1-y\left[\frac{\partial^2}{\partial x^2}\right]_q\right]^{-\nu} x^n=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu-1}H_n(x,yt;q)d_qt,3in(2.7)$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu-1}[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rt^rx^{n-r}}{[r]_q![n-r]_q!}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu+r-1}[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rx^{n-r}}{[r]_q![n-r]_q!}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{\Gamma_q(\nu+r)y^rx^{n-r}}{[r]_q![n-r]_q!}4in$$

since

$$\frac{\Gamma_q(\nu+r)}{\Gamma_q(\nu)}=\prod_{k=0}^{r-1}[\nu+k]_q=\frac{(q^\nu;q)_r}{(1-q)^r}$$

$$=[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r}{(1-q)^r}\frac{y^rx^{n-r}}{[r]_q![n-r]_q!}={}_\nu H_n(x,y;q),3in(2.8)$$

where $(q^\nu;q)_r$ is q-analogue of the pochhammer symbol.

By introducing the notation

$$\frac{(q^\nu;q)_r y^r}{(1-q)^r}=\left[\frac{(q^\nu;q)y}{(1-q)}\right]^r (2.9)$$

and the operators, q-analogue to the operators [3], are $[\overset{\square}{Y}_\nu]_q$ and $[\overset{\square}{D}_{y,\nu}]_q$ we have

$$[\overset{\square}{Y}_\nu]_q\frac{(q^\nu;q)_r y^r}{(1-q)^r}=\left[\frac{(q^\nu;q)y}{(1-q)}\right]^{r+1}=\frac{(q^\nu;q)_{r+1}y^{r+1}}{(1-q)^{r+1}}(2.10)$$

$$[\overset{\square}{D}_{y,\nu}]_q\frac{(q^\nu;q)_r y^r}{(1-q)^r}=[r]_q\left[\frac{(q^\nu;q)y}{(1-q)}\right]^{r-1}=[r]_q\frac{(q^\nu;q)_{r-1}y^{r-1}}{(1-q)^{r-1}}.(2.11)$$

It is not difficult to realize that the polynomial (2.8) is q-analogue of an umbral image [6] of the ordinary Hermite polynomials and that satisfy the recurrences

$$\left[\frac{\partial}{\partial x}\right]_q {}_\nu H_n(x,y;q)=\left[\frac{\partial}{\partial x}\right]_q\left[[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r}{(1-q)^r}\frac{y^rx^{n-r}}{[r]_q![n-r]_q!}\right]4in$$

$$=[n]_q[n-1]_q!\sum_{r=0}^{[\frac{n-1}{2}]}\frac{(q^\nu;q)_r}{(1-q)^r}\frac{y^rx^{n-1-r}}{[r]_q![n-1-r]_q!}=[n]_q\ {}_\nu H_{n-1}(x,y;q)4in$$

or

$$\left[\frac{\partial}{\partial x}\right]_q({}_\nu H_n(x,y;q))=[n]_q\ {}_\nu H_{n-1}(x,y;q).3in(2.12)$$

and

$$\left[2[\overset{\square}{Y}_\nu]_q\left[\frac{\partial}{\partial x}\right]_q+x\right]{}_\nu H_n(x,y;q)4.5in$$

$$=2[n]_q!\sum_{r=0}^{[\frac{n-1}{2}]}\frac{[\overset{\square}{Y}_\nu]_q(q^\nu;q)_ry^r}{(1-q)^r}\frac{x^{n-1-2r}}{[r]_q![n-1-2r]_q!}+[n]_q!\sum_{r=0}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{x^{n+1-2r}}{[r]_q![n-2r]_q!}4.5in$$

$$=2[n]_q!\sum_{r=0}^{[\frac{n-1}{2}]}\frac{(q^\nu;q)_{r+1}y^{r+1}}{(1-q)^{r+1}}\frac{x^{n-1-2r}}{[r]_q![n-1-2r]_q!}+[n]_q!\sum_{r=0}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{[n+1-2r]_qx^{n+1-2r}}{[r]_q![n+1-2r]_q!}4.5in$$

$$=2[n]_q!\sum_{r=1}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{x^{n+1-2r}}{[r-1]_q![n+1-2r]_q!}+[n]_q!\sum_{r=0}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{[n+1-2r]_qx^{n+1-2r}}{[r]_q![n+1-2r]_q!}4.5in$$

$$=[n]_q!\sum_{r=0}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{2[r]_qx^{n+1-2r}}{[r]_q![n+1-2r]_q!}+[n]_q!\sum_{r=0}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{[n+1-2r]_qx^{n+1-2r}}{[r]_q![n+1-2r]_q!}6.5in$$

