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Bounds of Eigen Values of Two-Parameter Problems

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Abstract: ***In this paper first we decouple the matrix form of two- parameter eigenvalue problems using Kronecker product. Then apply Gerschgorin theorem on the pair of matrix equations, rough bounds of eigenvalues are obtained and is exemplified by giving example.***

Keyword: Multiparameter, Eigenvalue, Eigenvector, Kronecker product

1. Introduction

Multiparameter eigenvalue problems are generalization of one-parameter eigenvalue problems and occur quite naturally when the method of separation of variables is applied to certain boundary value problems associated with partial differential equations. Typical examples are provided, for example by a vibrating membrane [Roach, 12] and a dynamical problem of homogeneous beam loaded by a vertical load [Collatz, 7].

A two -parameter eigenvalue problem in matrix Form is

$$A_1X = \lambda B_1X + \mu C_1X \quad (1.10)$$

$$A_2Y = \lambda B_2Y + \mu C_2Y \quad (1.11)$$

Where $\lambda, \mu \in \mathbb{C}$, X& Y are complex vectors and A_i, B_i, C_i are given n×n complex matrices for i=1,2.

A typical example of where a two-parameter eigenvalue problem occurs is in the three point boundary problem. This could be a problem of the form

- $-(p(x)\,y'(x))' + q(x)\,y(x) = (\lambda\, r(x) + \mu\, s(x))\,y(x)$
- $-(p(x)\,y'(x))' + q(x)\,y(x) = (\lambda\, r(x) + \mu\, s(x))\,y(x)$

subject to the three boundary conditions y(a) = y(b) = y(c) = 0, where a <b < c and x lies in the range a < x < c. This can then be treated as a two parameter eigenvalue problem when we consider the two separate cases

$$-(p(x)\,y_1'(x))' + q(x)\,y_1(x) = (\lambda r(x) + \mu s(x))\,y(x) \quad (1.12)$$

with boundary values $y_1(a) = y_1(b) = 0$ for x in the range a < x < b and

$$-(p(x)\,y_2'(x))' + q(x)\,y_2(x) = (\lambda r(x) + \mu s(x))\,y(x) \quad (1.13)$$

with boundary values $y_2(b) = y_2(c) = 0$ for x in the range b < x < c.

Many of these problems are examples of two parameter Sturm-Liouville problems in the form

$$-(p_i(x_i)\,y_i'(x_i))' + q_i(x_i)\,y_i(x_i) = (\lambda a_{i1}(x_i) + \mu a_{i2}(x_i))\,y_i(x_i)$$

with given boundary conditions. This form occurs when the separation constants cannot be decoupled. These equations can be discretised to give a two-parameter eigenvalue problem in the matrix form (1.10) & (1.11).

Another example where two -parameter eigenvalue problems arise is from the separation of variables technique. If we consider the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^2$ that arises in the modelling of the vibration of a fixed membrane and perform separation of variables on an elliptic domain we obtain the Mathieu and modified Mathieu equations

$$y_1''(x_1) + (2\lambda \cosh 2x_1 - \mu)\,y_1(x_1) = 0 \quad (1.14)$$

$$y_2''(x_2) + (2\lambda \cosh 2x_2 - \mu)\,y_2(x_2) = 0 \quad (1.15)$$

that have to be solved simultaneously and hence form a two-parameter eigenvalue problem. Here λ is the eigenvalue corresponding to the physical value k^2, whereas μ is a somewhat artificial eigenvalue as it arises as a separation parameter.

Two- parameter eigenvalue problems are also found in matrix form in varying circumstances. For instance, when estimating electrical properties of a material from measurements or interdigital dielectrometry sensors, the properties of a material that has two layers will be the eigenvalues from an appropriate two-parameter eigenvalue problem [1].

