International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

Finite Volume Numerical Grid Technique for Multidimensional Problems

J.S.V.R. Krishna Prasad[1], Parag V. Patil[2]

[1]Department of Mathematics, M. J. College, Jalgaon 425001, Maharashtra, India

[2]The Research Scholar, Department of Mathematics, M. J. College, Jalgaon 425001, Maharashtra, India

Abstract: ***In this paper numerical technique has been used to solve multidimensional steady state heat flow equation with Dirichlet boundary conditions. We focus on finite volume numerical technique for solving heat equation in two and three dimensional problems using TDMA solver. Finally, the efficiency of this technique is tested for some heat flow problem with known analytical solutions and the numerical results obtained show that the technique produces accurate results.***

Keywords: Finite Volume Technique, Steady State heat flow Equation, Dirichlet boundary conditions, TDMA Solver

1. Introduction

Computational Fluid Dynamics (CFD) is the branch of fluid dynamics providing a cost effective mean of simulating real flow by the numerical solution of the governing equations. The computational techniques replace the governing partial differentials equations with systems of algebraic equation that are much easier to solve using computers. The steady improvement in computer technology has led to development of numerous computational grid techniques for solving numerical solution of three dimensional problems, for more detailed the reader may consult [1,2]. As mathematical modelling became an integral part of analysis of engineering problems, a variety of numerical grid techniques such as finite difference method (FDM) which is the simplest procedure used to convert the differential equations to discrete form [3,4]. The finite element method (FEM) which was developed at first for structural dynamics problems and then by CFD developers to facilitate solving more complex geometry problems that are difficult to discretise using Finite difference method [3,5,6] . The finite volume method (FVM) which is now a days, the most popular technique for CFD. This technique can be viewed as a subset of the finite element method. Each of these methods has its own merits and demerits depending on the problem to be solved, for more detailed the reader may consult [7, 8]. The new advance technique is grid less technique plays important role for solving numerical solutions [9].Out of the available numerical gird techniques, finite volume technique is one of the most flexible and versatile technique for solving the problems in CFD.

The remainder of the paper is organised as follows. In Section 2, a short review of finite volume techniques with the help of TDMA (Tri-Diagonal Matrix Algorithm) solver is given. In Section 3, formulation the two and three dimensional heat flow problems with Dirichlet boundary conditions. In section 4, Numerical examples are presented to illustrate the efficiency of the developed scheme. In Section 5, the numerical solutions obtained by this technique are compared with exact solution. Finally, Section 6 concludes the paper.

2. Finite Volume Grid Technique

Finite Volume Method is an increasing popular numerical technique for the approximate solution of partial differential equations. For more detailed the reader may consult [10]. The Finite Volume analysis involves three basic steps.

- The problem domain is defined and divided the solution domain into discrete control volume. Let us place a numbers of nodal points in the given space and domain is divided in such way that, each node is surrounded by the control volume or grid and the physical boundaries coincide with the control volume boundaries.
- The integration of the governing equation over the control volume to yield a discretised equation at its nodal point.
- Solve the set of discretised equations using TDMA solver.

2.1 Finite Volume Discretizations

The General form of discretised equations for multidimensional steady state heat flow problems are given by equation (1).

$$a_p\theta_p = \sum a_i\theta_i + S_\theta \quad (1)$$

$$a_p = \sum a_i - S_p \quad (2)$$

$$a_i = \frac{kA}{\Delta} \quad (3)$$

Where a_i are the neighbouring coefficient $a_W, a_E, a_N, a_S, a_B, a_T$ for two and three dimensional problems, θ_i are the values of the function θ at the neighbouring nodes, Δ is the grid size $\Delta x, \Delta y$ and $\Delta x, \Delta y, \Delta z$ in two and three dimensional respectively and S_θ *and* S_p are the values obtained from the linear source term $S_\theta + S_p\theta_p$ which is the function of the dependent variable.