$$=[n]_q!\sum_{r=0}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{2[r]_qx^{n+1-2r}}{[r]_q![n+1-2r]_q!}\{2[r]_q+[n+1-2r]_q\}4.5in$$

$$=[n]_q!\sum_{r=0}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{2[r]_qx^{n+1-2r}}{[r]_q![n+1-2r]_q!}\{[n+1]_q\}4.5in$$

$$=[n+1]_q!\sum_{r=0}^{[\frac{n+1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{2[r]_qx^{n+1-2r}}{[r]_q![n+1-2r]_q!}={}_\nu H_{n+1}(x,y;q)4.5in$$

that is

$$\left[2[\overset{\square}{Y}_\nu]_q\left[\frac{\partial}{\partial x}\right]_q+x\right]{}_\nu H_n(x,y;q)={}_\nu H_{n+1}(x,y;q)4.5in$$

and consider the following q-analogue of the differential equations, we have

$$\left[2[\overset{\square}{Y}_\nu]_q\left[\frac{\partial^2}{\partial x^2}\right]_q\oplus x\left[\frac{\partial}{\partial x}\right]_q![n]_q\right]{}_\nu H_n(x,y;q)4.5in$$

$$=2[n]_q!\sum_{r=0}^{[\frac{n-2}{2}]}\frac{[\overset{\square}{Y}_\nu]_q(q^\nu;q)_ry^r}{(1-q)^r}\frac{x^{n-2-2r}}{[r]_q![n-2-2r]_q!}\oplus x[n]_q!\sum_{r=0}^{[\frac{n-1}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{x^{n-1-2r}}{[r]_q![n-1-2r]_q!}4.5in$$

$$![n]_q[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_ry^r}{(1-q)^r}\frac{x^{n-2r}}{[r]_q![n-2r]_q!}4.5in$$

$$=2[n]_q!\sum_{r=0}^{[\frac{n-2}{2}]}\frac{(q^\nu;q)_{r+1}y^{r+1}}{(1-q)^{r+1}}\frac{x^{n-2-2r}}{[r]_q![n-2-2r]_q!}\oplus[n]_q!\sum_{r=0}^{[\frac{n-1}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2r}}{[r]_q![n-1-2r]_q!}4.5in$$

$$![n]_q[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2r}}{[r]_q![n-2r]_q!}4.5in$$

$$=2[n]_q!\sum_{r=1}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2r}}{[r-1]_q![n-2r]_q!}\oplus[n]_q!\sum_{r=0}^{[\frac{n-1}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{[n-2r]_q x^{n-2r}}{[r]_q![n-2r]_q!}4.5in$$

$$![n]_q[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2r}}{[r]_q![n-2r]_q!}4.5in$$

$$=[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{2[r]_q x^{n-2r}}{[r]_q![n-2r]_q!}\oplus[n]_q!\sum_{r=0}^{[\frac{n-1}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{[n-2r]_q x^{n-2r}}{[r]_q![n-2r]_q!}4.5in$$

$$![n]_q[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2r}}{[r]_q![n-2r]_q!}4.5in$$

$$=[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2r}}{[r]_q![n-2r]_q!}\{2[r]_q\oplus[n-2r]_q![n]_q\}=04.5in$$

that is

$$\left[2[\hat{Y}_\nu]_q\left[\frac{\partial^2}{\partial x^2}\right]_q\oplus x\left[\frac{\partial}{\partial x}\right]_q![n]_q\right]{}_\nu H_n(x,y;q)=04.5in(2.14)$$

and q-analogue of the operator [3] is written as $[\hat{D}_{y;\nu}]_q$

$$[\hat{D}_{y,\nu}]_q[{}_\nu H_n(x,y;q)]=[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{[\hat{D}_{y,\nu}]_q(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2r}}{[r]_q![n-2r]_q!}3in$$

$$=[n]_q!\sum_{r=1}^{[\frac{n}{2}]}\frac{[r]_q(q^\nu;q)_{r-1}y^{r-1}}{(1-q)^{r-1}}\frac{x^{n-2r}}{[r]_q![n-2r]_q!}3in$$

$$=[n]_q!\sum_{r=1}^{[\frac{n}{2}]}\frac{(q^\nu;q)_{r-1}y^{r-1}}{(1-q)^{r-1}}\frac{x^{n-2r}}{[r-1]_q![n-2r]_q!}3in$$

$$=[n]_q!\sum_{r=0}^{[\frac{n-2}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2-2r}}{[r]_q![n-2-2r]_q!}4in$$

$$=\left[\frac{\partial^2}{\partial x^2}\right]_q[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r}\frac{x^{n-2r}}{[r]_q![n-2r]_q!}=\left[\frac{\partial^2}{\partial x^2}\right]_q{}_\nu H_n(x,y;q)3in$$

$$[\hat{D}_{y,\nu}]_q\ {}_\nu H_n(x,y;q)=\left[\frac{\partial^2}{\partial x^2}\right]_q{}_\nu H_n(x,y;q).3in(2.15)$$

