International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

A Note on Finite Rank Quadratic Operators

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Abstract: ***The representation result of a finite rank quadratic operator is given. It has been shown that the space FQ(H) of finite rank bounded quadratic operators on a real separable Hilbert space H is equal to the cross spaces L(H) ⊗H and SL(H) ⊗H for a space SL(H) of self-adjoint bonded linear operators on H and the space L(H) of bounded linear operators on H. As an application of the concept of finite rank quadratic operator, the numerical range of an operator A ∈SL(H) is derived.***

Keywords: quadratic operator, bilinear operator, symmetric, bilinear functional

1. Introduction

Amson and Gopal Reddy [1] has developed an H*-Algebra of Hilbert Schmidt quadratic operators on H. In this note we develop the cross space SL(H) ⊗H for the space SL(H) of self-adjoint operators on H analogue of Schatten [3] theory of the linear operator space H ⊗ H and to show that the space FQ(H) of finite rank bounded quadratic operators on H is closed with

$$\|A\otimes x\|_{op} = \|A\|_{op}\,\|x\|_{v}$$

An operator $Q: H \to H$ is a quadratic operator if there exists a bilinear operator $\hat{Q}: H \times H \to H$ such that for all $x\in H, Qx=\hat{Q}(x,x)$ identically, $\hat{Q}$ is unique if it is symmetric, it is then called the polarization of the quadratic operator Q. The space H will be assumed to be a real separable Hilbert space with Hilbert basis $\{b_n\}$ and let $\|.\|_v$ and $\|.\|_{op}$ be the norms of vector and operator respectively. The operator Q has finite rank if its range has finite dimension, and then that its rank is the dimension of its range. Every bounded quadratic functional is of finite rank.

2. Preliminaries

An operator B: $H \times H \to H$ sending $(x, y) \to B(x, y)$ is called bilinear if it is linear in each variable separately and symmetric if B(x,y) = B(y,x) for all x,y belongs to real separable Hilbert space H. An operator $Q: H \to H$ sending $x\in H$ to $Q(x)\in H$ is called quadratic if there exists a bilinear operator B: $H \times H \to H$ such that $Q(x) = B(x, x)$ for all $x\in H$. A bilinear operator B : $H \times H \to H$ is said to be bounded if there exists a constant C > O such that

$$\|B(x,y)\|_v \le C\|x\|_v\,\|y\|_v \ \ for\ all\ (x,y)\in H\times H$$

and

$\|B\|_{op} = \sup\{\|B(x,y)\|_v : \|x\|_v, \|y\|_v \le 1\}$ is called the uniform norm of B.

A quadratic operator $Q: H \to H$ is said to be bounded if there exists a constant C > O such that

$$\|Q(x)\|_v \le C\,\|x\|_v^2\ \ for\ all\ x\in X\ and$$

$$\|Q\|_{op} = \sup\{\|Q(x)\|_v : \|x\|_v \le 1\}$$

is said to be uniform norm of Q.

The following propositions are easy, so, left to reader.

Proposition 2.1: An operator $\hat{Q}: H \times H \to H$ satisfying

$$Q(C_1x_1+C_2x_2)=C_1^2Q(x_1)+2C_1C_2\,\hat{Q}(x_1,x_2)+C_2^2\,Q(x_2)$$

For all $x_1, x_2 \in H, C_1, C_2 \in F$ (Scalar field).

Proposition 2.2: If H is a real or complex Hilbert space. Q is a quadratic operator on H and $\hat{Q}$ is its polarized operator (i.e. symmetric bilinear) then

$$\|\hat{Q}\|_{op} = \|Q\|_{op}$$

Proof: We can see easily by using the parallelogram law for the norm in a Hilbert space.

3. Main Results

In this section we obtain some results on finite rank bilinear and quadratic operators on real separable Hilbert space H.

LEMMA 3.1: Let H be a real separable Hilbert space, let $A\in SL(H)$ be fixed, $b\in H$ be fixed vector and define a mapping

$$T = A\hat{\otimes} b : (x, y) \to \langle x / Ay\rangle b\ \ for\ all\ x, y\in H,$$

$A\hat{\otimes} b$ is a bounded bilinear operator from $H \times H \to H$ and

$$\|A\hat{\otimes} b\|_{op} = \|A\|_{op}\,\|b\|_v$$

Proof: We have T(x,y) = <x/Ay> b (1)

As A is bounded and b is fixed vector, clearly T= $A\hat{\otimes} b$ bounded. By definition of boundedness of bilinear operator we obtain $\|T\|_{op} \le \|A\|_{op}\,\|b\|_v$. By taking x = Ay in (1) we get reverse inequality.

LEMMA 3.2 Let $L(H)\otimes H$ be the vector space generated by all bilinear operators of the form $A\otimes b$, i.e. $T\in L(H)\otimes H$ if and only if $T=\sum_{n=1}^{N} A_n\otimes b_n$ for some $N\geq 1$ and $A_n\in SL(H)\subset(H), b_n\in H$. Let FB(H) be stand for the space of all bounded bilinear operators $B:H\,x\,H\to H$ such that the dimension of the smallest vector subspace containing range (B) is finite, i.e. rank $(B)<\infty$. Then $SL(H)\otimes H=FB(H)$.