Note that, to obtain the values S_θ *and* S_p from the linear source term $S_\theta + S_p\theta_p$ with boundary B. For Fixed value θ_B ,

$$S_\theta = \frac{2kA}{\Delta}\theta_B \text{ and } S_p = -\frac{2kA}{\Delta}$$

For Fixed Flux q,

$$S_\theta = q \times A \text{ and } S_p = 0$$

2.2 TDMA (Tri-Diagonal Matrix Algorithm)

The tri diagonal matrix algorithm (TDMA), also known also Thomas algorithm, is a simplified form of Gaussian elimination that can be used to solve tri diagonal system of equations

$$-a_i\theta_{i-1} + b_i\theta_i - c_i\theta_{i+1} = d_i \quad (4)$$
$$i = 1, - - - -, n$$

The TDMA is based on the Gaussian elimination procedure and consist of two parts-a forward elimination phase and a backward substitution phase. The TDMA is actually a direct method for 1D situation, but it can be applied iteratively in a line-by-line fashion, to solve multidimensional problems and is widely used in CFD programs. Let us consider the system for $i = 1, - - - -, n$ and we use the general form of the TDMA solver is given by

$$\theta_i = A_i\theta_{i+1} + B_i \quad (5)$$

Where

$$A_i = \frac{c_i}{b_i - a_iA_{i-1}} \text{ and } B_i = \frac{a_iB_{i-1} + d_i}{b_i - a_iA_{i-1}}$$

To solve the above system TDMA is applied for two and three dimensional problems respectively, the discretised equation is re-arranged in the form

$$-a_S\theta_S + a_P\theta_P - a_N\theta_N = a_W\theta_W + a_E\theta_E + S_\theta \quad (6)$$

$$-a_S\theta_S + a_P\theta_P - a_N\theta_N =$$
$$a_W\theta_W + a_E\theta_E + a_B\theta_B + a_T\theta_T + S_\theta \quad (7)$$

3. Problem Formulations

3.1 For two dimensional problems

Consider two dimensional steady state heat transfers in the plate with Dirichlet boundary conditions; the mathematical formulation of this problem is given by

$$\frac{\partial}{\partial x}\left(k\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial\theta}{\partial y}\right) = 0 \text{ in } 0 \le x, y \le 1 \quad (8)$$

Subject to the Dirichlet boundary conditions,

$$\theta(0, y) = f_1(y), 0 \le y \le 1$$
$$\theta(1, y) = f_2(y), 0 \le y \le 1$$

$$\theta(x, 0) = g_1(x), 0 \le x \le 1$$
$$\theta(x, 1) = g_2(x), 0 \le x \le 1$$

Where f_1, f_2, g_1 and g_2 are known functions. The solution region with boundary sides is shown in figure1.

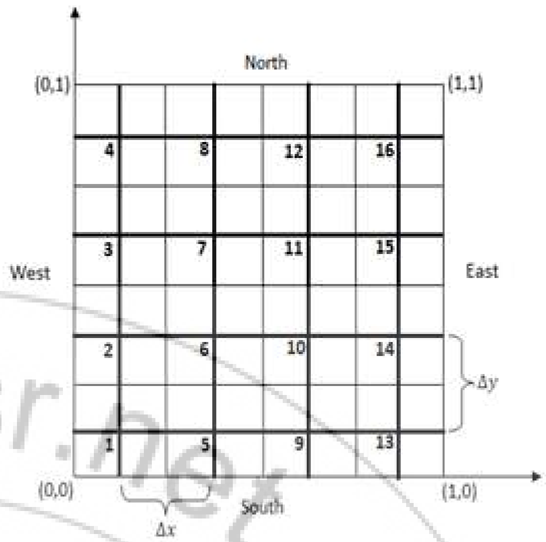


Figure 1: Solution region with boundary sides for two dimensional problems

3.2 For three dimensional problems

Consider three dimensional steady state heat transfers in the cube with Dirichlet boundary conditions; the mathematical formulation of this problem is given by

$$\frac{\partial}{\partial x}\left(k\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial\theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial\theta}{\partial z}\right) = 0 \quad (9)$$
$$0 \le x, y, z \le 1$$

