Fuzzy Metric on Fuzzy Linear Spaces

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Abstract: ***In this paper, we introduce metric on a subset of a fuzzy linear space and some of its properties are discussed. In the sequel, we proved that a norm on a fuzzy linear space (in sense of C. P. Santhosh and T. V. Ramakrishnan [1]) induces a metric of fuzzy linear spaces (in our sense).***

Keywords: Fuzzy field, fuzzy linear space, fuzzy metric space (linear space), norm on a fuzzy linear space.

1. Introduction

How to define a fuzzy metric is one of the fundamental problems in fuzzy mathematics which is widely used in fuzzy optimization and pattern recognition. Different authors introduced different notion of metric on a fuzzy set from different view point. K.C. Wong [2] defined fuzzy point and discussed some topological properties. Zike Dong [3] defined Pseudo- metric spaces with metric defined between fuzzy points rather than between fuzzy sets. Nai-Hung Hsu [4] introduced fuzzy metric space with metric defined between fuzzy points. Gu Wenxiang and Tu Lu [5] introduced notions of fuzzy field and fuzzy linear spaces over fuzzy field. Thereafter, C. P. Santhosh and T.V. Ramakrishna [1] introduced the concept of norm and inner product on fuzzy linear spaces. This paper is an attempt to define a metric of fuzzy set (fuzzy linear space over fuzzy field) contained in fuzzy linear spaces so that a norm defined by [1] induces a metric on fuzzy linear spaces.

2. Brief summary of Fuzzy Field and Fuzzy Linear Spaces

In this paper, F represents $\Re$ the set of all real numbers, or $\mathbb{C}$ the set of all complex numbers.

Definition 2.1 [5] Let F be a field and let K be fuzzy set in F with membership function μ . Suppose the following conditions hold

(1) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$
(2). $\mu(-x) \geq \mu(x)$
(3). $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ (4). $\mu(x^{-1}) \geq \mu(x)$

Then we call K is a fuzzy field in F(fuzzy field of F) and it is denoted by (K, F)

Proposition 2.2 [5] If (K, F) is a fuzzy field of F, then

(1). $\mu(0) \geq \mu(x), x \in F$ (2). $\mu(1) \geq \mu(x), x \neq 0$

Proposition 2.3 [1] If (K, F) is a fuzzy field of F, then

(1). $\mu(x) = \mu(-x), x \in F$ (2). $\mu(x^{-1}) = \mu(x), x \neq 0$

Proposition 2.4 [5] Let K and F be fields and $f: F \longrightarrow K$ be homomorphism. Suppose (X, F) is fuzzy a field of F and (Y, K) is a fuzzy field of K. Then

(i) $(f(X), K)$ is a fuzzy filed of K. (ii) $(f^{-1}(Y), F)$ a fuzzy field of F.

Definition 2.5 [5] Let F be a field and let K be fuzzy set in F with membership function μ.

Let X be a linear space over field F and U be a fuzzy set in X with membership function T. Suppose the following conditions hold:

(1) $T(x+y) \geq \min\{T(x), T(y)\}$ (2) $T(-x) \geq T(x)$
(3) $T(\lambda x) \geq \min\{\mu(\lambda), T(x)\}$ (4) $\mu(1) \geq T(0)$

Then we call (U, X) fuzzy linear space over fuzzy field (K, F).

Proposition 2.6 [1,5] If (U, X) is fuzzy linear space over fuzzy field (K,F).Then
(1) $\mu(0) \geq T(x)$ (2) $T(-x) = T(x)$ (3) $T(0) \geq T(x)$

Proposition 2.7 [5] Let X and Y be linear spaces over field F, let $f: X \longrightarrow Y$ be linear transformation. If (U, X) and (V, Y) are fuzzy linear spaces over fuzzy field (K, F),then

(i) $(f(U), Y)$ is a fuzzy linear spaces over fuzzy field of (K, F).
(ii) $(f^{-1}(V), X)$ a fuzzy linear space over fuzzy field of (K, F).