Proof: We can see that $T=\sum_{n=1}^{N} A_n\otimes b_n \in L(H)\otimes H$ is bilinear and bounded on H x H. We know from lemma 3.1, that each expression is bilinear and bounded. Since, each $A_n\otimes b_n$ is finite rank operator (i.e. its rank is 0 or 1), so, rank of $T\leq$ sum of finite ranks, so, T is finite rank operator and that implies T belongs to FB (H).

To prove converse, let $T\in FB(H)$. So there exists a finite Hilbert basis $\{b_1, b_2,\ldots,b_N\}$, Let $z\in$ range (T), so $z = T(x,y)$ for vectors $x, y\in H$. By Fourier representation, we have

$$T(x,y)=\sum_{n=1}^{N}<T(x,y)/b_n>b_n\ \forall n$$

$$<T(x,y)/b_n>=T_n(x,y)$$

where $T_n(x,y)$ is "Co-ordinate wise bilinear functional". Since T is symmetric, so each coordinate bilinear functional is symmetric, hence there exists a corresponding self-adjoint linear operator such that $T_n(x, y) = <A_n x/y>$. Hence on using the definition of $A\otimes b$, we get

$$T(x,y)=\sum_{n=1}^{N}<A_n x/y>b_n=\sum_{n=1}^{N}(A_n\otimes b_{n_i})(x,y)$$

Therefore $T\cong\sum_{n=1}^{N}A_n\otimes b_n$ with respect to the chosen basis.

Theorem 3.1: Let H be a real Hilbert space, let $A\in L(H), b\in H$ and define a mapping

$$Q=A\otimes b: x\to<x/Ax>b\ \forall x\in H, \text{ then}$$

$Q=A\otimes b$ is a bounded quadratic operator on H and

$$\|A\otimes b\|_{op}=\|A\|_{op}\,\|b\|_v$$

Proof: Let $Q=A\otimes b:H\to H$ be an operator, defined by

$$Qx=(A\otimes b)x=<x/Ax>b \quad \forall$$

$x\in H$.

Let $B:H\,x\,H\to H$ by $B(x,y)=<x/Ax>b$. From lemma 3.1 B is bounded and symmetric. Now, $B(x,x)=Qx, \forall x\in H$. hence Q is quadratic on H. Q is obviously bounded and $\|Q\|_{op}\leq\|A\|_{op}\,\|b\|_v$. To prove reverse inequality we polarize a quadratic operator Q as follows. Let $\hat{Q}$ be the polarized operator of Q, and be defined by

$$\hat{Q}(x,y)=A\hat{\otimes}b(x,y)=<x/Ay>b$$

on setting x = Ay, taking norm on both sides and using proposition 2.2, we get

$$\|A\|_{op}\,\|b\|_v\leq\|\hat{Q}\|_{op}=\|Q\|_{op}$$

Hence required result.

Theorem 3.2: Let $L(H)\otimes H$ be the vector space generated by all quadratic operators of the form $A\otimes b$ i.e. $T\in L(H)\otimes H$ if and only if $T=\sum_{n=1}^{N}A_n\otimes b_n$ for some $N\geq 1$ and $A_n\in SL(H), b_n\in H$, we write FQ(H) for the set of all quadratic operators $Q:H\to H$ such that the rank of Q is finite, then FQ(H) is a normed linear space and $SL(H)\otimes H=FQ(H)$.

PROOF : Let $T\,\varepsilon\,L(H)\otimes H$ implies $T=\sum_{n=1}^{N}A_n\otimes b_n$.

For every $x\in H$, we define $Tx=\left(\sum_{n=1}^{N}A_n\otimes b_n\right)x$. We proved in theorem 3.1, each $A_n\otimes b_n\ (n=1,2,\ldots,N)$ is quadratic and bounded. Hence $T=\sum_{n=1}^{N}A_n\otimes b_n$ is bounded quadratic operator. As each $A_n\otimes b_n\ (n=1,2,\ldots,N)$ is finite rank operator, T is finite rank operator. So, $T\in FQ(H)$. We can see that FQ(H) is a vector space and is normed space with $\|Q\|=\inf\Sigma\,\|A_n\|\,\|b_n\|\ \forall n$.

To prove converse, let $T\in FQ(H)$, That implies that Range $(T)=N<\infty$. So, there exists a finite Hilbert basis $\{b_1, b_2, \ldots, b_N\}$. Let $z\in$ Range (T). So, $z=Tx$ *for* $x\in H$. By fourier representation we have

$$Tx=\sum_{n=1}^{N}<Tx/b_n>b_n$$

On polarizing T to $\hat{T}$, we get $\hat{T}(x,y)=\sum_{n=1}^{N}<\hat{T}(x,y)/b_n>b_n$. Since $\hat{T}$ is bilinear, hence by lemma 3.2. there exists self adjoint linear operators $A_1,\ldots, A_N$ such that

$$\hat{T}(x,y)=\sum_{n=1}^{N}<A_n x/y>b_n$$

$$\hat{T}(x,y)=\sum_{n=1}^{N}< A_n\, x / y > b_n$$

Since T is quadratic, so $\hat{T}(x,x)=Tx$.

$$Tx=\sum_{n=1}^{N} A_n \otimes b_n)x.$$

$$T \simeq \sum_{n=1}^{N} A_n \otimes b_n$$

Hence the theorem.

4. Application

Here we use the concept of finite rank quadratic operator to derive the numerical range of self-adjoint operator. By using theorem 3.1 we can easily prove that the numerical range of an operator is equal to the norm of an operator.

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