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A Fully Polynomial Time Approximation Scheme for Weight Constrained BTSP with Two Linear Constraints on Halin Graphs

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Abstract: ***In this paper we show that the weight constrained version of BTSP i.e. WCBTSP on a Halin graph with n nodes can be solved in O(nlogn) time. We also show that WCBTSP with two linear constraints on a Halin graph can be solved in $O(n\ (W_1+1)^2 \log n)$ time, where W_1 denotes the first right hand side constant.***

Keywords: Bottleneck Travelling Salesman Problem, Halin graph, NP-Complete, Threshold algorithm, polynomial time, approximation scheme.

1. Introduction

The bottleneck travelling salesman problem (BTSP) is to find a hamilton cycle (tour) T on a Halin graph H such that the largest cost of the edges in the tour T is as small as possible. In this paper we are interested in computing a tour T subject to the constraint w(T) ≤ W such that the largest cost of edges in the tour is as small as possible. This problem is called the Weight Constrained Bottleneck Travelling Salesman Problem (WCBTSP).

The BTSP on a general graph is known to be NP-hard. Thus, WCBTSP on a Halin graph is also NP-hard. Cornuejol et.el. [1] have given *O(n)* algorithm for solving TSP on a Halin graph with n nodes. Using this algorithm as a subroutine within binary search version of threshold algorithm, Edomond, Fulkerson [3], BTSP on Halin graph can be solved in *O(nlogn)* time. In this paper we show that the weight constrained version of BTSP i.e. WCBTSP on a Halin graph with a *n* nodes can also be solved in *O(nlogn)* time. Using our $O(n(W_1+1)^2)$ algorithm, Gahir [7] to determine a minimum cost hamilton cycle with weight no more than a given bound W_1, we also show that WCBTSP with two linear constraints on a Halin graph can be solved in $O(n\ (W_1+1)^2 \log n)$ time, where W_1 denotes the first right hand side constant.

This paper is organized as follows. In section 1.2, we introduce some notation and give preliminary results. In section 1.3, we present a threshold algorithm for BTSP on a Halin graph. Section 1.4, considers an *O(nlogn)* threshold algorithm for WCTSP and in section 1.5, we give $O(n\ (W_1+1)^2 \log n)$ threshold algorithm for WCTSP with two linear constraints making concluding remarks in the final section 1.6.

2. Notation and Basic Results

A Halin graph H is obtained by embedding a tree T having no nodes of degree 2 in the plane, and then adding a cycle C to join the leaves of H in such a way that the resulting graph is planar. We write $H = T \cup C$. These graphs are edge minimal 3-connected, and in general have a large number of hamilton cycles. The bottleneck travelling salesman problem can be defined as follows:

BTSP:

$$\min imize: \quad \max_{T \in F} imize \quad \{c(e) : e \in T\}$$

Where F is the family of hamilton cycles on H.

The BTSP on a general graph is known to be NP-hard but BTSP on a Halin graph can be solved in *O*(nlogn) time using binary search version of threshold algorithm Edmonds [3]. Phillips [2] have improved on this result and have given an *O*(n) algorithm for solving BTSP on a Halin graph.

In this paper we have considered Weight Constrained Bottleneck Travelling Salesman Problem (WCTSP) on a Halin graph $H = (V, E)$ and have given *O*(nlogn) threshold algorithm for solving it. Let $\{ c(e), w(e)\}$ be the set of costs and weights respectively on edges $e \in E$. The WCBTSP can be defined as:

WCBTSP:

$$\min imize: \quad \max_{T \in F} imize \quad \{c(e) : e \in T\} \quad and \quad \sum_{e \in T} w(e) \le W$$

Where F is the family of hamilton cycles on H and W is the weight constant.

1.1 The Threshold algorithm for BTSP on a Halin graph H

begin
set Q = E
while |Q| ≠ 1do
begin

Let $M_0 = Median\ of\ \{c(e): e \in Q\}$,
$F(M_0) = \{T \in F : e \in T \Rightarrow c(e) \le M_0\}$
if $F(M_0) \ne \phi$ then
$Q = \{e: c(e) \le M_0, e \in Q\}$
else
$Q = \{e: c(e) > M_0, e \in Q\}$
endif
end
end

When we estimate the median M_0 for the edge set Q

Iteratively, we every time halve its size and |Q| can be at most $n(n-1)/2 \approx n^2$ .So if *f* is the number of iterations then $n^2 = 2f$. Thus $f = 2 \log n$ or $O(\log n)$. Again we can check whether $F(M_0)$ is ϕ or not using *O*(n) time given in Cornuejols [1]. Therefore the time complexity for BTSP threshold algorithm on a Halin graph becomes $O(n\log n)$

1.2 The Threshold Algorithm for the WCBTSP on a Halin Graph

Consider the Halin graph $H = (V, E)$. Let BIG be a very large number say L where $L > W+1$. Define the modified weight $w'(e)$ for edges $e \in E$ as follows:

if $c(e) > M_0$ then
$w'(e) = BIG$
else
$w'(e) = w(e)$
/* call the modified problem WCBTSP(W') */

Define $F(M_0) = \left\{T \in F : \sum_{e \in T} w'(e) \le W\right\}$, i.e. $F(M_0)$ is the family of hamilton cycles such that for any tour $T \in F(M_0)$,T does not contain edges with weight equal to BIG. Now $F(M_0) \ne \phi$ if there exist a tour T in $F(M_0)$ such that $\sum_{e \in T} w'(e) \le W$ .The feasibility condition can be checked in O(n) time using Cornuejol's algorithm [1], for obtaining minimum weight tour with respect to costs w'(e). $F(M_0) = \phi$ implies there does not exist a tour T such that $\sum_{e \in T} w'(e) \le W$ and $c(e) \le M_0$ for all $e \in T$.