2. Reduction of Two-Parameter Eigenvalue Problem

The matrix equation (1.10) & (1.11) are equivalent to

$$A_1X - \lambda B_1X - \mu C_1X = 0 \quad (2.10)$$

$$A_2Y - \lambda B_2Y - \mu C_2Y = 0 \quad (2.11)$$

So Kronecker multiplying (2.10) on the right by C_2Y and (2.11) on the left by C_1X and equating we get

$$(A_1X-\lambda B_1X-\mu C_1X)\otimes C_2Y=C_1X\otimes(A_2Y-\lambda B_2Y-\mu C_2Y)$$
$$\Rightarrow(A_1X-\lambda B_1X)\otimes C_2Y=C_1X\otimes(A_2Y-\lambda B_2Y)$$
$$\Rightarrow A_1X\otimes C_2Y-\lambda(B_1X\otimes C_2Y)=C_1X\otimes A_2Y-\lambda(C_1X\otimes B_2Y)$$
$$\Rightarrow A_1X\otimes C_2Y-C_1X\otimes A_2Y=\lambda(B_1X\otimes C_2Y-C_1X\otimes B_2Y)$$
$$\Rightarrow(A_1\otimes C_2-C_1\otimes A_2)(X\otimes Y)=\lambda(B_1\otimes C_2-C_1\otimes B_2)(X\otimes Y)$$
$$\Rightarrow\Delta_1Z=\lambda\Delta_0Z$$

(2.12)

Similarly multiplying (2.10) on the right by B_2Y and (2.11) on the left by B_1X we get

$$(A_1X-\lambda B_1X-\mu C_1X)\otimes B_2Y=B_1X\otimes(A_2Y-\lambda B_2Y-\mu C_2Y)$$
$$\Rightarrow(A_1X-\mu C_1X)\otimes B_2Y=B_1X\otimes(A_2Y-\mu C_2Y)$$
$$\Rightarrow B_1X\otimes A_2Y-A_1X\otimes B_2Y=\mu(B_1X\otimes C_2Y-C_1X\otimes B_2Y)$$
$$\Rightarrow(B_1\otimes A_2-A_1\otimes B_2)(X\otimes Y)=\mu(B_1\otimes C_2-C_1\otimes B_2)(X\otimes Y)$$
$$\Rightarrow\Delta_2Z=\mu\Delta_0Z$$

(2.13)

Hence equation (1.1) and (1.2) imply

$$\Delta_1Z=\lambda\Delta_0Z$$
$$\Delta_2Z=\mu\Delta_0Z$$

where $Z=X\otimes Y$

$\Delta_0,\Delta_1,\Delta_2$ are $(n^2)\times(n^2)$ dimensional matrices defined as

$$\Delta_0=B_1\otimes C_2-C_1\otimes B_2$$

$$\Delta_1=A_1\otimes C_2-C_1\otimes A_2$$

$$\Delta_2=B_1\otimes A_2-A_1\otimes B_2$$

and $Z=X\otimes Y$

3. Bounds of Eigenvalues

To estimate rough bounds of the parameters λ and μ of (2.12) and (2.13) we go through Gerschgorin theorem due to Burden and Faires[3]. Following Burden and Faires we have that in (2.13), λ is an eigenvalue of $A=\Delta_0^{-1}\Delta_1=[a_{ij}]_{n^2\times n^2}$ with associated eigenvector Z where $\|Z\|_\infty=\max_{i\le n}|Z_i|=1$.

Therefore, the equivalent component representation of (2.12) is

$$\sum_{j=1}^{m}a_{ij}Z_j=\lambda Z_i$$ for each i=1,2,......,m (say $m=n^2$) (3.10)

If k is an integer with $|Z_k|=\|Z\|_\infty=1$, the above equation (3.10), with i=k implies that

$$\sum_{j=1}^{m}a_{kj}Z_j=\lambda Z_k \quad (3.11)$$

Thus

$$\sum_{\substack{j=1\\j\neq k}}^{m}a_{kj}Z_j=\lambda Z_k-a_{kk}Z_k=(\lambda-a_{kk})Z_k$$ (3.12) and

$$|\lambda-a_{kk}||Z_k|=|(\lambda-a_{kk})Z_k|=\left|\sum_{\substack{j=1\\j\neq k}}^{m}a_{kj}Z_j\right|\le\sum_{\substack{j=1\\j\neq k}}^{m}|a_{kj}||Z_j|$$

