Duality for vector optimization problems with cone constraints involving support functions

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Abstract

In this work, first and higher order duality is discussed for vector optimization problems with cone constraint where every component of the objective function contains a term involving the support function of a compact convex set. It is an attempt to remove certain omissions and inconsistencies in the work of Kim and Lee (Nonlinear Anal. Theory Meth. Appl. 71 (2009),2474-2480).
Keywords: vector optimization, cones, Mond-Weir dual, invexity, quasi-(pseudo)invexity, higher order type I functions, higher order pseudoquasi-type I functions, higher order (F, ρ) type I functions, higher order (F, ρ)-pseudoquasi-type I functions.

1 Introduction

Duality theory has played an important role in the development of optimization theory. Duality in linear programming was first introduced by John Von Neuman [16] and was later studied by Dantzig and Ordan [3]. Isermann ([4], [5]) developed multiobjective duality in linear case, while the results for the nonlinear case have been given by Jahn [6], Luc [9] and others. The study of higher order duality is important due to computational advantage over first order duality as it provides better bounds for the value of the objective function when approximations are used because there are more parameters involved. Many researchers like Mangasarian [11], Mond and Weir [14] established higher order duality for a vector optimization problem with non-negative orthant as the cone. Mishra and Rueda [13] formulated a number of higher order duals to a nondifferentiable programming problem and established duality under the higher order generalized invexity conditions introduced in ([12], [13]). In this paper, certain shortcomings in definitions and dual models discussed by Kim and Lee in "Nondifferentiable higher order duality in multiobjective programming involving cones, Nonlinear Anal. Theory Meth. Appl. 71 (2009),2474-2480." is pointed out. Modified (Corrected) version of these definitions and dual formulations are presented. Similar omissions in [7] have been corrected.

2 Notations and Definitions

If $x, y \in \mathbb{R}^n$, then $x \geqq y \iff x_i \geq y_i, i = 1, 2, ..., n$; $x \geq y \iff x \geqq y$ and $x \neq y$; $x > y \iff x_i > y_i, i = 1, 2, ..., n$. $x \nleq u$ is the negation of $x \leq u$. All vectors shall be considered as column vectors.

Definition 2.1. A set $K \subseteq \mathbb{R}^n$ is said to be a *cone* if $\alpha x \in K, \quad \forall x \in K, \alpha \in \mathbb{R}$ and $\alpha \geq 0$.

A cone is said to be a convex cone if it is also a convex set apart from being a cone.
Consider the following multiobjective programming

problem:

$$\begin{aligned}\text{Minimize } & f(x) + A(x) = (f_1(x) + s(x \mid D_1) \\ & , f_2(x) + s(x \mid D_2), \ldots, f_l(x) + s(x \mid D_l)) \\ & \text{subject to } - g(x) \in C^*,\ x \in \mathbb{R}^n \end{aligned} \tag{NMP}$$

where,

(i) $f = (f_1, f_2, \ldots, f_l) : \mathbb{R}^n \to \mathbb{R}^l$, $g = (g_1, \ldots, g_m) : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions.

(ii) C is a closed convex cone with nonempty interior in $\mathbb{R}^m$ and C^* is the negative polar cone of C.

(iii) D_i are compact convex sets in $\mathbb{R}^n$.

(iv) $A(x)$ is a notation for the vector $(s(x \mid D_1), s(x \mid D_2), \ldots, s(x \mid D_l))^T$.

Let $C \subseteq \mathbb{R}^m$ be a closed convex pointed ($C \cap -C = \{0\}$) cone with vertex at the origin with nonempty interior. The negative polar cone C^* is defined as follows:

$$C^* = \{y \in \mathbb{R}^m : z^T y \leq 0, \forall z \in C\}$$

Definition 2.2. A feasible point $\bar{x}$ is a weakly efficient (efficient) solution of (NMP), if there exists no feasible point x such that $f(x) + A(x) < f(\bar{x}) + A(\bar{x}) \quad (f(x) + A(x) \leq f(\bar{x}) + A(\bar{x}))$.

Definition 2.3. A differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be

(i) invex at $u \in \mathbb{R}^n$ if there exists an n-dimensional vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall\ x \in \mathbb{R}^n$,

$$\phi(x) - \phi(u) \geq \eta(x, u)^T \nabla\phi(u).$$

(ii) quasi-invex at u if there exists an n-dimensional vector function
$\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that$\forall\ x \in \mathbb{R}^n$,

$$\phi(x) \leq \phi(u) \Longrightarrow \eta(x, u)^T \nabla\phi(u) \leq 0$$

(iii) pseudoinvex at u if there exists an n-dimensional vector function
$\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall x \in \mathbb{R}^n$,

$$\eta(x, u)^T \nabla\phi(u) \geq 0 \Longrightarrow \phi(x) \geq \phi(u)$$

.

$\nabla\phi$ is the gradient of ϕ which is taken as $n \times 1$ vector.