It is worth noting that q-analogue of the Hermite polynomial ${}_\nu H_n(x,y;q)$ can be derived from the operational rule

$$e_q^{[\hat{Y}_\nu]_q\left[\frac{\partial^2}{\partial x^2}\right]_q}x^n=\sum_{r=0}^{\infty}\frac{\left([\hat{Y}_\nu]_q\left[\frac{\partial^2}{\partial x^2}\right]_q\right)^r}{[r]_q!}x^n 3in$$

$$=\sum_{r=0}^{\infty}\frac{([\hat{Y}_\nu]_q^r(q^\nu;q)_0 y^0}{(1-q)^0[r]_q!}\left[\frac{\partial^{2r}}{\partial x^{2r}}\right]_q x^n=\sum_{r=0}^{\infty}\frac{(q^\nu;q)^r y^r}{(1-q)^r[r]_q!}\frac{[n]_q!}{[n-2r]_q!}x^{n-2r}3in$$

$$=[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r[r]_q![n-2r]_q!}=[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{(q^\nu;q)_r y^r}{(1-q)^r[r]_q![n-2r]_q!}\frac{x^{n-2r}}{}3in$$

or

$$e_q^{[\hat{Y}_\nu]_q\left[\frac{\partial^2}{\partial x^2}\right]_q}x^n={}_\nu H_n(x,y;q).3in(2.16)$$

q-analogous results can be obtained for a new family of polynomials, which can be viewed as an umbral image of the q-Laguerre family.

Now consider the operational definition in the following manner

$$\left(1+y\left[\frac{\partial}{\partial x}\right]_q x\left[\frac{\partial}{\partial x}\right]_q\right)^{-\nu}\left[\frac{(-1)^n x^n}{[n]_q!}\right]3.5in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu-1}e_q^{-ty\left[\frac{\partial}{\partial x}\right]_q x\left[\frac{\partial}{\partial x}\right]_q}\left[\frac{(-1)^n x^n}{[n]_q!}\right]d_q t3in$$

the under-brace has been determined in eq. (1.10), therefore

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu-1}L_n(x,ty;q)d_q t3.5in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu-1}[n]_q!\sum_{r=0}^{n}\frac{(-1)^r x^r(ty)^{n-r}}{[n-r]_q!([r]_q!)^2}d_q t3in$$

$$=\frac{1}{\Gamma_q(\nu)}[n]_q!\sum_{r=0}^{n}\frac{(-1)^r x^r y^{n-r}}{[n-r]_q!([r]_q!)^2}\int_0^\infty e_q^{-t}t^{\nu+n-r-1}d_q t3in$$

$$=\frac{1}{\Gamma_q(\nu)}[n]_q!\sum_{r=0}^{n}\frac{(-1)^r x^r y^{n-r}}{[n-r]_q!([r]_q!)^2}\Gamma_q(\nu+n-r)3in$$

$$=[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}x^r}{[n-r]_q!([r]_q!)^2}={}_\nu L_n(x,y;q).3in$$

or

$$\left(1+y\left[\frac{\partial}{\partial x}\right]_q x\left[\frac{\partial}{\partial x}\right]_q\right)^{-\nu}\frac{(-1)^n x^n}{[n]_q!}={}_\nu L_n(x,y;q).3in(2.17)$$

The use of the same operators as before allows the derivation of the following as well

$$[\hat{D}_{y,\nu}]_q\;{}_\nu L_n(x,y;q)=[n]_q!\sum_{r=0}^{n}\frac{(-1)^r[\hat{D}_{y,\nu}]_q(q^\nu;q)_{n-r}\,y^{n-r}}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,3in$$

$$=[n]_q!\sum_{r=0}^{n-1}\frac{(-1)^r[n-r]_q(q^\nu;q)_{n-r-1}\,y^{n-r-1}}{(1-q)^{n-r-1}([r]_q!)^2[n-r]_q!}x^r\,4in$$

$$=[n]_q!\sum_{r=0}^{n-1}\frac{(-1)^r(q^\nu;q)_{n-r-1}\,y^{n-r-1}}{(1-q)^{n-r-1}([r]_q!)^2[n-r-1]_q!}x^r\,4in$$

