Table 1: Rigid body dynamic parameters for linear and rotational motion

Parameter	*Linear motion*	*Angular (rotational) motion*
Position	Linear position is presented by the radius-vector of the center of mass $\mathbf{r}_c$. All linear variables are given with respect to the space reference frame, while the origin of the body reference frame coincides with the center of mass $\mathbf{r}_c = \overrightarrow{OO'}$.	Angular position or orientation is expressed by the rotation matrix $\mathbf{R}$ or any of its reduction derivatives, such as Euler angles, rotation quaternion, etc. The rotation matrix connects the space and the body reference frames by the *forward frame transform operator* $\mathbf{X}'\mathbf{R} = \mathbf{X}$ and the *reverse frame transform operator* $\mathbf{X}\mathbf{R}^{\sim} = \mathbf{X}'$.
Velocity	Linear velocity vector of the center of mass $\mathbf{v}_c = \dot{\mathbf{r}}_c$.	Angular velocity vector $\boldsymbol{\omega}$, where $\boldsymbol{\omega}^* = \dot{\mathbf{R}}^{\sim}\mathbf{R}$ [3] and $\boldsymbol{\omega} = \begin{bmatrix}\omega^*_{32} & \omega^*_{13} & \omega^*_{21}\end{bmatrix}$. In the body reference frame the angular velocity is $\boldsymbol{\omega}' = \boldsymbol{\omega}\mathbf{R}^{\sim}$.
Acceleration	Linear acceleration vector of the center of mass $\mathbf{a}_c = \ddot{\mathbf{r}}_c$.	Angular acceleration vector $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$ or $\boldsymbol{\varepsilon}^* = \ddot{\mathbf{R}}^{\sim}\mathbf{R} + \dot{\mathbf{R}}^{\sim}\dot{\mathbf{R}}$ [3] and $\boldsymbol{\varepsilon} = \begin{bmatrix}\varepsilon^*_{32} & \varepsilon^*_{13} & \varepsilon^*_{21}\end{bmatrix}$. In the body reference frame the angular acceleration is $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\mathbf{R}^{\sim}$.
Inertia	Total mass of the rigid body $m = \iiint_V \rho(\mathbf{r})dV =$ $\iiint_V \rho(x,y,z)dxdydz$ Note: $\mathbf{r} = \begin{bmatrix}x & y & z\end{bmatrix}$.	Moment of inertia tensor in the body reference frame $\mathbf{I}' = \begin{bmatrix} I'_{xx} & I'_{xy} & I'_{xz} \\ I'_{xy} & I'_{yy} & I'_{yz} \\ I'_{xz} & I'_{yz} & I'_{zz} \end{bmatrix} = \iiint_{V'} \left(r'^2\mathbf{1} - \mathbf{r}' \otimes \mathbf{r}'\right)dm =$ $= -\iiint_{V'} \mathbf{r}'^{*2} \rho(\mathbf{r}')dV' = -\iiint_{V'} \mathbf{r}'^{*2} \rho(x',y',z')dx'dy'dz'$ $\mathbf{r}'^{*2}$ is a symmetric matrix and its integral is also a symmetric matrix. Hence, tensor $\mathbf{I}'$ is a symmetric matrix and has only six degrees of freedom. In the space reference frame, the moment of inertia tensor is $\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} = \mathbf{R}^{\sim}\mathbf{I}'\mathbf{R}$. Here $\mathbf{I} = \mathbf{R}^{\sim}\mathbf{I}'\mathbf{R}$ and $\mathbf{I}' = \mathbf{R}\mathbf{I}\mathbf{R}^{\sim}$ are *similarity transformations*. Note that, while $\mathbf{I}'$ is constant, $\mathbf{I}$ depends on the current body orientation. It is also obvious that $\mathbf{I} = \mathbf{R}^{\sim}\mathbf{I}'\mathbf{R} = -\iiint_{V'} \mathbf{R}^{\sim}\mathbf{r}'^{*}\mathbf{R}\mathbf{R}^{\sim}\mathbf{r}'^{*}\mathbf{R}\rho(\mathbf{r}')dV' =$ $= -\iiint_{V} \mathbf{r}^{*2} \rho(\mathbf{r})dV$ Matrix $\mathbf{I}$ is also symmetric, because rotation preserves symmetry and anti-symmetry. The moment of inertia tensor $\mathbf{I}$ is a tensor of second rank that relates vector $\mathbf{L}$ to vector $\boldsymbol{\omega}$ (see below). $\mathbf{I}$ is equivalent to a 3x3 matrix.
Momentum	Linear momentum of the center of mass $\mathbf{p}_c = m\mathbf{v}_c$.	Angular momentum. Its differential form is $d\mathbf{L}(\mathbf{r}) = \mathbf{r}d(\mathbf{p}(\mathbf{r}))^* = \mathbf{r}(\mathbf{v}(\mathbf{r})dm)^* = \mathbf{r}\left(\boldsymbol{\omega}\mathbf{r}^* dm\right)^* = -\boldsymbol{\omega}\mathbf{r}^{*2}dm$ and its integral form is