Definition 2.4. A functional $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is sublinear in its third component, if $\forall\ x, u \in \mathbb{R}^n$ following holds:

(i) $F(x, u, a_1 + a_2) \leq F(x, u, a_1) + F(x, u, a_2), \forall\ a_1, a_2 \in \mathbb{R}^n$.

(ii) $F(x, u, \alpha a) = \alpha F(x, u, a), \forall\ \alpha \geq 0, a \in \mathbb{R}^n, \alpha \in \mathbb{R}$.

Following functions are used to establish higher order duality. Let $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ and $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ be differentiable vector valued functions.

Definition 2.5. [13]

(a) $(f + (.)^T w, g_j)$ is said to be higher order type I at u with respect to η if $\forall\ x$, the following holds:
$f(x) + x^T w - f(u) - u^T w \geqq \eta(x, u)^T \left[\nabla_p h(u, p) + w\right] + h(u, p) - p^T \nabla_p h(u, p)$
and,
$-g_j(u) \geq \eta(x, u)^T \nabla_p k_j(u, p) + k_j(u, p) - p^T \nabla_p k_j(u, p), \quad j = 1, \cdots, m.$
where,

(i) $x^T w = \left[x^T w_1, \cdots, x^T w_l\right]^T, \quad w_i \in \mathbb{R}^n, \quad \forall i.$

(ii) $\eta(x, u)^T \left[\nabla_p h(u, p) + w\right] = [\eta(x, u)^T (\nabla_p h_1(u, p) + w_1), \cdots, \eta(x, u)^T (\nabla_p h_l(u, p) + w_l)]^T.$

(iii) $p^T \nabla_p h(u, p) = \left[p^T \nabla_p h_1(u, p), \cdots, p^T \nabla_p h_l(u, p)\right]^T.$

(b) $(f + (.)^T w, g_j)$ is said to be higher order pseudoquasi-type I at u with respect to η if $\forall x$, the following holds:
$\eta(x, u)^T \left[\nabla_p h(u, p) + w\right] \geqq 0$
$\Longrightarrow f(x) + x^T w - f(u) - u^T w - h(u, p) + p^T \nabla_p h(u, p) \geqq 0.$
and,

$-g_j(u) - k_j(u,p) + p^T\nabla_p k_j(u,p) \leq 0$
$\Longrightarrow \eta(x,u)^T(\nabla_p k_j(u,p)) \leq 0, \quad j = 1,2,\ldots,m$

Definition 2.6. Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a sublinear functional,
$\rho^1 \in \mathbb{R}^l, \rho^1 = (\rho^1_1,\ldots,\rho^1_l)^T$ and $\rho^2 \in \mathbb{R}^m, \rho^2 = (\rho^2_1,\ldots,\rho^2_m)^T$ and $d(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

(a) $(f + (.)^T w, g_j)$ is said to be Higher order (F,ρ) type I at u if $\forall x$
$f(x) + x^T w - f(u) - u^T w \geqq F(x,u,\nabla_p h(u,p) + w) + h(u,p) - p^T\nabla_p h(u,p) + \rho^1 d^2(x,u)$
and,
$-g_j(u) + k_j(u,p) - p^T\nabla_p k_j(u,p) \geq F(x,u,-\nabla_p k_j(u,p)) + \rho^2_j d^2(x,u), \quad j = 1,2,\ldots,m.$
where $F(x,u,\nabla_p h(u,p) + w)$ denotes the vector component
$(F(x,u,\nabla_p h_1(u,p) + w_1,$
$\cdots, F(x,u,\nabla_p h_l(u,p) + w_l)^T$.

(b) $(f + (.)^T w, g_j)$ is said to be higher order (F,ρ)-pseudoquasi-type I at u , if $\forall x$
$F(x,u,\nabla_p h(u,p) + w) \geqq -\rho^1 d^2(x,u)$
$\Longrightarrow f(x) + x^T w - f(u) - u^T w - h(u,p) + p^T\nabla_p h(u,p) \geqq 0.$
and,
$-g_j(u) - k_j(u,p) + p^T\nabla_p k_j(u,p) \leq 0$
$\Longrightarrow F(x,u,\nabla_p k_j(u,p)) \leq -\rho^2_j d^2(x,u), \quad j = 1,2,\ldots,m.$

For any set $B \subseteq \mathbb{R}^n$, the support function $s(x|B)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that $s(y|B) \geq s(x|B) + z^T(y-x)$ for all $y \in B$. Equivalently, $z^T x = s(x|B)$. The subdifferential of $s(x|B)$ is given by $\partial s(x|B) := \{z \in B : z^T x = s(x|B)\}$.
For any set $S \subseteq \mathbb{R}^n$ the normal cone to S at a point $x \in S$ is defined by $N_S(x) := \{y \in \mathbb{R}^n : y^T(z-x) \leq 0 \text{ for all } z \in S\}$.
It is readily verified that for a compact convex set B, y is in $N_B(x)$ if and only if $s(y|B) = x^T y$, or equivalently, x is in the subdifferential of s at y