Theorem 1:
The Weight Constrained Bottleneck Travelling Salesman Problem (WCBTSP) on a Halin graph $H = (V, E)$ can be solved in $O(n\log n)$ time, where n is the number of vertices in the Halin graph.

Proof:
In order to prove the theorem we first find out the time required to check if $F(M_0) = \phi$ or not. $F(M_0) \ne \phi$ if and only if there exist a tour T in $F(M_0)$ such that $\sum_{e \in T} w'(e) \le W$ .This can be checked in *O(n)* time using Cornuejol's algorithm, for obtaining minimum weight tour with respect to cost coefficients as $w'(e), e \in E$. *O(log n)* iterates will be required till we get |Q| = 1 . Thus threshold algorithm has time complexity O(nlogn) for solving WCTSP.

1.3 The Threshold Algorithm for WCBTSP with Two Linear Constraints

The problem can be defined as:

$$\min imize: \quad \max imize \quad \{c(e): e \in T\}$$
$$T \in F \qquad and$$
$$\sum_{e \in T} w(e) \le W_1$$
$$\sum_{e \in T} u(e) \le W_2$$

where F is the family of hamilton cycles on the Halin graph H and W_1, W_2 are right hand side constants.

Let BIG be a very large number say L where $L > W_2 + 1$. Define the modified u(e) values for $e \in E$ as follows:

if $c(e) > M_0$ then
$u'(e) = BIG$
else
$u'(e) = u(e)$

We now define the transformed WCBTSP as follows:

$$\min imize: \quad \max imize \quad \{c(e): e \in T\}$$
$$T \in F \qquad and$$
$$\sum_{e \in T} w(e) \le W_1 ,$$
$$\sum_{e \in T} u'(e) \le W_2$$

Consider the WCTSP(*):

$$\min \sum_{e \in T} u'(e)$$

such that $\sum_{e \in T} w(e) \le W_1$

$T \in F$

To solve WCTSP(*), we have an $O(n(W_1+1)^2)$ algorithm Gahir[7], to determine a minimum cost hamilton cycle with weight no more than a given bound W_1, where n is the number of vertices on H.

Define

$$F(M_0) = \left\{T \in F : \sum_{e \in T} w(e) \le W_1; \sum_{e \in T} u'(e) \le W_2\right\}$$

Now $F(M_0) \neq \phi$ if and only if there exist a tour T in $F(M_0)$ such that $\sum_{e \in T} w(e) \leq W_1$ and $\sum_{e \in T} u'(e) \leq W_2$

We can test $F(M_0) = \phi$ or not in $O(n\,(M+1)^2)$ time using Theorem 1 for WCBTSP(*). If the optimal value of WCBTSP(*) $\leq$ W_2 $\Rightarrow$ $F(M_0) \neq \phi$, otherwise $F(M_0) = \phi$. $F(M_0) \neq \phi$ implies there exist a tour T in $F(M_0)$ such that T does not contain edges e with u(e) equal to BIG. $F(M_0) = \phi$ $\Rightarrow$ all tours satisfying $\sum_{e \in T} w(e) \leq W_1$ do not satisfy the transformed constraint $\sum_{e \in T} u'(e) \leq W_2$

Threshold Algorithm for WCBTSP with Two Linear Constraints

begin
let Q = E
while $|Q| \neq 1$ do
begin
Let M_0=Median of { c(e): e Q }
for each edge $e \in E$
if $c(e) > M_0$ then
u'(e) = BIG
else
u'(e) = u(e)
/*BIG a very large number say L */
endif
endfor
Let $F(M_0) = \left\{ T \in F : \sum_{e \in T} w(e) \leq W_1; \sum_{e \in T} u'(e) \leq W_2 \right\}$
/* $F(M_0) \neq \phi$, implies there exist a tour T in $F(M_0)$ which does not contain an edge with u(e) equal to BIG and T satisfies weight constraint and $c(e) \leq M_0$ for all $e \in T$. */
if $F(M_0) \neq \phi$, then
$Q = \{ e : c(e) \leq M_0 , e \in Q \}$
else
$Q = \{ e : c(e) > M_0 , e \in Q \}$
endif
end
end

Theorem 2:
Two linear constrained bottleneck travelling salesman problem on Halin graph with *n* nodes, can be solved in $O(n(W_1+1)^2 \log n)$ where W_1 is a right hand side constant.

Proof:
The time complexity of the threshold algorithm is $O(n(W_1+1)^2 \log n)$ because $O(n(W_1+1)^2)$ is the time complexity of the algorithm to solve a WCBTSP that verifies $F(M_0) \neq \phi$ or not. $O(\log n)$ iterations will be required in binary search till |Q| = 1. Thus a threshold algorithm for solving WCBTSP with two linear constraints is a pseudo polynomial time algorithm.

1.4 Concluding Remarks

In this paper, we propose an $O(\log n)$ time threshold algorithm to solve WCBTSP on a Halin graph. We then provide an $O(n(W_1+1)^2 \log n)$ time threshold algorithm for WCBTSP with two linear constraints on a Halin graph. An attempt was made to see if an *O*(n) algorithm for WCBTSP with one constant on the line of the *O*(n) Phillips et.el.[2] can be constructed but without success. On the positive side, we have presented a fully polynomial time approximation scheme for this problem, which has application to many optimization problems.

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