Proposition 2.8 [1] let $\{(K_i, F)\}$ be a fuzzy field over F, let $\{(V_i, X_i)\}_{i=1}^{n}$ be sequence of fuzzy linear spaces over (K_i, F), then $(V_1 x V_2 x \ldots x V_n, X_1 x X_2 x \ldots x X_n)$ is fuzzy linear space.

Proposition 2.9 [6] Let $U_1, U_2, \ldots, U_n$ be fuzzy sets in $X_1, X_2, \ldots, X_n$ respectively, then the Cartesian product is a fuzzy set in the product space $X_1 x X_2 x \ldots x X_n$, with membership function $T_{U x U_2 x \ldots x U_n}(x) = \min\{T_{U_i}(x_i)\}$: Where $x = (x_1, x_2, \ldots, x_n), x_i \in X_i$

3. Fuzzy Metric of Fuzzy Linear Spaces

In this section, a metric will be defined on a set contained in fuzzy linear space.

Notation: Throughout this section, the following notations will be used:

(i) (U, X) a fuzzy linear spaces over fuzzy field (K, F) with membership functions of U and K, T and μ respectively
(ii) . A is non empty fuzzy subset of X, we mean that $A \subseteq X$ and $T(x) \neq 0$ for every $x \in A$.

Definition3.2 Let (K, F) be fuzzy field in F , X be linear spaces over F, and let (U, X) be fuzzy linear spaces over (K, F).

Let $\emptyset \neq A \subseteq U$. A function, $d: Ax\, A \longrightarrow [0, \infty)$ satisfying the following conditions:
(1) $\mu(d(x, y)) \geq TAx\, A(x, y)$
(2) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
(3) $d(x, y) = d(y, x)$ for all $x, y \in A$
(4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in A$.

Then d is said to be fuzzy metric on (A, U) (fuzzy metric on A) and$((A, U), d)$ is called fuzzy metric space.

Example 3.3 Let X be a linear space over F, and let (U,X) be a fuzzy linear spaces over a fuzzy field (K, F). Let A be a nonempty subset of X. Consider a discrete metric $d, d: A\, x\, A \longrightarrow [0, \infty)$ given by $d(x, y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$. Then $((A, U), d)$ is a fuzzy metric space.

Proof: Clearly, d is metric on A , and hence it satisfies conditions (2)-(4) of definition 3.2. So, it suffices to verify definition 3.2(1). But,
$\mu(d(x, y)) = \begin{cases} \mu(1) \text{if } x \neq y \\ \mu(0) \text{ if } x = y \end{cases}$ .By definition 2.5(4) and proposition 2.6(1), we have $\mu(1) \geq T(x)$ and $\mu(0) \geq T(x)$. Thus, $\mu(1) \geq T_{Ax\, A}(x, y)$ and $\mu(0) \geq T_{Ax\, A}(x, y)$. Therefore, $((A, U), d)$ is fuzzy metric space.

A fuzzy metric as in example 3.3 will be referred as a discrete fuzzy metric on (A, U).

Example 3.4 Let $(F, \mathcal{R})$ be a fuzzy filed in $\mathcal{R}$. If $d: \mathcal{R}x\mathcal{R} \longrightarrow [0, \infty)$ is a mapping defined by
$d(x, y) = |x - y|$,then $((F,\mathcal{R}),d)$ is a fuzzy metric space.

Proof: Since d satisfies (2)-(4) of definition 3.2, we need to verify definition 3.2(1).But
$$\mu(d(x, y)) = \begin{cases} \mu(|x - y|) = \mu(x - y) \text{ if } x \geq y \\ \quad \mu(y - x) \text{ if } x < y \end{cases}$$
$$= \mu(x - y) \geq \min\{\mu(x), \mu(-y)\}$$
$$= \min\{\mu(x), \mu(y)\} = \mu_{Kx\, K}(x, y)$$

Therefore, $((F, \mathcal{R}), d)$ is fuzzy metric space.