3 First Order Duality

We now associate the following Mond-Weir Type dual programming problem (MWD) to (NMP):

$$\text{Maximize} \quad f(u) + u^T w \tag{MWD}$$

$$\text{subject to} \quad \nabla(\lambda^T f)(u) + \sum_{i=1}^{l}\lambda_i w_i = \nabla(y^T g)(u), \tag{1}$$

$$y^T g(u) \leq 0, \tag{2}$$

$$w_i \in D_i, i = 1,\ldots,l,$$

$$y \in C, \lambda \geq 0, \lambda^T e = 1,$$

where

(i) $e = (1,\ldots,1)^T \in \mathbb{R}^l$,

(ii) $u^T w = [u^T w_1,\ldots,u^T w_l]^T$.

Remark 3.1. *In comparison to dual proposed in [7] , first constraint is slightly change to ensure that vectors on both the sides of equality are comparable.Also, the second constraint in[7] is taken as: $y^T g(u) \in C_2^*$ and $x \in C_1$ where C_1 and C_2 are closed convex cones. However, with these conditions strong duality does not hold as $[\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T\nabla g(\overline{x})]^T x \geqq 0$, for all $x \in C_1$, $\nRightarrow \overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T\nabla g(\overline{x}) = 0$. Moreover, $\overline{y}^T g(\overline{x}) \leqq 0 \nRightarrow g(\overline{x}) \in C_2^*$. Therefore, the above model was considered and duality results were established.*

Theorem 3.1. *(Weak Duality):*
Let x and $(u,y,\lambda,w_1,w_2\cdots,w_l)$ be the feasible solutions of (NMP) and (MWD), respectively. Assume that one of the following holds:

(a) $f_i(.) + (.)^T w_i, \quad i \in \{1,2,\cdots,l\}$ and $-y^T g(.)$ is invex at u with respect to same η.

(b) $\sum_{i=1}^{l}\lambda_i f_i(.) + (.)^T\sum_{i=1}^{l}\lambda_i w_i$ is pseudoinvex at u and $-y^T g(.)$ is quasi-invex at u with respect to the same η.

Then,

$$f(x) + A(x) \nleq f(u) + u^T w$$

Proof. Assume on contrary that
$f(x) + A(x) < f(u) + u^T w$
$\implies f_i(x) + s(x \mid D_i) < f_i(u) + u^T w_i, \qquad \forall i.$
Multiplying above equations by corresponding λ_i, for each i and adding them, we get

$$\sum_{i=1}^{l} \lambda_i (f_i(x) + s(x \mid D_i)) < \sum_{i=1}^{l} \lambda_i f_i(u) + u^T \sum_{i=1}^{l} \lambda_i w_i. \tag{3}$$

(a) Suppose $f_i(.) + (.)^T w_i, i \in \{1, 2, \cdots, l\}$ and $-y^T g(.)$ is invex at u.
Then,

$$f_i(x) + x^T w_i - f_i(u) - u^T w_i \geq \eta(x,u)^T (\nabla f_i(u) + w_i), \quad \forall i \tag{4}$$

$$-y^T g(x) + y^T g(u) \geq \eta(x,u)^T \nabla(-y^T g)(u). \tag{5}$$

Multiplying (4) for each i, by λ_i and adding, we get

$$\sum_{i=1}^{l} \lambda_i f_i(x) + x^T \sum_{i=1}^{l} \lambda_i w_i - \sum_{i=1}^{l} \lambda_i f_i(u) - u^T \sum_{i=1}^{l} \lambda_i w_i \geq \eta(x,u)^T \left(\sum_{i=1}^{l} \lambda_i \nabla f_i(u) + \sum_{i=1}^{l} \lambda_i w_i \right) \tag{6}$$

Adding (5) and (6), we get
$$\sum_{i=1}^{l} \lambda_i f_i(x) + x^T \sum_{i=1}^{l} \lambda_i w_i - y^T g(x) - \sum_{i=1}^{l} \lambda_i f_i(u) - u^T \sum_{i=1}^{l} \lambda_i w_i + y^T g(u) \geq \eta(x,u)^T \left(\sum_{i=1}^{l} \lambda_i \nabla f_i(u) + \sum_{i=1}^{l} \lambda_i w_i - \nabla(y^T g)(u) \right)$$

Using (1), (2) and the fact $-g(x) \in C^*$, it follows that

$$\sum_{i=1}^{l} \lambda_i f_i(x) + x^T \sum_{i=1}^{l} \lambda_i w_i - \sum_{i=1}^{l} \lambda_i f_i(u) - u^T \sum_{i=1}^{l} \lambda_i w_i \geq 0.$$

Since,

$$s(x \mid D_i) \geq x^T w_i, \quad \forall\, i, \tag{7}$$

we get,
$\sum_{i=1}^{l} \lambda_i (f_i(x) + s(x \mid D_i)) \geq \sum_{i=1}^{l} \lambda_i f_i(u) + u^T \sum_{i=1}^{l} \lambda_i w_i$, which contradicts (3).
Hence $f(x) + A(x) \nless f(u) + u^T w$.