Subject to the Dirichlet boundary conditions,

$$\theta(0, y, z) = f_1(y, z), 0 \le y, z \le 1$$
$$\theta(1, y, z) = f_2(y, z), 0 \le y, z \le 1$$

$$\theta(x, 0, z) = g_1(x, z), 0 \le x, z \le 1$$
$$\theta(x, 1, z) = g_2(x, z), 0 \le x, z \le 1$$

$$\theta(x, y, 0) = h_1(x, y), 0 \le x, y \le 1$$
$$\theta(x, y, 1) = h_2(x, y), 0 \le x, y \le 1$$

Where f_1, f_2, g_1, g_2, h_1 and h_2 are known functions. The solution region with boundary sides is shown in figure 2.

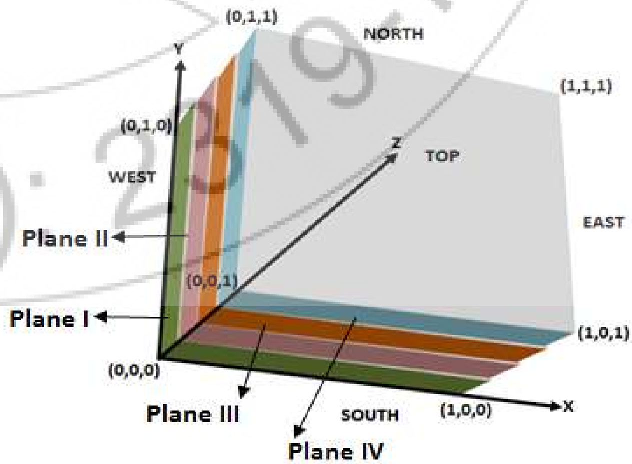
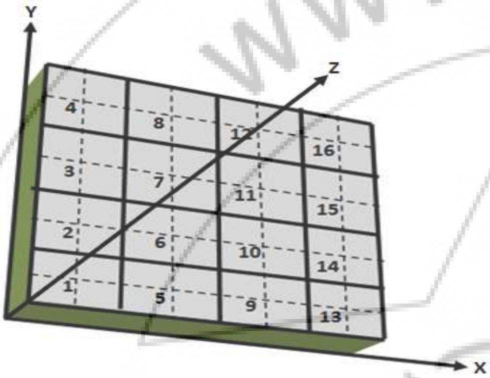


Figure 2: Solution region with boundary sides for three dimensional problems

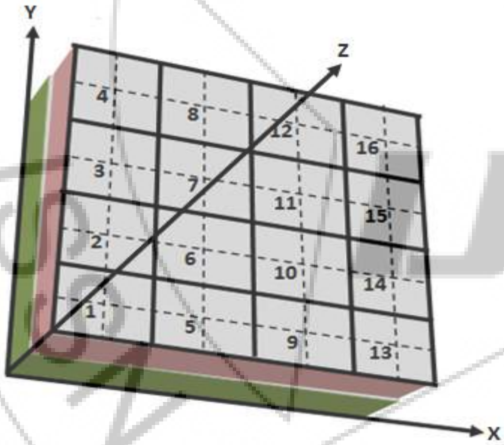
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

For three dimensional problems, the TDMA solver is applied line by line on the selected plane and then the calculation is moved to the next plane, scanning the solution region plane by plane. In this problem, there are four planes. These planes are numbered as I, II, III, and IV from bottom to top as shown in figure 2.