(3.13)

Since $|Z_k|\ge|Z_j|$ for all j=1,2,....,m.

$$|\lambda-a_{kk}|\le\sum_{\substack{j=1\\j\neq k}}^{m}|a_{kj}|\left|\frac{Z_j}{Z_k}\right|\le\sum_{\substack{j=1\\j\neq k}}^{m}|a_{kj}| \quad (3.14)$$

The inequality (3.14) implies that λ is in

$$R1_k=\{x\in C\,|\,|x-a_{kk}|\le\sum_{\substack{j=1\\j\neq k}}^{m}|a_{kj}|\}$$ (3.15)

Where C denotes the complex plane. And hence λ lies in

$$R1=\bigcup_{k=1}^{m}R1_k \quad (3.16)$$

Proceeding in a similar way we have in (2.13), μ is an eigenvalue of $B=\Delta_0^{-1}\Delta_1=[b_{ij}]_{m\times m}$ where $m=n^2$ with associated eigenvector Z where $\|Z\|_\infty=\max_{i\le n}|Z_i|=1$.

Therefore, the equivalent component representation of (2.13) is

$$\sum_{j=1}^{m}b_{ij}Z_j=\mu Z_i$$ for each i=1,2,....,m. (3.16)

If k is an integer with $|Z_k|=\|Z\|_\infty=1$, the above equation (3.16), with i=k implies that

$$\sum_{j=1}^{m}b_{kj}Z_j=\mu Z_k \quad (3.17)$$

Thus

$$\sum_{\substack{j=1\\j\neq k}}^{m}b_{kj}Z_j=\mu Z_k-b_{kk}Z_k=(\mu-a_{kk})Z_k$$ (3.18) and

$$|\mu-b_{kk}||Z_k|=|(\mu-b_{kk})Z_k|=\left|\sum_{\substack{j=1\\j\neq k}}^{m}b_{kj}Z_j\right|\le\sum_{\substack{j=1\\j\neq k}}^{m}|b_{kj}||Z_j|$$

(3.19)

Since

$|Z_k|\ge|Z_j|$ for all j=1,2,....,n.

$$|\mu-b_{kk}|\le\sum_{\substack{j=1\\j\neq k}}^{m}|b_{kj}|\left|\frac{Z_j}{Z_k}\right|\le\sum_{\substack{j=1\\j\neq k}}^{m}|b_{kj}| \quad (3.20)$$

The inequality (3.20) implies the μ lies in

$$R2_k = \{x \in C \mid |x - b_{kk}| \leq \sum_{\substack{j=1 \\ j \neq k}}^{m} |b_{kj}|\} \quad (3.21)$$

Where C denotes the complex plane. And μ lies in

$$R2 = \bigcup_{k=1}^{m} R2_k \quad (3.22)$$

4. A Numerical Example

Consider the two parameter eigen value problem

$$\begin{pmatrix} -.1290 & 2.4839 \\ -.9677 & 3.1290 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1.2258 & 3.9032 \\ -11.2258 & 21.0968 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu \begin{pmatrix} 2.9032 & 4.6129 \\ -17.4194 & 32.3226 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 2.9677 & -2.1290 \\ -4.2581 & 6.9677 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} 5.4194 & 5.6774 \\ -26.7097 & 49.1613 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \mu \begin{pmatrix} 7.0968 & 6.3871 \\ -32.9032 & 60.3871 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Using Kronecker product the above equations can be written as

$$\begin{pmatrix} -9.5313 & 5.3570 & 3.9380 & 25.6858 \\ 16.6066 & -28.0186 & -62.0861 & 117.8542 \\ 44.8280 & -43.2667 & -73.7179 & 88.8001 \\ -42.3331 & 62.9366 & 34.6788 & -36.2629 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \lambda \begin{pmatrix} -7.0343 & -8.6533 & 2.7011 & -1.2591 \\ 37.2109 & -68.7026 & -5.2186 & 8.9268 \\ 14.7354 & 27.1966 & -25.4493 & -48.7610 \\ -95.9022 & 178.4668 & 169.1747 & -315.0465 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \quad 4.10)$$