We may define convergence of sequences in $((A, U), d)$ as follows.

Definition 3.5 Let $((A, U), d)$ be fuzzy metric spaces. A sequence $\{x_n\}$ is said to be convergent to $\{x_0\}$ (denoted by $min_{n \to \infty} x_n = x_o$) with respect to fuzzy metric d if and only if given $\epsilon > 0$,there exists a positive integer N such that for all $n \geq N, \mu(d(x_0, x_n)) \geq T_{AxA}(x_0, x_n)$ and $d(x_0, x_n) < \epsilon$

Remark 3.6 If limit of a sequence exists it is unique.

Definition 3.7 Let $((A, U), d)$ be a fuzzy metric spaces. A sequence $\{x_n\}$ is said to be Cauchy sequence with respect to d if and only if given $\in > 0$, there is a positive integer N such that for all $m, n \geq N, d(x_m, x_n) < \epsilon$ and $\mu(d(x_m, x_n)) \geq T_{AxA}(x_m, x_n)$.

Definition 3.8 A fuzzy metric space $((A, U), d)$ is said to be complete if and only if every Cauchy sequence of $((A, U), d)$ has a convergent subsequence.

Theorem 3.9 If a fuzzy metric space $((A, U), d)$ is complete then (A, d) is complete metric space

Proof: The result follows from definition 3.2.

Theorem 3.10 Suppose $((V_i, X_i), d_i)_{i=1}^n$ is the sequence fuzzy metric spaces over fuzzy fields (K_i, F) for each i=1,2,3,...,n, then $(V_1 x \ldots x\, V_n, X_1 x \ldots x\, X_n)$ is a fuzzy metric space.

Proof: Consider a mapping $d: X_1 x \ldots x\, X_n \longrightarrow [0, \infty)$ given by
$d(x, y) = \sum_{i=1}^n d_i(x_i, y_i)$, Where $x = (x_1, x_2, \ldots, x_n)$,$y=(y_1, y_2, \ldots, y_n)$. Then d is metric on $X_1 x \ldots x\, X_n$. Hence, it satisfies (2)-(4) of definition 3.2. Therefore, it suffices to verify definition 3.2(1). Now suppose μ_{K_i} and T_{V_i} are membership functions of K_i and V_i for all $i = 1, 2, \ldots, n$ respectively. Then,
$$\mu(d(x, y)) = \mu(\textstyle\sum_{i=1}^n d_i(x_i, y_i))$$
$$\geq \min\{\mu(d_1(x_1, y_1)), \mu(d_2(x_2, y_2)), \ldots, \mu(d_n(x_n, y_n))\}$$
$$\geq \min\{T_{V_1 x V_1}(x_1, y_1), T_{V_2 x V_2}(x_2, y_2), \ldots, T_{V_n x V_n}(x_n, y_n)\}$$
$$\geq \min\{\min\{T_{V_1}(x_1), T_{V_1}(y_1)\}, \ldots, \min\{T_{V_n}(x_n), T_{V_n}(y_n)\}\}$$
$$= \min\{\min\{T_{V_1}(x_1), \ldots, T_{V_n}(x_n)\}, \min\{T_{V_1}(y_1), \ldots, \mu_{V_n}(y_n)\}\}$$
$$= \min\{T_{V_1 x V_2 x \ldots x V_n}(x), T_{V_1 x V_2 x \ldots x V_n}(y)\} = T_{V_1 x V_2 x \ldots x V_n}(x, y)$$

Hence, $((V_1 x V_2 x \ldots x V_n, X x X_2 x \ldots x X_n), d)$ is a fuzzy metric space.

Example 3.11 Let $(K, \mathcal{R})$ be a fuzzy field of $\mathcal{R}$. A function $d{:}\mathcal{R}^n \longrightarrow [0, \infty)$ given by
$d(x, y) = \sum_{i=1}^n |x_i - y_i|$, where $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$ defines a metric on $(K_1, K_2, \ldots, K_n, \mathcal{R}^n)$.