$$=-[n]_q!\sum_{r=0}^{n-1}\frac{(-1)^{r+1}(q^\nu;q)y^{n-r-1}[r+1]_q^2}{(1-q)^{n-r-1}([r+1]_q!)^2[n-r-1]_q!}x^r\,4in$$

$$=-[n]_q!\sum_{r=1}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}([r]_q)^2}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^{r-1}\,4in$$

$$=-\left[\frac{\partial}{\partial x}\right]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,4in$$

$$=-\left[\frac{\partial}{\partial x}\right]_q x[n]_q!\sum_{r=1}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^{r-1}\,3in$$

$$=-\left[\frac{\partial}{\partial x}\right]_q x\left[\frac{\partial}{\partial x}\right]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,4in$$

or

$$[\hat{D}_{y,\nu}]_q\;{}_\nu L_n(x,y;q)=-\left[\frac{\partial}{\partial x}\right]_q x\left[\frac{\partial}{\partial x}\right]_q\left[{}_\nu L_n(x,y;q)\right]4in(2.18)$$

with the use of the following operators, we find

$$\left[[\hat{Y}_\nu]_q x\left[\frac{\partial^2}{\partial x^2}\right]_q\oplus([\hat{Y}_\nu]_q!x)\left[\frac{\partial}{\partial x}\right]_q\oplus[n]_q\right]{}_\nu L_n(x,y;q)3.5in4in$$

$$=[\hat{Y}_\nu]_q x\left[[n]_q!\sum_{r=2}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q[r-1]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^{r-2}\right]4in$$

$$\oplus([\hat{Y}_\nu]_q!x)[n]_q!\sum_{r=1}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^{r-1}\oplus[n]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,4in$$

$$=[n]_q!\sum_{r=1}^{n}\frac{(-1)^r[\hat{Y}_\nu]_q(q^\nu;q)_{n-r}\,y^{n-r}[r]_q[r-1]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^{r-1}\,4in$$

$$\oplus[n]_q!\sum_{r=1}^{n}\frac{(-1)^r[\hat{Y}_\nu]_q(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^{r-1}![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,4.5in$$

$$\oplus[n]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,3.5in$$

$$=[n]_q!\sum_{r=1}^{n}\frac{(-1)^r(q^\nu;q)_{n-r+1}\,y^{n-r+1}[r]_q[r-1]_q}{(1-q)^{n-r+1}([r]_q!)^2[n-r]_q!}x^{r-1}3.5in$$

$$\oplus[n]_q!\sum_{r=1}^{n}\frac{(-1)^r(q^\nu;q)_{n-r+1}\,y^{n-r+1}[r]_q}{(1-q)^{n-r+1}([r]_q!)^2[n-r]_q!}x^{r-1}![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,4.5in$$

$$\oplus[n]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,3.5in$$

$$=[n]_q!\sum_{r=0}^{n}\frac{(-1)^{r+1}(q^\nu;q)_{n-r}\,y^{n-r}[r+1]_q[r]_q}{(1-q)^{n-r}([r+1]_q!)^2[n-r-1]_q!}x^r\,3.5in$$

$$\oplus[n]_q!\sum_{r=0}^{n}\frac{(-1)^{r+1}(q^\nu;q)_{n-r}\,y^{n-r}[r+1]_q}{(1-q)^{n-r}([r+1]_q!)^2[n-r-1]_q!}x^r![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,4.5in$$

$$\oplus[n]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,3.5in$$

$$=![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r+1]_q[r]_q}{(1-q)^{n-r}([r+1]_q!)^2[n-r-1]_q!}x^r\,3.5in$$

$$![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r+1]_q}{(1-q)^{n-r}([r+1]_q!)^2[n-r-1]_q!}x^r![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,4.5in$$

$$\oplus[n]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}}{(1-q)^{n-r}[r!]_q^2[n-r]_q!}x^r\,3.5in$$

$$=![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[n-r]_q[r+1]_q^2}{(1-q)^{n-r}([r+1]_q!)^2[n-r]_q!}x^r![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\,3.5in$$

$$\oplus[n]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}}{(1-q)^{n-r}([r]_q!)^2(n-r)_q!}x^r\,3.5in$$

$$=![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[n-r]_q}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r![n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}\,y^{n-r}[r]_q}{([r]_q!)^2[n-r]_q!}x^r\,3.5in$$

$$\oplus[n]_q[n]_q!\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}y^{n-r}}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r 4in$$

$$=\sum_{r=0}^{n}\frac{(-1)^r(q^\nu;q)_{n-r}y^{n-r}}{(1-q)^{n-r}([r]_q!)^2[n-r]_q!}x^r\{![n-r]_q![r]_q\oplus[n]_q\}=04in$$

or

$$\left[[\hat{Y}_\nu]_q x\left[\frac{\partial^2}{\partial x^2}\right]_q\oplus([\hat{Y}_\nu]_q!x)\left[\frac{\partial}{\partial x}\right]_q\oplus[n]_q\right]{}_\nu L_n(x,y;q)=0.2.5in(2.19)$$