(b) Suppose $\sum_{i=1}^{l} \lambda_i f_i(.) + (.)^T \sum_{i=1}^{l} \lambda_i w_i$ is pseudoinvex and $-y^T g(.)$ is quasi-invex at u. Using (7) ,(3) implies that
$\sum_{i=1}^{l} \lambda_i f_i(x) + x^T \sum_{i=1}^{l} \lambda_i w_i < \sum_{i=1}^{l} \lambda_i f_i(u) + u^T \sum_{i=1}^{l} \lambda_i w_i.$

Since $\sum_{i=1}^{l} \lambda_i f_i(.) + (.)^T \sum_{i=1}^{l} \lambda_i w_i$ is a real valued pesudoinvex function at u, therefore,
$\eta(x,u)^T \left[\nabla(\lambda^T f)(u) + \sum_{i=1}^{l} \lambda_i w_i \right] < 0$
From (1), we have $\eta(x,u)^T (\nabla(y^T g)(u)) < 0$.
$\implies \eta(x,u)^T (\nabla(-y^T g)(u)) > 0$.
By quasi-invexity of $-y^T g(.)$, we get

$$-y^T g(x) > -y^T g(u) \tag{8}$$

From (2) and $-g(x) \in C^*$, we obtain,
$-y^T g(x) \leq -y^T g(u)$ which contradicts (8) .
Hence $f(x) + A(x) \nless f(u) + u^T w$.

□

In order to prove strong duality we need the following lemma ([2], [15], [10]).

Lemma 3.1. *If $\bar{x}$ is a weakly efficient solution (resp. efficient) of (NMP) at which generalized Slater constraint qualification is satisfied. Then there exist $\overline{w_i} \in D_i, i \in \{1, 2, \cdots, l\}, \bar{\lambda} \geq 0$ (resp.$\bar{\lambda} > 0$) and $\bar{y} \in C$ such that*

$$(\bar{\lambda}^T \nabla f(\bar{x}) + \sum_{i=1}^{l} \bar{\lambda}_i \bar{w_i}^T - \bar{y}^T \nabla g(\bar{x}))$$
$$(x - \bar{x}) \geq 0, \qquad \forall x \in \mathbb{R}^n$$
$$\bar{y}^T g(\bar{x}) = 0,$$
$$s(\bar{x} \mid D_i) = \bar{x}^T \bar{w}_i, \quad i \in 1, 2 \cdots, l.$$

Theorem 3.2. *(Strong Duality):*
If $\bar{x}$ is a weakly efficient solution of (NMP) at which generalized Slater constraint qualification holds. Then there exist $\bar{\lambda} \geq 0, \bar{y} \in C$ and $\bar{w}_i \in D_i\{i = 1, 2 \cdots, l\}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_l)$ is feasible for (MWD) and the corresponding values of (NMP) and (MWD) are equal. If the assumptions of the Theorem 3.1 are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_l)$ is weakly efficient for (MWD).

Proof. Since $\bar{x}$ is a weakly efficient solution of (NMP), by Lemma 3.1 there exist $\bar{w}_i \in D_i, i \in \{1, 2, \cdots, l\}, \bar{\lambda} \geq 0$ and $\bar{y} \in C$ such that

$$\left[\sum_{i=1}^{l} \bar{\lambda}_i (\nabla f_i(\bar{x})^T + \bar{w}_i{}^T) - \bar{y}^T \nabla g(\bar{x})\right]$$
$$(x - \bar{x}) \geq 0, \quad \forall x \in \mathbb{R}^n, \tag{9}$$
$$\bar{y}^T g(\bar{x}) = 0, \tag{10}$$
$$s(\bar{x} \mid D_i) = \bar{x}^T \bar{w}_i \quad i \in \{1, \cdots, l\}, \tag{11}$$
$$\bar{\lambda} \geq 0, \bar{\lambda}^T e = 1.$$

Since $\langle u, v \rangle \geq 0, \forall\, v \in \mathbb{R}^n$ implies $u = 0$.
Therefore (9) implies,

$$\left[\sum_{i=1}^{l} \bar{\lambda}_i (\nabla f_i(\bar{x})^T + \bar{w}_i{}^T) - \bar{y}^T \nabla g(\bar{x})\right] = 0$$

that is, $\left[\sum_{i=1}^{l} \bar{\lambda}_i (\nabla f_i(\bar{x}) + \bar{w}_i) - \nabla g(\bar{x})^T \bar{y}\right] = 0.$

Now,

$$\begin{aligned}
\nabla g(\bar{x})^T \bar{y} &= \sum_{i=1}^{m} \bar{y}_i \nabla g_i(\bar{x}) \\
&= \nabla \left(\sum_{i=1}^{m} \bar{y}_i g_i(\bar{x})\right) \\
&= \nabla (\bar{y}^T g)(\bar{x}).
\end{aligned}$$

Therefore, $\nabla(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{l} \bar{\lambda}_i \bar{w}_i = \nabla(\bar{y}^T g)(\bar{x})$.

