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Common Fixed Point for Intimate Mappings in Banach Space

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Abstract: ***In this paper, common fixed point theorems for intimate mapping in Banach space are obtained. The presented results in this paper generalize and improve the corresponding results of Savita [6].***

Keywords: Intimate mapping, compatible mapping of type (A).

1. Introduction

The concept of compatible maps of type (A) introduced by Jungck, Murthy and Cho [2] in metric and Banach space by motivating the concept of compatible maps see Jungck [3] . Recently Sahu, Dhagat and Shrivastava [5] generalized the concept of intimate mapping in metric space. In this paper, we present a common fixed point theorem for intimate mappings in Banach space.

Definition 1

Let A and S be mapping of Banach space X into itself. Then {A, S} is said to be compatible pair of type (A) if

$\lim_{n\to\infty} \| ASx_n - SSx_n \| = \lim_{n\to\infty} \|SAx_n - AAx_n\| = 0$
whenever {xn} is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Definition 2

Let A and S be mapping of Banach space X into itself. Then {A, S} is said to be S-intimate if and only if

$\alpha\| SAx_n - Sx_n \| \le \alpha\|ASx_n - Ax_n\|$,

where $\alpha = \lim_{\sup}$ or $\lim_{\inf}$, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

__Example__: Let X = [0,1] and A, S are self mappings on X defined as follows $Ax = 2/(x+2)$ and $Sx = 1/(1+x)$ for all x in [0, 1]. Now the sequence $\{x_n\}$ in X defined by $x_n = 1/n$, $n\in N$. Then we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$.

Again $\|ASx_n - Ax_n\| \to 1/3$ and $\|SSx_n - Sx_n\| \to 1/2$ as $n \to\infty$. The clearly, we have

$\lim_{n\to\infty} \| ASx_n - Ax_n \| \le \lim_{n\to\infty} \|SSx_n - Sx_n\|$, . This {A, S} is A-intimate.

But {A, S} is not compatible mapping of type (A).

__**Preposition 1:**__ If the pair {A, S} is compatible of type (A) then it is both A and S-intimate.

Proof: Since
$\| ASx_n - Ax_n\| \le \| ASx_n - SSx_n \| + \| SSx_n - Sx_n \|$

For $n \in N$ therefore,

$\propto\| ASx_n - Ax_n \| \le \propto \| ASx_n - SSx_n\| + \propto \| SSx_n - Sx_n \|$
$\propto \| ASx_n - Ax_n \| \le \propto \| SSx_n - Sx_n \|$

whenever $\{x_n\}$ is a sequence in Banach space X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t\in X$.

Thus the pair {A, S} is A – intimate.

Similarly we have show that the pair {A, S} is S – intimate. But its converse need not be true. See example.

__**Preposition 2:**__ Let A and S be a mappings of Banach space X into itself. If the pair {A, S} is S – intimate and $At = St = p \in X$,

Then $\| Sp - p \| \le \| Ap - p \|$.

Proof: Suppose $x_n = t$ for all $n \ge 1$, so $Ax_n = Sx_n = At = St = p$. Since the pair { A, S } is S – intimate, then

$\| SAt - St \| = \lim_{n\to\infty} \| SAx_n - Sx_n \|$
$\le \lim_{n\to\infty} \| AAx_n - Ax_n \|$
$= \| AAt - At \|$
$\Rightarrow \| Sp - p \| \le \| Ap - p \|$

LEMMA 1: (Singh and Meade 1977). For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n (t) = 0$ where γ^n denotes the n times composition of γ.

We now suppose that A, B. S and T be the mapping from Banach space X into itself such that

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$
(2) $\| Ax - By \| \le c \max \{ \|Sx - Ty \| , \|Sx - Ax \|, \| Ty - By \|, [\| Sx - By \| + \|Ty - Ax \|]/2\}$, for all x, y in X and $0 < c < 1$.

Let x_0 be an arbitrary point in X. Then by (1) there exists a point x_1 in X such that $Ax_0 = Tx_1$ and then a point x_2 in X such that $Bx_1 = Sx_2$ and soon, we obtain a sequence $\{y_n\}$ in X such that

(3) $\begin{cases} y_{2n} = Sx_{2n} = Bx_{2n-1} \\ y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ for } n = 1, 2, 3,\ldots \end{cases}$

Now prove the following lemma:

LEMMA 2: suppose that A, B. S and T be the mapping from Banach space X into itself satisfying condition (1) and (2). Then the sequence $\{y_n\}$ as defined by (3) is a Cauchy sequence.

Proof: Using (2) and (3), we get

$\| y_{2n+1} - y_{2n} \| = \| Ax_{2n} - Bx_{2n-1} \|$
$\le c \max \{ \|Sx_{2n} - Tx_{2n-1}\| , \|Sx_{2n} - Ax_{2n}\|,$
$\|Tx_{2n-1} - Bx_{2n-1}\|, [\|Sx_{2n} - Bx_{2n-1}\|$
$+ \|Tx_{2n-1} - Ax_{2n}\|]/2\}$
$= c \max \{ \|y_{2n} - y_{2n-1}\| , \| y_{2n} - y_{2n+1} \|,$
$\| y_{2n-1} - y_{2n} \|, [\| y_{2n} - y_{2n} \|$
$+ \|y_{2n-1} - y_{2n+1}\|]/2\}$
$= c \max \{ \|y_{2n} - y_{2n-1}\| , \| y_{2n} - y_{2n+1} \|,$
$\| y_{2n-1} - y_{2n} \|, [\| y_{2n-1} - y_{2n} \|$
$+ \|y_{2n} - y_{2n+1}\|]/2\}$

If $\| y_{2n+1} - y_{2n} \| > \| y_{2n} - y_{2n-1}\|$

Then $\| y_{2n+1} - y_{2n} \| \le c \| y_{2n+1} - y_{2n} \|$ which is contradiction.