A particularly interesting case arises when the highest order of the derivative appearing in the fractional operator is 1, namely, when

$$\left[1-y\left[\frac{\partial}{\partial x}\right]_q\right]^\nu x^n=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu-1}\underbrace{e_q^{yt\left[\frac{\partial}{\partial x}\right]_q}x^n}d_qt4in$$

for the value of the under-brace we use the eq. (1.7), we have

$$\left[1-y\left[\frac{\partial}{\partial x}\right]_q\right]^\nu x^n=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-t}t^{\nu-1}(x+yt)^n d_qt,4in(2.20)$$

it is evident that the corresponding polynomial

$${}_\nu H_n^{(1)}(x,y;q)=[n]_q!\sum_{r=0}^{n}\frac{(q^\nu;q)_r y^r x^{n-r}}{(1-q)^r[r]_q![n-r]_q!}=[n]_q!\sum_{r=0}^{n}\frac{(q^\nu;q)_r y^r x^{n-r}}{(1-q)^r[r]_q![n-r]_q!}.4in(2.21)$$

is an umbral image of the ordinary binomials. The above example provides an indication of the implications offered by the method we have proposed. In the next concluding section, we will discuss further examples aimed at proving that the combined use of operational rules and integral representations may provide unsuspected links between apparently disconnected fields.

3. The Remarks

One of the advantages offered by the use of the q-integral representation in dealing with q-differential operators is the possibility of giving a meaning to apparently meaningless operations. This is indeed the case of the Riemann– Liouville definition of fractional derivative (see [14]; see also [15, Chapter 5]).

By taking the advantage from the definition of the Euler Γ_q function (see[9], [10]) and use of the identity (see[3, p. 153, eq. (12)]) allows to determine the q-analogue of the results obtained by Dattoli et. al [3], we conclude that

$$\left[x\left[\frac{d}{dx}\right]_q\right]^{-\nu}f(x)=\frac{1}{\Gamma_q(\nu)}\int_0^\infty t^{\nu-1}\underbrace{e_q^{-xt\left[\frac{d}{dx}\right]_q}f(x)}d_qt4in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty t^{\nu-1}f(e_q^{-t+\ln_q(x)})d_qt5in$$

or

$$\left[x\left[\frac{d}{dx}\right]_q\right]^{-\nu}f(x)=\frac{1}{\Gamma_q(\nu)}\int_0^\infty t^{\nu-1}f(xe_q^{-t})d_qt.4in(3.1)$$

If we set the q-analogue of $\frac{1}{(1-x)^a}$ in place of $f(x)$ in eq. (3.1) we find that

$$\left[x\left[\frac{d}{dx}\right]_q\right]^{-\nu}\frac{1}{(1-x)_q^a}=\frac{1}{\Gamma_q(\nu)}\int_0^\infty t^{\nu-1}e_q^{-xt\left[\frac{d}{dx}\right]_q}\frac{1}{(1-x)_q^a}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty t^{\nu-1}e_q^{-xt\left[\frac{d}{dx}\right]_q}\sum_{n=0}^\infty\frac{(a;q)_n}{(q;q)_n}x^n d_qt=\frac{1}{\Gamma_q(\nu)}\int_0^\infty t^{\nu-1}\sum_{n=0}^\infty\frac{(a;q)_n}{(q;q)_n}\underbrace{e_q^{-xt\left[\frac{d}{dx}\right]_q}x^n}d_qt1.3in$$

use of the eq. (3.1), yields

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty t^{\nu-1}\sum_{n=0}^\infty\frac{(a;q)_n}{(q;q)_n}(xe_q^{-t})^n d_qt=\frac{1}{\Gamma_q(\nu)}\sum_{n=0}^\infty\frac{(a;q)_n}{(q;q)_n}x^n\int_0^\infty t^{\nu-1}e_q^{-nt}d_qt2in$$

$$=\frac{1}{\Gamma_q(\nu)}\sum_{n=0}^\infty\frac{(a;q)_n}{(q;q)_n}\frac{\Gamma_q(\nu)}{[n]^\nu}x^n=\sum_{n=0}^\infty\frac{(a;q)_n}{(q;q)_n}\frac{x^n}{[n]^\nu}1.3in$$

or

$$\left[x\left[\frac{d}{dx}\right]_q\right]^{-\nu}\left(\frac{1}{1-x}\right)_q=\sum_{n=0}^\infty\frac{(a;q)_n}{(q;q)_n}\frac{x^n}{[n]^\nu}=\zeta(x,\nu),\quad |x|<1.3in(3.2)$$