Thus from (10) we get $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_l)$ is feasible for (MWD).
Using (11), the objective function value of (NMP) is

$$\begin{aligned}
f(\bar{x}) + A(\bar{x}) &= f(\bar{x}) + (s(\bar{x} \mid D_1), \cdots, s(\bar{x} \mid D_l))^T \\
&= f(\bar{x}) + \left(\bar{x}^T \bar{w}_1, \cdots, \bar{x}^T \bar{w}_l\right)^T \\
&= f(\bar{x}) + \bar{x}^T \bar{w}.
\end{aligned}$$

where, $f(\bar{x}) + \bar{x}^T \bar{w}$ is the objective function value of (MWD).
Hence, the corresponding values of (NMP) and (MWD) are equal.
Let the assumptions of Theorem 3.1 hold and assume on contrary that
$(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_l)$ is not weakly efficient for (MWD). Then, there exists a feasible solution of (MWD) say $(\hat{x}, \hat{y}, \hat{\lambda}, \hat{w}_1, \hat{w}_2, \cdots, \hat{w}_l)$, such that

$$\begin{aligned}
f(\hat{x}) + \hat{x}^T \hat{w} &> f(\bar{x}) + \bar{x}^T \bar{w} \\
&= f(\bar{x}) + A(\bar{x})
\end{aligned}$$

which contradicts Weak Duality 3.1. Therefore, $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_1, \bar{w}_2, \cdots, \bar{w}_l)$ is weakly efficient for (MWD). □

4 Higher Order Duality

In this section, we formulate higher order duals to (NMP) which have great computational advantage over the first order duals. Throughout the section $\eta(., .)$ is a vector valued function taking values in $\mathbb{R}^n$. We propose the following Mond-Weir higher order multiobjective dual problem (MMCD) to (NMP):

$$\begin{aligned}\text{Maximize}\quad & f(u)+u^Tw+(\lambda^Th)(u,p)e \\ & -(p^T\nabla_p(\lambda^Th)(u,p))e \end{aligned} \tag{MMCD}$$

$$\text{subject to}\quad \nabla_p(\lambda^Th)(u,p) + \sum_{i=1}^{l}\lambda_i w_i = \nabla_p(y^Tk)(u,p), \tag{12}$$

$$y^T(g(u)+k(u,p)-p^T\nabla_p k(u,p))\leq 0, \tag{13}$$

$$w_i\in D_i, i=1,\ldots,l,$$

$$y\in C, \lambda>0, \lambda^Te=1.$$

where

(i) $e=(1,\ldots,1)^T\in\mathbb{R}^l$,

(ii) $u^Tw=[u^Tw_1,\ldots,u^Tw_l]^T$.

(iii) $h:\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^l$ and $k:\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^m$ are differentiable functions with $\nabla_p h_j(u,p)$ and $\nabla_p(y^Tk)(u,p)$ as the $n\times 1$ gradient of h_j and y^Tk with respect to p, respectively.

(iv) $p^T\nabla_p k(u,p)$ denotes the vector $\left[p^T\nabla_p k_1(u,p),\cdots,p^T\nabla_p k_m(u,p)\right]^T$.

Remark 4.1. *Again, in comparison to dual proposed in [8], first constraint is slightly change to ensure that vectors on both the sides of equality are comparable.Also, the second constraint in [8] is taken as:* $g(u)+k(u,p)-p^Tk(u,p)\in C_2^*$ *and* $x\in C_1$ *where* C_1 *and* C_2 *are closed convex cones. However, with these conditions strong duality does not hold as* $[\overline{\lambda}^T(\nabla f(\overline{x})+w)-\overline{y}^T\nabla g(\overline{x})]^Tx\geqq 0$*, for all* $x\in C_1$ *,*
$\not\Rightarrow \overline{\lambda}^T(\nabla f(\overline{x})+w)-\overline{y}^T\nabla g(\overline{x})=0$*. Moreover,* $\overline{y}^Tg(\overline{x})\leqq 0\not\Rightarrow g(\overline{x})\in C_2^*$*. Therefore, the above model was considered and duality results were established.*

Weak and strong duality theorems are proved below.

Theorem 4.1. *(Weak Duality):*
Let x *and* $(u,y,\lambda,w_1,\cdots,w_l,p)$ *be feasible solutions of (NMP)and (MMCD) respectively. Assume that one of the following holds:*

(a) $\left(\sum_{i=1}^{l}\lambda_i f_i(.)+(.)^T\sum_{i=1}^{l}\lambda_i w_i, -y^Tg(.)\right)$ *is higher order pseudoquasi-type I at* u.