		$\mathbf{L}=\iiint_V d\mathbf{L}(\mathbf{r})=-\iiint_V \boldsymbol{\omega}\mathbf{r}^{*2}dm=-\boldsymbol{\omega}\iiint_V \mathbf{r}^{*2}\rho(\mathbf{r})dV=\boldsymbol{\omega}\mathbf{I}$. In the body reference frame the angular momentum is $\mathbf{L}'=\mathbf{L}\mathbf{R}^{\sim}=\boldsymbol{\omega}\mathbf{I}\mathbf{R}^{\sim}=\boldsymbol{\omega}\mathbf{R}^{\sim}\mathbf{R}\mathbf{I}\mathbf{R}^{\sim}=\boldsymbol{\omega}'\mathbf{I}'$.
Force	Force $\sum \mathbf{F}_{ext}=m\mathbf{a}_c=m\dot{\mathbf{v}}_c=\dot{\mathbf{p}}_c$. The sum of all external forces, applied to the rigid body, changes the linear momentum of the center of mass in respect to time as follows: $\dot{\mathbf{p}}_c=\sum \mathbf{F}_{ext}$ or $\Delta\mathbf{p}_c=\int_0^{\Delta t}\sum \mathbf{F}_{ext}dt$.	Torque (moment of force) $\boldsymbol{\tau}=\mathbf{r}\mathbf{F}^{*}$. The sum of all external torques, applied to the rigid body, changes the angular momentum in respect to time as follows: $\dot{\mathbf{L}}=\sum \boldsymbol{\tau}_{ext}=\sum \mathbf{r}\mathbf{F}_{ext}^{*}$ or $\Delta\mathbf{L}=\int_0^{\Delta t}\sum \boldsymbol{\tau}_{ext}dt=\int_0^{\Delta t}\sum \mathbf{r}\mathbf{F}_{ext}^{*}dt$. In the body reference frame the torque is $\boldsymbol{\tau}'=\boldsymbol{\tau}\mathbf{R}^{\sim}$.
Kinetic energy	Kinetic energy of the linear motion $E_K=\frac{mv_c^2}{2}$.	Kinetic energy of the rotational motion $E_{Krot}=\iiint_V \frac{(dm)(v_{rot}(\mathbf{r}))^2}{2}=\frac{1}{2}\iiint_V \boldsymbol{\omega}\mathbf{r}^{*}(\boldsymbol{\omega}\mathbf{r}^{*})^{\sim}\rho(\mathbf{r})dV=$ $=-\frac{1}{2}\iiint_V \boldsymbol{\omega}\mathbf{r}^{*2}\boldsymbol{\omega}^{\sim}\rho(\mathbf{r})dV=\frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2}=\frac{\mathbf{n}\mathbf{I}\mathbf{n}^{\sim}\omega^2}{2}=\frac{I\omega^2}{2}$ Note: vector $\mathbf{n}=\frac{\boldsymbol{\omega}}{\omega}$ and $I=\mathbf{n}\mathbf{I}\mathbf{n}^{\sim}$.

4. Benefits

The student benefits from the rigid body motion table in matrix form in a number of situations. Here, some of these situations are disclosed in order to unveil the erroneous approach of presenting the rigid body motion table using only vector notation.

Example 1. Particle Kinematics in Inertial and Non-inertial Reference Frames

We shall examine the equations describing particle kinematics using matrix formalism instead of vector formalism. Let O be the inertial and not moving reference frame and O' be a non-inertial, moving and rotating reference frame (Figure 2.). All quantities in regard to the inertial reference frame are non-primed, while all quantities in regard to the non-inertial reference frame are primed.