Clearly, the fractional forms of the operator $x[ddx]_q$ can be used as an operational definition of the Riemann $\zeta-$ function [1]. Next example in this direction is provided by

$$\left[\alpha+x\left[\frac{\partial}{\partial x}\right]_q\right]^{-\nu}f(x)=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}\underbrace{e_q^{-tx\left[\frac{\partial}{\partial x}\right]_q}f(x)}d_qt3in$$

or

$$\left[\alpha+x\left[\frac{\partial}{\partial x}\right]_q\right]^{-\nu}f(x)=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}f(xe_q^{-t})d_qt.3in(3.3)$$

and if $f(x)=e_q^x$, we end up with

$$\left[\alpha+x\left[\frac{\partial}{\partial x}\right]_q\right]^{-\nu}e_q^x=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}\underbrace{e_q^{-tx\left[\frac{\partial}{\partial x}\right]_q}e_q^x}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}e_q^{xe_q^{-t}}d_qt=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}\sum_{n=0}^\infty\frac{x^n e_q^{-nt}}{[n]_q!}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}\sum_{n=0}^\infty\frac{x^n}{[n]_q!}\int_0^\infty t^{\nu-1}e_q^{-\alpha t}e_q^{-nt}d_qt=\frac{1}{\Gamma_q(\nu)}\sum_{n=0}^\infty\frac{x^n}{[n]_q!}\int_0^\infty t^{\nu-1}e_q^{-(\alpha+n)t}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}\sum_{n=0}^\infty\frac{x^n}{[\alpha+n]^\nu[n]_q!}\Gamma_q(\nu)=\sum_{n=0}^\infty\frac{x^n}{[\alpha+n]^\nu[n]_q!}3in$$

or

$$\left[\alpha+x\left[\frac{\partial}{\partial x}\right]_q\right]^{-\nu}e_q^x=\sum_{n=0}^\infty\frac{x^n}{[\alpha+n]^\nu[n]_q!}.3in(3.4)$$

eq. (3.4) suggests further consequences too. By replacing $f(x)$ with $L_n(x,y;q)$ in eq. (3.3), we find that

$$\left[\alpha+x\left[\frac{\partial}{\partial x}\right]_q\right]^{-\nu}L_n(x,y;q)=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}e_q^{-tx\left[\frac{\partial}{\partial x}\right]_q}L_n(x,y;q)d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}[n]_q!\sum_{r=0}^n\frac{(-1)^r y^{n-r}}{([r]_q!)^2[n-r]_q!}\underbrace{e_q^{-tx\left[\frac{\partial}{\partial x}\right]_q}x^r}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}[n]_q!\sum_{r=0}^n\frac{(-1)^r(xe_q^{-t})^r y^{n-r}}{([r]_q!)^2[n-r]_q!}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}[n]_q!\sum_{r=0}^n\frac{(-1)^r x^r y^{n-r}}{([r]_q!)^2[n-r]_q!}\int_0^\infty e_q^{-(\alpha+r)t}t^{\nu-1}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}[n]_q!\sum_{r=0}^n\frac{(-1)^r x^r y^{n-r}}{([r]_q!)^2[n-r]_q!}\frac{\Gamma_q(\nu)}{[\alpha+r]^\nu}=[n]_q!\sum_{r=0}^n\frac{(-1)^r x^r y^{n-r}}{[\alpha+r]^\nu([r]_q!)^2[n-r]_q!}3in$$

or

$$\left[\alpha+x\left[\frac{\partial}{\partial x}\right]_q\right]^{-\nu}L_n(x,y;q)=[n]_q!\sum_{r=0}^n\frac{(-1)^r x^r y^{n-r}}{[\alpha+r]^\nu([r]_q!)^2[n-r]_q!}.3in(3.5)$$