(b) $\left(\sum_{i=1}^{l}\lambda_i f_i(.)+(.)^T\sum_{i=1}^{l}\lambda_i w_i, -y^Tg(.)\right)$ *is higher order* (F,ρ) *type I at* u *with* $\rho^1+\rho^2\geq 0$.

(c) $\left(\sum_{i=1}^{l}\lambda_i f_i(.)+(.)^T\sum_{i=1}^{l}\lambda_i w_i, -y^Tg(.)\right)$ *is higher order* (F,ρ)*-pseudoquasi-type I at* u *with* $\rho^1+\rho^2\geq 0$.

Then,
$f(x)+A(x)\nleq f(u)+u^Tw+(\lambda^Th)(u,p)e-(p^T\nabla_p(\lambda^Th)(u,p))e$.

Remark 4.2. *The functions in above conditions are all real-valued functions defined on* $\mathbb{R}^n$*, therefore the inequalities in Definitions 2.5 and 2.6 reduce to general ordering in* $\mathbb{R}$*. Also,* ρ^1 *and* ρ^2 *involved are not vectors in* $\mathbb{R}^l$ *and* $\mathbb{R}^m$ *respectively but mere real numbers.*

Proof. Assume to the contrary that
$f(x)+A(x)\leq f(u)+u^Tw+(\lambda^Th)(u,p)e-(p^T\nabla_p(\lambda^Th)(u,p))e$.
$\Longrightarrow$
$f_i(x)+s(x\mid D_i)\leq f_i(u)+u^Tw_i+\lambda^Th(u,p)-p^T\nabla_p(\lambda^Th)(u,p), \forall\, i\in\{1,\cdots,l\}$
and
$f_r(x)+s(x\mid D_r)<f_r(u)+u^Tw_r+\lambda^Th(u,p)-p^T\nabla_p(\lambda^Th)(u,p)$, for atleast one $r\in\{1,\cdots,l\}$.
Multiplying each of the above equations with corresponding λ_i and summing up, we get

$$\begin{aligned}\sum_{i=1}^{l}\lambda_i(f_i(x)+s(x\mid D_i)) &< \\ \sum_{i=1}^{l}\lambda_i f_i(u)+u^T\sum_{i=1}^{l}\lambda_i w_i & \\ +\lambda^Th(u,p)-p^T\nabla_p(\lambda^Th)(u,p). &\end{aligned} \tag{14}$$

(a) Suppose $\left(\sum_{i=1}^{l}\lambda_i f_i(.)+(.)^T\sum_{i=1}^{l}\lambda_i w_i, -y^Tg(.)\right)$ is higher order pseudoquasi-type I at u.

By (13), we have
$-(-y^Tg(u)) - (-y^Tk(u,p)) + p^T\nabla_p(-(y^Tk)(u,p)) \leq 0.$
$\Longrightarrow \eta(x,u)^T\left[\nabla_p(-(y^Tk)(u,p))\right] \leq 0.$
$\Longrightarrow \eta(x,u)^T\left[\nabla_p(y^Tk)(u,p)\right] \geq 0.$
Using (12), we get

$$\eta(x,u)^T\left[\nabla_p(\lambda^Th)(u,p) + \sum_{i=1}^{l}\lambda_i w_i\right] \geq 0.$$

$$\Longrightarrow \sum_{i=1}^{l}\lambda_i f_i(x) + x^T\sum_{i=1}^{l}\lambda_i w_i - \sum_{i=1}^{l}\lambda_i f_i(u) - u^T\sum_{i=1}^{l}\lambda_i w_i - \lambda^Th(u,p) + p^T\nabla_p(\lambda^Th)(u,p) \geq 0$$

Using (7) and rearranging the terms , we have

$$\sum_{i=1}^{l}\lambda_i(f_i(x) + s(x \mid D_i)) \geq \sum_{i=1}^{l}\lambda_i f_i(u) + u^T\sum_{i=1}^{l}\lambda_i w_i + \lambda^Th(u,p) - p^T\nabla_p(\lambda^Th)(u,p)$$

which contradicts (14)
Hence $f(x) + A(x) \nleq f(u) + u^Tw + (\lambda^Th)(u,p)e - (p^T\nabla_p(\lambda^Th)(u,p))e.$

(b) Suppose $\left(\sum_{i=1}^{l}\lambda_i f_i(.) + (.)^T\sum_{i=1}^{l}\lambda_i w_i, -y^Tg(.)\right)$ is higher order (F,ρ) type I at u with $\rho^1+\rho^2 \geq 0$.