Thus
$\| y_{2n+1} - y_{2n} \| \le c \| y_{2n} - y_{2n-1} \|$

Similarly we have
$\| y_{2n+2} - y_{2n+1} \| \le c \| y_{2n+1} - y_{2n} \|$
Now $\| y_{n+1} - y_n \| \le c \| y_n - y_{n-1} \|$
$\le c^2 \| y_{n-1} - y_{n-2} \| \ldots\ldots$
$\ldots\ldots \le c^n \| y_1 - y_0 \|$

For every integer $p > 0$, we get
$\| y_n - y_{n+p} \| \le \| y_n - y_{n+1} \| + \| y_{n+1} - y_{n+2} \|$
$\ldots\ldots \ldots\ldots \| y_{n+p-1} - y_{n+p} \|$
$\le (1+ c + c^2 + \ldots\ldots\ldots\ldots c^{p-1}) \| y_n - y_{n+1} \|$
$\le \{c^p/(1-c)\} \| y_n - y_{n+1} \| \to 0$ as $n \to\infty$.
Thus $\| y_n - y_{n+p} \| \to 0$.

Therefore $\{ y_n\}$ is a Cauchy sequence. Now, we shall prove our main result:

THEOREM: Let A, B, S and T be mapping for a Banach space X into itself satisfying (1), (2), (3) and following condition:

(4) The pair {A, S} is S – intimate and { B, T } is T – intimate.

(5) S(X) is complete.

Then A, B, S and T have a unique common fixed point in X.
PROOF: Define the sequence $\{ y_n\}$ (as above by lemma 2) is a Cauchy sequence and has a limit u in X. Since S(X) is complete and $\{ Sx_{2n}\}$ is Cauchy sequence. Then it converges to a point p = Su for some u in X . Also $\{ Ax_{2n}\}$, $\{ Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{ Tx_{2n+1}\}$ are subsequence of $\{y_n\}$, this subsequence converge to p. Hence Ax_{2n}, Sx_{2n}, Bx_{2n-1}, Tx_{2n+1} $\to$ p.

Now from (2), we get

$\| Au - Bx_{2n+1} \| \le c \max \{ \|Su - Tx_{2n+1} \| ,$
$\| Su - Au \|, \| Tx_{2n+1} - Bx_{2n+1} \|,$
$[\|Su - Bx_{2n+1}\| + \|T_{2n+1} - Au \|]/2\}$, as $n \to \infty$
$\| Au - p \| \le c \max \{ \|Su - p \| , \| Su - Au \|,$
$\|p- p \|, [\| Su - p \| + \|p - Au \|]/2\}$
$\| Au - p \| \le c \max \{ \|p - p \| , \| p - Au \|,$
$\| p - p \|, [\| p - p \| + \|p - Au \|]/2\}$
$\| Au - p \| \le c \| Au - p \|$

which is contradiction. So, Au = p = Su.

Since $A(X) \subseteq T(X)$ $\exists$ $v \in X$ such that Tv = p. Hence for (2), we have
$\| p - Bv \| = \| Au - Bv \|$
$\le c \max \{ \|Su - Tv\| , \|Su - Au\|, \|Tv - Bv\|,[\|Su - Bv\| + \|Tv-Au\|]/2\}$
$\| p - Bv \| \le c \|p - Bv\|$

which is contradiction. So, Bv = p = Tv.

Since Au = Su = p and the pair { A, S} is S – intimate. Then by Preposition 2, we have

$\| Sp - p \| \le \| Ap - p\|,$

Suppose $Ap \neq p$ then from (2), we get

$\| Ap - p \| = \| Ap - Bv \|$
$\le c \max \{ \|Sp - Tv\| , \|Sp - Ap\|,$
$\|Tv - Bv\|,[\|Sp - Bv\| + \|Tv-Ap\|]/2\}$
$= c \max \{ \|Sp - p\| , \|Sp - Ap\|,$
$\|p - p\|,[\|Sp - p\| + \|p-Ap\|]/2\}$
$\le c \max \{\|Ap - p\|,[\|Ap- p \|+ \|p- Ap\|],$
$0, [\|Ap - p\| + \|p-Ap\|]/2\}$
$\le c \|Au - p\|$

Which is contradiction. So, Ap = p and also Sp = p.

Hence Ap = Sp = p.

Similarly, we get Bp = Tp = p.

UNIQUENESS:

Let us consider q is another common fixed point A, B, S and T such that $p \neq q$, therefore
$\| p - q \| = \| Ap - Bq \|$
$\le c \max \{ \|Sp - Tq\| , \|Sp - Ap\|,$
$\|Tq - Bq\|,[\|Sp - Bq\| + \|Tq-Ap\|]/2\}$
$= c \max \{ \|p - q\| , \|p - p\|, \|q - q\|,$
$[\|p - q\| + \|q-p\|]/2\}$
$\le c \|p - q\|$
a contradiction. This show that p = q.

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