Denoting the polynomial $L_n(x,y;\alpha,\nu;q)$ on the right hand side of the above eq. (3.5) we find the following recurrences

$$\left[\frac{\partial}{\partial y}\right]_q L_n(x,y;\alpha,\nu;q)=[n]_q!\sum_{r=0}^{n-1}\frac{(-1)^r x^r y^{n-1-r}}{[\alpha+r]^\nu([r]_q!)^2[n-1-r]_q}4in$$

$$=[n]_q[n-1]_q!\sum_{r=0}^{n-1}\frac{(-1)^r x^r y^{n-1-r}}{[\alpha+r]^\nu([r]_q!)^2[n-1-r]_q!}=[n]_qL_{n-1}(x,y;\alpha,\nu;q)4in$$

or

$$\left[\frac{\partial}{\partial y}\right]_q L_n(x,y;\alpha,\nu;q)=[n]_qL_{n-1}(x,y;\alpha,\nu;q)4in(3.6)$$

and

$$-\left[\frac{\partial}{\partial x}\right]_q x\left[\frac{\partial}{\partial x}\right]_q L_n(x,y;\alpha,\nu;q)=-\left[\frac{\partial}{\partial x}\right]_q x\left[\frac{\partial}{\partial x}\right]_q[n]_q!\sum_{r=0}^n\frac{(-1)^r x^r y^{n-r}}{[\alpha+r]^\nu([r]_q!)^2[n-r]_q!}4in$$

$$=-\left[\frac{\partial}{\partial x}\right]_q[n]_q!\sum_{r=0}^n\frac{(-1)^r[r]_q x^r y^{n-r}}{[\alpha+r]^\nu([r]_q!)^2[n-r]_q!}=-[n]_q!\sum_{r=1}^n\frac{(-1)^r[r]_q^2x^{r-1}y^{n-r}}{[\alpha+r]^\nu([r]_q!)^2[n-r]_q!}4in$$

$$=[n]_q[n-1]_q!\sum_{r=1}^n\frac{(-1)^{r-1}x^{r-1}y^{n-1-r+1}}{[\alpha+[r-1]_q]^\nu([r-1]_q!)^2[n-1-r+1]_q!}4in$$

$$=[n]_q[n-1]_q!\sum_{s=0}^{n-1}\frac{(-1)^s x^s y^{n-1-s}}{[\alpha+[s]_q]^\nu([s]_q!)^2[n-1-s]_q!}4in$$

or

$$-\left[\frac{\partial}{\partial x}\right]_q x\left[\frac{\partial}{\partial x}\right]_q L_n(x,y;\alpha,\nu;q)=[n]_qL_{n-1}(x,y;\alpha+\nu;q).4in(3.7)$$

Again by replacing $f(x)$ in the similar fashion as above with $H_n(x,y;q)$ in eq. (3.3), we find that

$$\left[\alpha+x\left[\frac{\partial}{\partial x}\right]_q\right]^{-\nu}H_n(x,y;q)=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}e_q^{-tx\left[\frac{\partial}{\partial x}\right]_q}H_n(x,y;q)d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r}{[r]_q![n-2r]_q!}\underbrace{e_q^{-tx\left[\frac{\partial}{\partial x}\right]_q}x^{n-2r}}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rx^{n-2r}}{[r]_q![n-2r]_q!}\int_0^\infty e_q^{-\alpha t}t^{\nu-1}e_q^{-(n-2r)t}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rx^{n-2r}}{[r]_q![n-2r]_q!}\int_0^\infty e_q^{-(\alpha+n-2r)t}t^{\nu-1}d_qt3in$$

$$=\frac{1}{\Gamma_q(\nu)}[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rx^{n-2r}}{[r]_q![n-2r]_q!}\frac{\Gamma_q(\nu)}{[\alpha+n-2r]^\nu}3in$$

or

$$\left[\alpha+x\left[\frac{\partial}{\partial x}\right]_q\right]^{-\nu}H_n(x,y;q)=[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rx^{n-2r}}{[\alpha+n-2r]^\nu[r]_q![n-2r]_q!}.4in(3.8)$$

Denoting by $H_n(x,y;\alpha,\nu;q)$ the polynomial defined on the right hand side of eq. (3.8), we find the recurrences

$$\left[\frac{\partial}{\partial x}\right]_q H_n(x,y;\alpha,\nu;q)=\left[\frac{\partial}{\partial x}\right]_q\left[[n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rx^{n-2r}}{[\alpha+[n-2r]_q]^\nu[r]_q![n-2r]_q!}\right]4in$$

$$=[n]_q\left[[n-1]_q!\sum_{r=0}^{[\frac{n-1}{2}]}\frac{y^r x^{n-1-2r}}{[\alpha+[n-1-2r]_q]^{\nu}[r]_q![n-1-2r]_q!}\right]4in$$

or

$$\left[\frac{\partial}{\partial x}\right]_q H_n(x,y;\alpha,\nu;q)=[n]_q H_{n-1}(x,y;\alpha,\nu;q)4in(3.9)$$