$$\Longrightarrow \sum_{i=1}^{l}\lambda_i f_i(x) + x^T\sum_{i=1}^{l}\lambda_i w_i - \sum_{i=1}^{l}\lambda_i f_i(u) - u^T\sum_{i=1}^{l}\lambda_i w_i \geq F(x,u,\nabla_p(\lambda^Th)(u,p) + \sum_{i=1}^{l}\lambda_i w_i) + \lambda^Th(u,p) - p^T\nabla_p(\lambda^Th)(u,p) + \rho^1 d^2(x,u).$$

and

$$-(-y^Tg(u)) + y^Tk(u,p) - p^T\nabla_p(y^Tk)(u,p) \geq F(x,u,-\nabla_p(y^Tk)(u,p)) + \rho^2 d^2(x,u).$$

Adding above two inequalities and using sublinearity of F, we get

$$\sum_{i=1}^{l}\lambda_i f_i(x) + x^T\sum_{i=1}^{l}\lambda_i w_i - \sum_{i=1}^{l}\lambda_i f_i(u) - u^T\sum_{i=1}^{l}\lambda_i w_i - \lambda^Th(u,p) + p^T\nabla_p(\lambda^Th)(u,p) - [-y^Tg(u) - y^Tk(u,p) + p^T\nabla_p(y^Tk)(u,p)]$$

$$\geq F(x,u,\nabla_p(\lambda^Th)(u,p) + \sum_{i=1}^{l}\lambda_i w_i) + F(x,u,-\nabla_p(y^Tk)(u,p)) + (\rho^1+\rho^2)d^2(x,u)$$

$$\geq F(x,u,\nabla_p(\lambda^Th)(u,p) + \sum_{i=1}^{l}\lambda_i w_i - \nabla_p(y^Tk)(u,p)) + (\rho^1+\rho^2)d^2(x,u)$$

$\geq 0,$ because (12) holds.
From (13), we obtain
$y^T\left[g(u) + k(u,p) - p^T\nabla_p k(u,p)\right] \leq 0.$
$\Longrightarrow$

$$\sum_{i=1}^{l}\lambda_i f_i(x) + x^T\sum_{i=1}^{l}\lambda_i w_i - \sum_{i=1}^{l}\lambda_i f_i(u) - u^T\sum_{i=1}^{l}\lambda_i w_i - \lambda^Th(u,p) + p^T\nabla_p(\lambda^Th)(u,p)$$
$$\geq -y^Tg(u) - y^Tk(u,p) + p^T\nabla_p((y^Tk)(u,p))$$
$$\geq 0.$$

Again, using (7) we get,

$$\sum_{i=1}^{l}\lambda_i(f_i(x) + s(x \mid D_i)) \geq \sum_{i=1}^{l}\lambda_i f_i(u) + u^T\sum_{i=1}^{l}\lambda_i w_i + \lambda^Th(u,p) - p^T\nabla_p(\lambda^Th)(u,p).$$

which contradicts (14)
Hence $f(x) + A(x) \nleq f(u) + u^Tw + (\lambda^Th)(u,p)e - (p^T\nabla_p(\lambda^Th)(u,p))e.$

(c) Suppose $\left(\sum_{i=1}^{l}\lambda_i f_i(.) + (.)^T\sum_{i=1}^{l}\lambda_i w_i, -y^Tg(.)\right)$ is higher order (F,ρ)-pseudoquasi-type I at u with $\rho^1 + \rho^2 \geq 0$.
Using (7), (14) implies

$$\sum_{i=1}^{l}\lambda_i f_i(x) + x^T\sum_{i=1}^{l}\lambda_i w_i < \sum_{i=1}^{l}\lambda_i f_i(u) + u^T\sum_{i=1}^{l}\lambda_i w_i + \lambda^Th(u,p) - p^T\nabla_p(\lambda^Th)(u,p).$$

$$\Longrightarrow \sum_{i=1}^{l}\lambda_i f_i(x) + x^T\sum_{i=1}^{l}\lambda_i w_i - \sum_{i=1}^{l}\lambda_i f_i(u) -$$

$u^T \sum_{i=1}^{l} \lambda_i w_i - \lambda^T h(u,p) +$
$p^T \nabla_p (\lambda^T h)(u,p) < 0.$
By the given hypothesis, we get

$$F(x,u,\nabla_p(\lambda^T h)(u,p) + \sum_{i=1}^{l} \lambda_i w_i) < -\rho^1 d^2(x,u) \tag{15}$$

From (13), we get
$-(-y^T g)(u) - (-y^T k(u,p)) + p^T \nabla_p(-(y^T k)(u,p)) \leq 0$

$$\implies F(x,u,\nabla_p(-(y^T k)(u,p))) \leq -\rho^2 d^2(x,u) \tag{16}$$

Using sublinearity of F, (15) and (16) we get
$F(x,u,\nabla_p(\lambda^T h)(u,p) - (y^T k)(u,p)) + \sum_{i=1}^{l} \lambda_i w_i) < -(\rho^1+\rho^2) d^2(x,u).$
Since $\rho^1 + \rho^2 \geq 0$ and $d^2(x,u) \geq 0$, therefore $-(\rho^1+\rho^2)d^2(x,u) \leq 0$.
$\implies F(x,u,\nabla_p(\lambda^T h)(u,p) - (y^T k)(u,p)) + \sum_{i=1}^{l} \lambda_i w_i) < 0$, which contradicts (13).
Hence $f(x) + A(x) \nleq f(u) + u^T w + \lambda^T h(u,p)e - (p^T \nabla_p(\lambda^T h)(u,p))e$.