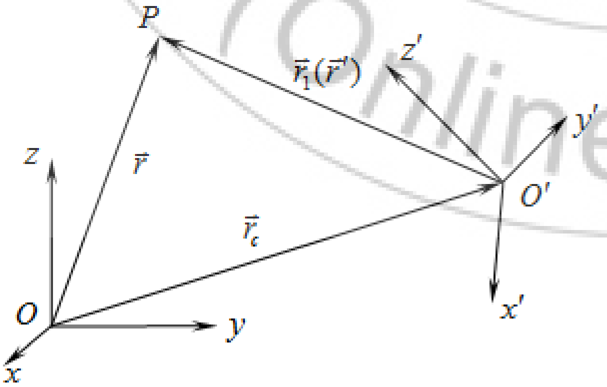

Figure 2: Transformation between frames O and O'.

We want to describe how the kinematic quantities of a particle in regard to the inertial reference frame depend on the kinematic quantities of this particle in regard to the non-inertial reference frame. I.e. we want to make the relation between non-primed and primed quantities. This relation is based on the kinematic quantities of the non-inertial reference frame in regard to the inertial reference frame. The non-inertial reference frame has position and orientation ($\mathbf{r}_C$, $\mathbf{R}$) in regard to the inertial reference frame. The latter quantities may be called linear and angular positions of the non-inertial reference frame. For vector $\mathbf{r}$ and its derivatives we have that:

(13.) $\mathbf{r}=\mathbf{r}_C+\mathbf{r}_1=\mathbf{r}_C+\mathbf{r}'\mathbf{R}$

(14.) $\dot{\mathbf{r}}=\dot{\mathbf{r}}_C+\dot{\mathbf{r}}_1=\mathbf{v}_C+\mathbf{v}'\mathbf{R}+\mathbf{r}'\dot{\mathbf{R}}$

(15.) $$\ddot{\mathbf{r}}=\ddot{\mathbf{r}}_C+\ddot{\mathbf{r}}_1=\mathbf{a}_C+\dot{\mathbf{v}}'\mathbf{R}+\mathbf{v}'\dot{\mathbf{R}}+\dot{\mathbf{r}}'\dot{\mathbf{R}}+\mathbf{r}'\ddot{\mathbf{R}}=$$
$$\mathbf{a}_C+\mathbf{a}'\mathbf{R}+2\mathbf{v}'\dot{\mathbf{R}}+\mathbf{r}'\ddot{\mathbf{R}}$$

The derivative and double derivative of the rotation matrix are found in [3]. Using (14.) and (15.) we obtain:
(16.)

$$\mathbf{v}=\dot{\mathbf{r}}=\dot{\mathbf{r}}_C+\dot{\mathbf{r}}_1=\mathbf{v}_C+\mathbf{v}'\mathbf{R}+\mathbf{r}'\dot{\mathbf{R}}=\mathbf{v}_C+\mathbf{v}'\mathbf{R}+\mathbf{r}'\mathbf{R}\boldsymbol{\omega}^{*\sim}$$

(17.) $$\mathbf{a}=\ddot{\mathbf{r}}=\mathbf{a}_C+\mathbf{a}'\mathbf{R}+2\mathbf{v}'\mathbf{R}\boldsymbol{\omega}^{*\sim}+\mathbf{r}'\ddot{\mathbf{R}}=$$
$$\mathbf{a}_C+\mathbf{a}'\mathbf{R}+2\mathbf{v}'\mathbf{R}\boldsymbol{\omega}^{*\sim}+\mathbf{r}'\mathbf{R}\boldsymbol{\omega}^{*\sim}\boldsymbol{\omega}^{*\sim}+\mathbf{r}'\mathbf{R}\boldsymbol{\varepsilon}^{*\sim}$$

The terms in (17.) are the different accelerations that appear when a particle is moving in a non-inertial reference frame. Let the student recognize how these accelerations are called.

Example 2. Free Rigid Body Motion.

The free rigid body motion will be observed as an example. Notably, the free rigid body motion is comprehended with difficulty by students. Here we show that the most difficulties come from the inadequate presentation of variables using only vector mathematical notation. The difficulties are easily overcome using Table 1. variables in matrix form.