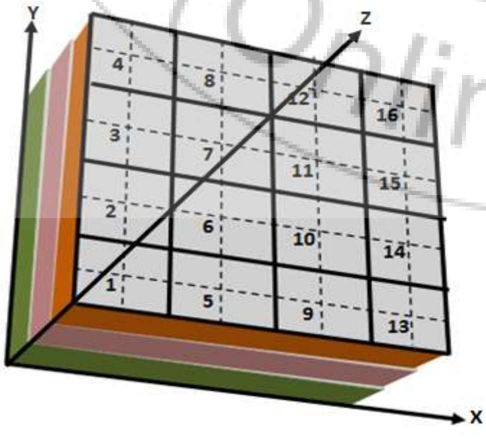
Using the TDMA procedure values of the θ along a selected north- south line are computed. The calculation is moved to the next line and subsequently swept through the whole plane until all unknown values on each line have been calculated. After completion the calculation of the plane-I, the process is moved on to the next plane-II and then continue up to plane-IV. The solution region is divided into four XY planes with girds as shown in figure 3.



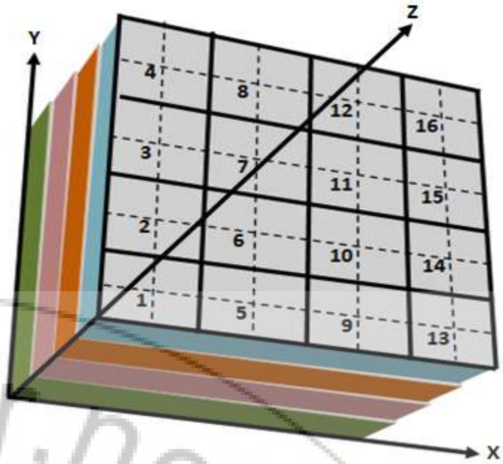
(a)



(b)



(c)



(d)

Figure 3: The solution region is divided into four XY planes with girds and nodes

4. Numerical Examples

The finite volume technique is apply on two selected examples in which the exact solutions of θ are known to us in order to test the efficiency and adaptability of the proposed technique. The computed solution is found for the entire interior grid points are given in Tables 1-4.

Example I

Let us consider the two dimensional steady state heat equations as shown by equation (8) with dirichlet boundary conditions

$$\theta(x,0) = -y^2, \theta(1,y) = 1 - y^2$$
$$\theta(x,1) = x^2 - 1, \theta(0,y) = x^2$$

The Exact solution of this problem is given by $\theta(x,y) = x^2 - y^2$ and its converged solution is obtained after 9th iterations as shown in table 1.

Example II

Let us consider the three dimensional steady state heat equations as shown by equation (9) with dirichlet boundary conditions

$$\theta(0,y,z) = \theta(x,0,z) = \theta(x,y,0) = 1$$
$$\theta(1,y,z) = \theta(x,1,z) = \theta(x,y,1) = 1$$

The Exact solution of this problem is given by $\theta(x,y,z) = 1$ and its converged solution is obtained after 17th iterations as shown in table 2.

Example III

Let us consider the three dimensional steady state heat equation as shown by equation (9) with dirichlet boundary conditions

$$\theta(0,y,z) = \theta(x,0,z) = \theta(x,y,0) = 0$$
$$\theta(1,y,z) = yz$$
$$\theta(x,1,z) = xz$$
$$\theta(x,y,1) = xy$$

The Exact solution of this problem is given by $\theta(x,y,z) = xyz$ and its converged solution is obtained after 7th iterations as shown in table 3-4.

Table 1 Comparison between Finite Volume solution and exact solution for example I

Nodes	Finite Volume	Exact
1	0.0000	0.0000
2	-0.1328	-0.1250
3	-0.3828	-0.3750
4	-0.7500	-0.7500
5	0.1328	0.1250
6	0.0000	0.0000
7	-0.2500	-0.2500
8	-0.6172	-0.6250
9	0.3828	0.3750
10	0.2500	0.2500
11	0.0000	0.0000
12	-0.3672	-0.3750
13	0.7500	0.7500
14	0.6172	0.6250
15	0.3672	0.3750
16	0.0000	0.0000

Table 2: Comparison between Finite Volume solution and exact solution for example II