□

Theorem 4.2. *(Strong duality):*
If $\bar{x}$ is an efficient solution of (NMP) at which generalized Slater constraint qualification is satisfied. Let $h(\bar{x},0) = k(\bar{x},0) = 0, \nabla_p h(\bar{x},0) = \nabla f(\bar{x})$ and $\nabla_p k(\bar{x},0) = \nabla g(\bar{x})$. Then there exist $\bar{\lambda} > 0, \bar{y} \in C$ and $\bar{w}_i \in D_i\{i = 1,2\cdots,l\}$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_l,\bar{p} = 0)$ is feasible for (MMCD) and the corresponding values of (NMP) and (MMCD) are equal. If the assumptions of the Theorem 4.1 are satisfied, then $(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_l,\bar{p} = 0)$ is efficient for (MMCD).

Proof. Since $\bar{x}$ is an efficient solution of (NMP), therefore by Lemma 3.1, there exist $\bar{\lambda} > 0, \bar{y} \in C$ and $\bar{w}_i \in D_i\{i = 1,2\cdots,l\}$ such that (9), (10) and (11) holds.
As done in Theorem 3.2, we get
$\sum_{i=1}^{l} \bar{\lambda}_i (\nabla f_i(\bar{x}) + \bar{w}_i) = \nabla \bar{y}^T g(\bar{x}).$
From the given hypothesis, we get
$\nabla_p(\bar{\lambda}^T h)(\bar{x},0) + \sum_{i=1}^{l} \bar{\lambda}_i \bar{w}_i = \nabla_p(\bar{y}^T k)(\bar{x},0)$
From (10), we have
$\bar{y}^T [g(\bar{x}) + k(\bar{x},0) - 0] = \bar{y}^T g(\bar{x}) = 0.$
Hence,$(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_l,\bar{p} = 0)$ is feasible for (MMCD).
Since $\bar{p} = 0, h(\bar{x},0) = 0$ and (11) holds, the objective function value of (MMCD) is

$$\begin{aligned} f(\bar{x}) + \bar{x}^T\bar{w} + (\bar{\lambda}^T h)(\bar{x},0)e - \bar{p}^T \nabla_p(\bar{\lambda}^T h)(\bar{x},0)e &= f(\bar{x}) + \bar{x}^T \bar{w} \\ &= f(\bar{x}) + \left[\bar{x}^T\bar{w}_1,\cdots,\bar{x}^T\bar{w}_l\right]^T \\ &= f(\bar{x}) + \left[s(\bar{x} \mid D_1),\cdots,s(\bar{x} \mid D_l)\right]^T \end{aligned}$$

where $f(\bar{x}) + \left[s(\bar{x} \mid D_1),\cdots,s(\bar{x} \mid D_l)\right]^T$ is the objective function value of (NMP).
Suppose that the assumptions of Theorem 4.1 are satisfied but on contrary
$(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_l,\bar{p} = 0)$ is not an efficient solution of (MMCD). Then there exists a feasible solution $(\hat{x},\hat{y},\hat{\lambda},\hat{w}_1,\hat{w}_2,\cdots,\hat{w}_l,\hat{p})$, to (MMCD) such that

$$f(\hat{u}) + \hat{u}^T\hat{w} + (\hat{\lambda}^T h)(\hat{u},\hat{p})e - (\hat{p}^T \nabla_p(\hat{\lambda}^T h)(\hat{u},\hat{p}))e \geq f(\bar{x}) + \bar{x}^T\bar{w}$$

where $f(\bar{x}) + \bar{x}^T\bar{w}$ is the objective function value of (NMP) which is a contradiction to Weak Duality Theorem 4.1 .
Hence, $(\bar{x},\bar{y},\bar{\lambda},\bar{w}_1,\bar{w}_2,\cdots,\bar{w}_l,\bar{p} = 0)$ is an efficient solution of (MMCD). □

Remark 4.3. *Throughout the above section we are using*

$$y^T p^T \nabla_p k(u,p) = p^T \nabla_p(y^T k)(u,p)$$

which is evident from our set notations.

5 Conclusion

In this paper, the main focus was to give the modified (corrected) versions of Mond-Weir type duals and established duality results. Wolfe type duals discussed in [7],[8] can be slightly modified to ensure vectors under consideration are comparable. This paper provides a base for studying unified first order and higher order dual for (NMP). Unified dual provides a common platform to study both Wolfe type as well as Mond-Weir type duals. For Unified dual considered in [1] strong duality does not hold.

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