The first invariant of free rigid body motion is the angular momentum vector $\mathbf{L}$. The student can easily find the relation between the angular momentum vector and the angular velocity vector $\boldsymbol{\omega}$ using Table 1.:

(18.) $\mathbf{L} = \boldsymbol{\omega}\mathbf{I}$ and $\mathbf{L}' = \boldsymbol{\omega}'\mathbf{I}'$

(19.) $\boldsymbol{\omega} = \mathbf{L}\mathbf{I}^{-1}$ and $\boldsymbol{\omega}' = \mathbf{L}'\mathbf{I}'^{-1}$

If the body reference frame is chosen along the principal axis of inertia, the moment of inertia tensor transforms to a diagonal matrix

(20.) $$\mathbf{I}' = \begin{bmatrix} I'_{xx} & 0 & 0 \\ 0 & I'_{yy} & 0 \\ 0 & 0 & I'_{zz} \end{bmatrix} = const$$

Its inverse is also diagonal and has the simple form of

(21.)

$$\mathbf{I}'^{-1} = \begin{bmatrix} I'^{-1}_{xx} & 0 & 0 \\ 0 & I'^{-1}_{yy} & 0 \\ 0 & 0 & I'^{-1}_{zz} \end{bmatrix} = \begin{bmatrix} 1/I'_{xx} & 0 & 0 \\ 0 & 1/I'_{yy} & 0 \\ 0 & 0 & 1/I'_{zz} \end{bmatrix} = const$$

The second invariant in free rigid body motion is the kinetic energy of rotational motion E_{Krot}. This conservation parameter leads to the following constraint of movement (Table 1.):

(22.)

$$E_{Krot} = \frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\boldsymbol{\omega}\mathbf{R}^{\sim}\mathbf{R}\mathbf{I}\mathbf{R}^{\sim}\mathbf{R}\boldsymbol{\omega}^{\sim}}{2} = \frac{\boldsymbol{\omega}'\mathbf{I}'\left(\boldsymbol{\omega}\mathbf{R}^{\sim}\right)^{\sim}}{2} = \frac{\boldsymbol{\omega}'\mathbf{I}'\boldsymbol{\omega}'^{\sim}}{2} = const$$

Analogously, the above equation can be transformed in respect to the angular momentum:

Here, vector $\mathbf{n_L} = \frac{\mathbf{L}}{L} = \mathbf{n}_{\gamma}$ is the normal vector of an invariable plane γ.

(23.) $$E_{Krot} = \frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\mathbf{L}\mathbf{I}^{-1}\mathbf{I}\left(\mathbf{L}\mathbf{I}^{-1}\right)^{\sim}}{2} = \frac{\mathbf{L}\mathbf{I}^{-1}\mathbf{I}\mathbf{I}^{-1\sim}\mathbf{L}^{\sim}}{2} = \frac{\mathbf{L}\mathbf{I}^{-1}\mathbf{L}^{\sim}}{2} = \frac{\mathbf{L}'\mathbf{I}'^{-1}\mathbf{L}'^{\sim}}{2} = const$$

Evolving these two equations by the vector components and the principal moments of inertia gives:

(24.)
$$\boldsymbol{\omega}'\mathbf{I}'\boldsymbol{\omega}'^{\sim} = \omega'^2_x I'_{xx} + \omega'^2_y I'_{yy} + \omega'^2_z I'_{zz} = 2E_{Krot} = const$$

(25.)
$$\mathbf{L}'\mathbf{I}'^{-1}\mathbf{L}'^{\sim} = L'^2_x I'^{-1}_{xx} + L'^2_y I'^{-1}_{yy} + L'^2_z I'^{-1}_{zz} = 2E_{Krot} = const$$

These are the equations of two ellipsoids which are static (invariant) in the body reference frame. These two ellipsoids depend solely on the inertial properties of the rigid body and will be denoted with $\varepsilon_{\boldsymbol{\omega}}$ and $\varepsilon_{\mathbf{L}}$ respectively. If the rotational kinetic energy is to stay constant, both vectors $\boldsymbol{\omega}'$ and $\mathbf{L}'$ are constrained to point on the surface of the two ellipsoids defined by equations (24.) and (25.) respectively. If either of vectors $\boldsymbol{\omega}'$ or $\mathbf{L}'$ points outside of its constraint ellipsoid, E_{Krot} will increase. Analogously, if either vector points inside its constraint ellipsoid, E_{Krot} will decrease.