Nodes	*Finite Volume Technique*				*Exact*
	I	*II*	*III*	*IV*	
1	1.000	1.000	1.000	1.000	1.000
2	1.000	1.000	1.000	1.000	1.000
3	1.000	1.000	1.000	1.000	1.000
4	1.000	1.000	1.000	1.000	1.000
5	1.000	1.000	1.000	1.000	1.000
6	1.000	1.000	1.000	0.999	1.000
7	1.000	1.000	0.999	0.999	1.000
8	1.000	1.000	0.999	0.999	1.000
9	1.000	1.000	1.000	1.000	1.000
10	1.000	1.000	0.999	0.999	1.000
11	1.000	1.000	0.999	0.997	1.000
12	1.000	1.000	0.999	0.995	1.000
13	1.000	1.000	1.000	1.000	1.000
14	1.000	1.000	0.999	0.999	1.000
15	1.000	1.000	0.999	0.995	1.000
16	1.000	1.000	0.998	0.984	1.000

Table 3: Comparison between Finite Volume solution of plane I and II with exact solution for example III

Planes	*I*		*II*	
Nodes	*Finite Volume*	*Exact*	*Finite Volume*	*Exact*
1	0.0019	0.0020	0.0064	0.0059
2	0.0055	0.0059	0.0176	0.0176
3	0.0101	0.0098	0.0283	0.0293
4	0.0123	0.0137	0.0395	0.0410
5	0.0055	0.0059	0.0197	0.0176
6	0.0174	0.0176	0.0546	0.0527
7	0.0305	0.0293	0.0867	0.0879
8	0.0400	0.0410	0.1186	0.1230
9	0.0090	0.0098	0.0329	0.0293
10	0.0305	0.0293	0.0920	0.0879
11	0.0529	0.0488	0.1463	0.1465
12	0.0600	0.0684	0.1985	0.2051
13	0.0165	0.0137	0.0435	0.0410
14	0.0400	0.0410	0.1260	0.1230
15	0.0700	0.0684	0.2057	0.2051
16	0.0901	0.0957	0.2819	0.2871

Table-4 Comparison between Finite Volume solution of plane III and IV with exact solution for example III

Planes	*III*		*IV*	
Nodes	*Finite Volume*	*Exact*	*Finite Volume*	*Exact*
1	0.0097	0.0098	0.0136	0.0137
2	0.0288	0.0293	0.0408	0.0410
3	0.0480	0.0488	0.0680	0.0684
4	0.0677	0.0684	0.0954	0.0957
5	0.0295	0.0293	0.0409	0.0410
6	0.0877	0.0879	0.1225	0.1230
7	0.1453	0.1465	0.2041	0.2051
8	0.2033	0.2051	0.2858	0.2871
9	0.0495	0.0488	0.0684	0.0684
10	0.1470	0.1465	0.2050	0.2051
11	0.2435	0.2441	0.3415	0.3418
12	0.3403	0.3418	0.4781	0.4785
13	0.0687	0.0684	0.0957	0.0957
14	0.2052	0.2051	0.2867	0.2871
15	0.3417	0.3418	0.4784	0.4785
16	0.4776	0.4785	0.6698	0.6699

5. Results and Discussions

All the numerical calculations obtained with control volume grids for two and three dimensional heat flow problems using Microsoft excel and the TDMA procedure is repeated until a converged solution is obtained. The comparison between the finite volume technique and exact solution for example-I as shown in figure 4 and for example III with respective to plane I, II, III and IV as shown in figure 5.