$$\begin{pmatrix} 4.3369 & -1.8773 & -1.8777 & -22.4120 \\ -8.6651 & 14.8828 & 49.7240 & -94.9154 \\ -28.0705 & 29.3937 & 45.6517 & -62.6797 \\ 21.9536 & -30.6446 & -6.2576 & -6.8295 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \mu \begin{pmatrix} -7.0343 & -8.6533 & 2.7011 & -1.2591 \\ 37.2109 & -68.7026 & -5.2186 & 8.9268 \\ 14.7354 & 27.1966 & -25.4493 & -48.7610 \\ -95.9022 & 178.4668 & 169.1747 & -315.0465 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \quad (4.11)$$

(4.10) Can be written as

$$\begin{pmatrix} .9066 & -.6480 & -.9118 & -1.1786 \\ .2525 & .0566 & .4592 & -2.4577 \\ -.4778 & .6549 & 1.0056 & -2.4834 \\ -.2552 & -.3813 & -.9676 & -2.2520 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

Or, $.9066z_1 - .6480z_2 - .9118z_3 - 1.1786z_4 = \lambda z_1$ (4.12)

$.2525z_1 + .0566z_2 + .4592z_3 - 2.4577z_4 = \lambda z_2$ (4.13)

$-.4778z_1 + .6549z_2 + 1.0056z_3 - 2.4834z_4 = \lambda z_3$ (4.14)

$-.2552z_1 + .3813z_2 + .9676z_3 - 2.2520z_4 = \lambda z_4$ (4.15)

Then by using (3.15) on the matrix equation represented by (4.10) we get

$R1_1 = \{\lambda \mid |\lambda| \leq 3.6450\}$

$R1_2 = \{\lambda \mid |\lambda| \leq 3.1694\}$

$R1_3 = \{\lambda \mid |\lambda| \leq 3.6161\}$

$R1_4 = \{\lambda \mid |\lambda| \leq -.6579\}$

Thus λ lies in $R1 = \bigcup_{k=1}^{4} R1_k$ and $|\lambda| \leq 3.8561$

Similarly (4.11) can be written as

$$\begin{pmatrix} -.3949 & .2316 & .6900 & 1.0297 \\ -.0902 & -.0912 & -.4012 & 2.0406 \\ .3840 & -.5277 & -.5043 & 1.7665 \\ .2056 & -.3082 & -.6883 & 1.8128 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \mu \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

Or, $-.3949z_1 + .2316z_2 + .6900z_3 + 1.0297z_4 = \mu z_1$

$-.0902z_1 - .0912z_2 - .4012z_3 + 2.0406z_4 = \mu z_2$

$.3840z_1 - .5277z_2 - .5043z_3 + 1.7665z_4 = \mu z_3$

$.2056z_1 - .3082z_2 - .6883z_3 + 1.8128z_4 = \mu z_4$

Then by using (3.21) on the matrix equation represented by (4.11) we get

$R2_1 = \{\mu \mid |\mu| \leq .5564\}$

$R2_2 = \{\mu \mid |\mu| \leq 2.5408\}$

$R2_3 = \{\mu \mid |\mu| \leq 2.1739\}$

$R2_4 = \{\mu \mid |\mu| \leq 3.0149\}$

Thus μ lies in $R2 = \bigcup_{k=1}^{4} R2_k$ and $|\mu| \leq 3.0149$.

5. Conclusion

The procedure discussed in this paper confirms that the rough bounds of the eigenvalues of (1.10), (1.11) can be estimated straightforwardly.

6. Future Scope

The importance of the above procedure of finding rough bounds of eigenvalues lies in the fact that the starting approximations of the eigenvalue pairs can be obtained more easily and it will definitely play significant roles for further research in tackling the two-parameter for matrices and the multiparameter problem in general

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