By implementing the $\mathbf{L}$ constraint in the E_{Krot} constraint, one observes that:

(26.) $$E_{Krot} = \frac{\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\mathbf{L}\mathbf{I}^{-1}\mathbf{I}\boldsymbol{\omega}^{\sim}}{2} = \frac{\mathbf{L}\boldsymbol{\omega}^{\sim}}{2} = \frac{\boldsymbol{\omega}\mathbf{L}^{\sim}}{2} = const \Rightarrow \boldsymbol{\omega}\mathbf{L}^{\sim} = 2E_{Krot} = const$$

In other words, $\boldsymbol{\omega}\mathbf{L}^{\sim} = \omega L\cos\alpha = 2E_{Krot}$, but $\mathbf{L} = const \Rightarrow \omega\cos\alpha = \frac{2E_{Krot}}{L} = const$. The projection vector $\boldsymbol{\omega}_{\mathbf{L}}$ of vector $\boldsymbol{\omega}$ in the direction of vector $\mathbf{L}$ is always the same (Figure 3.):

$$\boldsymbol{\omega}_{\mathbf{L}} = \frac{\boldsymbol{\omega}\mathbf{L}^{\sim}\mathbf{L}}{|\mathbf{L}|^2} = \frac{\omega L\cos\alpha}{L^2}\mathbf{L} = \frac{\omega\cos\alpha}{L}\mathbf{L} = \omega\cos\alpha\,\mathbf{n_L} = \frac{2E_{Krot}}{L}\mathbf{n_L}$$

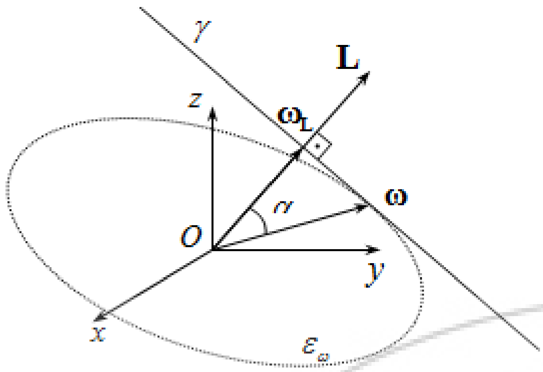

Figure 3: Relation between angular velocity and angular momentum in free rigid body motion.

5. Future Scope

Authors intend to conduct experiments with students in General physics, Analytical mechanics and Theoretical mechanics at different universities. The experiments would prove the effectiveness of the proposed systematic formalization for learning the rigid body dynamics. The understanding of the mathematical formalism and the physical and real phenomena by students using this table is expected to improve. This supposition should be proven by the planned experimental tests.

6. Conclusions

Understanding rigid body motion and manipulating the variables describing it may be harmed severely if only vector mathematical formalism is used. By applying matrix formalism the student is liberated from such burdens and is allowed to easily and clearly formalize and describe the studied phenomena. Solving problems becomes a systematic and pleasant pursuit. Furthermore, physical laws become consistently formalized, and formulas: easily derived from one another.

References

[1] Zabunov, S. (2010). Rigid body motion in stereo 3D simulation. Eur. J. Phys., 31, 1345
[2] Goldstein, H., Pool, C., and Safko, J. (2001). Classical mechanics third edition. Addison Wesley Miami
[3] Zabunov, S. (2010). Mathematical Methods in E-Learning Mechanics for Graduates – Rotation Matrix. Physics Education Journal, Vol. 27, Issue 4, pp. 221, (2010).
[4] Arnold, V. I. (1989). Mathematical Methods of Classical Mechanics Second Edition. Springer-Verlag

Author Profile

Svetoslav Zabunov is working at Space Research and Technology Institute at the Bulgarian Academy of Sciences. He is interested in theoretical mechanics, 3D simulations of physical phenomena, spacecraft and aircraft design.

Petar Getsov is professor at Space Research and Technology Institute at the Bulgarian Academy of Sciences and is director of the institute. He has a large experience in unmanned aircraft design, spacecraft design, analog electronics, analog computers, etc.

Maya Gaydarova is an associate professor at Sofia University, Bulgaria. She is head of Methodology of Physics Department at the Physics Faculty. She is specialized in tutoring and e-learning physics and has interests in new methods of tutoring such as 3D physics simulations.