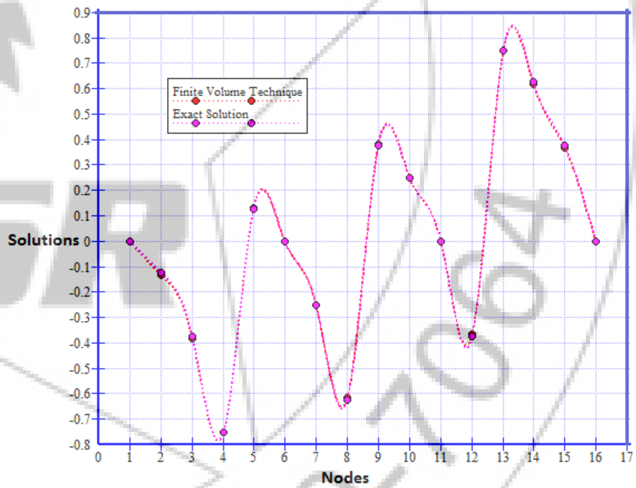
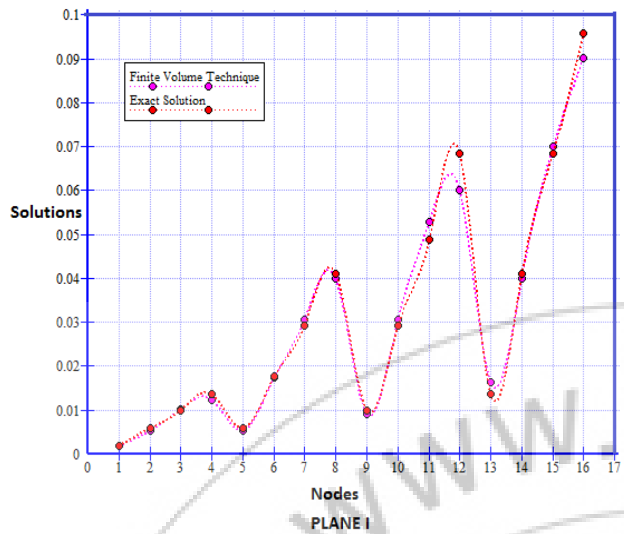
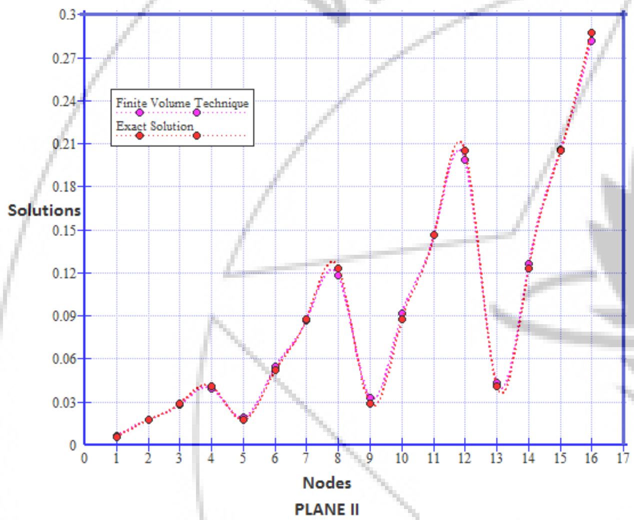
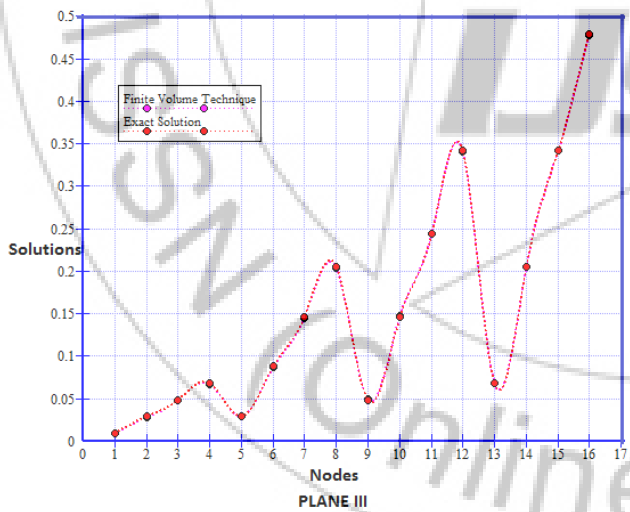


Figure 4: Graphical Comparison between Numerical and Exact Solution for example I