and

$$\left[\frac{\partial}{\partial y}\right]_q H_n(x,y;\alpha,\nu;q)=\left[\frac{\partial}{\partial y}\right]_q [n]_q!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{[\alpha+[n-2r]_q]^{\nu}[r]_q![n-2r]_q!}4in$$

$$=[n]_q!\sum_{r=1}^{[\frac{n}{2}]}\frac{y^{r-1}x^{n-2r}}{[\alpha+[n-2r]_q]^{\nu}[r-1]_q![n-2r]_q!}4in$$

$$=[n]_q[n-1]_q[n-2]_q!\sum_{r=0}^{[\frac{n-2}{2}]}\frac{y^r x^{n-2-2r}}{[\alpha+[n-2-2r]_q]^{\nu}[r]_q![n-2-2r]_q!}4in$$

or

$$\left[\frac{\partial}{\partial y}\right]_q H_n(x,y;\alpha,\nu;_q)=[n]_q[n-1]_q H_{n-2}(x,y;\alpha,\nu;q).3in(3.10)$$

We consider, therefore, the differential equations of the type

$$\sqrt{(\alpha)^2\left[\frac{d^2}{dx^2}\right]_q+1}f(x)=S(x)3in(3.11)$$

where $S(x)$ denotes a known function.

The formal solution of eq. (3.11) can be cast in the following form

$$f(x)=\frac{1}{\sqrt{(\alpha)^2\left[\frac{d^2}{dx^2}\right]_q+1}}S(x).3in(3.12)$$

By recalling from the theory of Laplace transforms (see [7])

$$\frac{1}{\sqrt{A^2+1}}=\int_0^{\infty}J_0(t)e_q^{-At}d_qt,3in(3.13)$$

and on replacing A with $\alpha[ddx]_q$, we find that

$$\frac{1}{\sqrt{(\alpha\left[\frac{d}{dx}\right]_q)^2+1}}=\int_0^{\infty}J_0(t)e_q^{-\alpha t\left[\frac{d}{dx}\right]_q}d_qt,3in$$

or

$$\frac{1}{\sqrt{(\alpha\left[\frac{d}{dx}\right]_q)^2+1}}S(x)=\int_0^{\infty}J_0(t)e_q^{-\alpha t\left[\frac{d}{dx}\right]_q}S(x)d_qt=\int_0^{\infty}J_0(t)S(x-\alpha t)d_qt,$$

i.e.

$$f(x)=\int_0^{\infty}J_0(t)e_q^{-\alpha t\left[\frac{d}{dx}\right]_q}S(x)d_qt=\int_0^{\infty}J_0(t)S(x-\alpha t)d_qt.1in(3.14)$$

The solution of eq. (3.11) in the form (3.14) holds only if the q-integral is convergent and can be viewed as a kind of convolution of $S(x)$ on the 0^{th} -order cylindrical Bessel function.

As a final example, we will consider the solution of the fractional diffusive equation

$$\begin{cases}\left[\frac{\partial}{\partial y}\right]_q f(x,y)=-\left[\frac{\partial^{1/2}}{\partial x^{1/2}}\right]_q f(x,y),\\ f(x,0)=g(x).\end{cases}\quad 3in(3.16)$$

Which can be treated using an identity valid within the framework of the Laplace transform theory [7], that can be written in the following form

$$e_q^{-y\sqrt{d}}=\frac{y}{2\sqrt{\pi}}\int_0^{\infty}\frac{e_q^{-\frac{y^2}{4t}-td}}{t\sqrt{t}}d_qt=\frac{y}{2\sqrt{\pi}}\int_0^{\infty}\frac{e_q^{-\frac{y^2}{4t}}e_q^{-td}}{t\sqrt{t}}d_qt.(3.17)$$

By replacing d with $\left[\frac{\partial}{\partial x}\right]_q$ and by proceeding as before, we find the solution of eq. (3.17) as

$$e_q^{-y\left[\frac{\partial}{\partial x}\right]_q^{1/2}}g(x)=\frac{y}{2\sqrt{\pi}}\int_0^{\infty}\frac{e_q^{-\frac{y^2}{4t}}}{t\sqrt{t}}e_q^{-t\left[\frac{\partial}{\partial x}\right]_q}g(x)d_qt=\frac{y}{2\sqrt{\pi}}\int_0^{\infty}\frac{e_q^{-\frac{y^2}{4t}}}{t\sqrt{t}}g(x-t)d_qt.(3.18)$$

This result can be viewed as the q-analogue of the Gauss transform for the solution of the heat diffusion equation. In this paper it has been shown that q - analogue of operational method can provide a fairly useful tool to solve a large number of problems including fractional propagation equations.

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