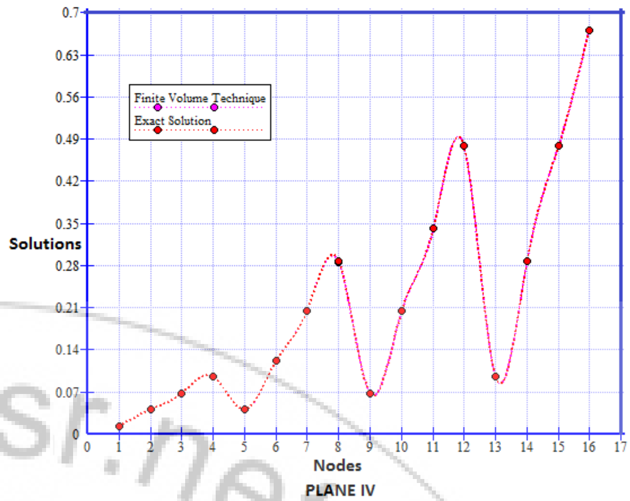


Figure 5: Graphical Comparison between Numerical and Exact Solution of planes I, II, III, and IV for example III

6. Conclusion

In this work, we have studied finite volume numerical grid technique for steady state heat flow problems and obtained the numerical solution for two and three dimensional heat flow equations with Dirichlet boundary conditions. We have used TDMA solver for solving algebraic equations and the results obtained by this technique are all in good agreement with the exact solutions and the total error less than 1 under study. Moreover this technique is efficient, reliable, accurate and easier to implement in Microsoft excel as compared to the other costly techniques.

References

[1] Cheniguel and M. Reghioua,"On the Numerical Solution of three- dimensional diffusion equation with an integral condition", WCECS2013, San Francisco, USA, 2013.

[2] Nissaya Chuathong and Wattana Toutip, "An accuracy comparison of solution between boundary element method and Meshless method for Laplace equation", (AMM2011), Khon Kaen University, Khon Kaen, Thailand, pp. 29-42, 2011.

[3] Parag V. Patil and Dr. J.S.V.R. Krishna Prasad, "Numerical Solution for Two Dimensional Laplace Equation with Dirichlet Boundary Conditions", *International Organization of Scientific Research-Journal of Mathematics,6* (4), pp 66-75, 2013.

[4] M.Necati Ozisik , Heat Transfer A Basic Approach (McGraw-Hill Book Company, first edition, 1985).

[5] Mark A. Lau, and Sastry P. Kuruganty, "Spreadsheet Implementations for Solving Boundary-Value Problems in Electromagnetic", *Spreadsheets in Education (eJSiE)*, 4(1), 2010.

[6] Parag V. Patil and Dr. J.S.V.R. Krishna Prasad, "Solution of Laplace Equation using Finite Element Method", Pratibha: International Journal of Science, Spirituality, Business and Technology, *2(1), pp* 40-46, 2013.

[7] M. N. O. Sadiku, *Elements of Electromagnetics* (New York: Oxford University Press, 4th edition, 2006).

[8] Erwin kreyszig, Advanced Engineering Mathematics (New York: John Wiley & Sons, 10th edition, 2011).
[9] Parag V. Patil and J.S.V.R. Krishna Prasad, "A numerical grid and grid less (Mesh less) techniques for the solution of 2D Laplace equation", Advances in Applied Science Research, Pelagia Research Library, 5(1), 150-155, 2014.
[10] H.K.Versteeg and W. Malalasekera, An Introduction to computational fluid dynamics: The finite volume method, (Longman Scientific and Technical, 1th edition, 1995).

Author Profile

Dr. J. S. V. R. Krishna Prasad received the Ph.D. degree in Mathematics from Indian Institute of Technology Kanpur, India. Currently he is an Associate Professor at Department of Mathematics, M. J. College, Jalgaon, Maharashtra, India and has published many articles in national and international journals.

Parag V. Patil received Master degree in Mathematics from M. J. College under North Maharashtra University, Jalgaon in 2007. He is pursuing PhD in Applied Mathematics from North Maharashtra University, Jalgaon, India.