Proof: The result follows from example 3.4 and theorem 3.10.

Theorem 3.12 Let X and Y be linear spaces over the field F. Let (U,X) be fuzzy linear spaces over fuzzy field (K,F), and let A be a non empty subset of X and B be a non empty subset of Y. If $f: A \longrightarrow B$ is a bijective mapping, then the following statements are equivalent

(1). (V,A) is fuzzy metric space.
(2). (f(V),B) is fuzzy metric space.

Proof:(1) $\Longrightarrow$ (2): Let $((V,A),d_A)$ be fuzzy metric space. Let $d_B: Bx\, B \longrightarrow [0,\infty)$ be given by $d_B(y_1,y_2) = d_A(x_1,x_2)$, where $y_i = f(x_i), i = 1,2$. Then clearly d_B defines a metric on B. Moreover,

$$\mu(d_B(w,z)) = \mu(d_A(x,y): w = fx, z = fy)$$
$$\geq T_{Ax\,A}((x,y) = min\,\{T(x),T(y)\}$$
$$= min\,\{T_{f(A)}(f(x)),T_{f(A)}(f(y))\}$$
$$= T_{f(A)x\,f(A)}(w,z) = T_{Bx\,B}(w,z)$$

(2) $\Longrightarrow$ 1: Let $(f(V),B,\,d_B)$ be a fuzzy metric space. Let $d_A: Ax\, A \longrightarrow [0,\infty)$ given by $d_A(x_1,x_2) = d_B(y_1,y_2), y_i = f(x_i), i = 1,2$. Clearly d_A defines metric on A. Moreover,

$$\mu(d_A(x_1,x_2)) = \mu(d_B(y_1,y_2): y_i = f(x_i), i = 1,2)$$
$$\geq T_{f(V)x\,f(V)}(y_1,y_2)$$
$$= min\,\{T_{f(A)}(y_1),T_{f(A)}(y_2)\}$$
$$= min\,\{T(x_1),T(x_2)\} = T_{Ax\,A}(x_1,x_2).$$

Now we will give an example of fuzzy linear spaces without non trivial metric on it; even though, the universal spaces are metric spaces.

Example 3.13 Let $(K,\mathcal{R})$ fuzzy field with membership function μ such that

$$\mu(x) = \begin{cases} 1\; if\; x = \pm 1, x = 0 \\ \frac{1}{2}\; if\; x \neq 0, \pm 1 \end{cases}$$

. Let X be a metric linear space over F. Let U be a fuzzy set with membership function T such that $T(x) = 1$ for all $x \in X$, then (U,X) is fuzzy linear space. However, there is no nontrivial fuzzy metric, d on (U,X) which satisfies definition 3.2(1).

C.P. Santhosh and T. V. Ramakrishan [1] introduced a norm on Fuzzy linear spaces. Now we will show that, this norm induces metric on the same fuzzy linear spaces in our sense.

Definition 3.14 [1] Let (K,F) be fuzzy field in F , X be linear spaces over F, and let (U,X) be fuzzy linear spaces over (K,F).

A norm on (U,X) is a function, $||.||: X \longrightarrow [0,\infty)$ satisfies the following conditions:

(1). $\mu(||x||) \geq\; T(x)$
(2). $||x|| \geq\; 0$ and $||x||\; = 0$ if and only if $x = 0$
(3) $||\alpha x|| = |\alpha|||x||$ for all $x \in X$
(4) $||x - y|| \leq\; ||x|| + ||y||$ for all $x, y \in x$.

A pair (U,X,|| ||) is called fuzzy normed linear space.

Theorem 3.15 Let (U,X) be a fuzzy normed linear space over a fuzzy field (K,F) .Then (U,X) is fuzzy linear metric space.

Proof: Let of $(U,X,||\,||)$ be a normed space, let T and μ be membership functions of fuzzy set U in X and K in F respectively.

Consider a mapping, $d: Xx\, X \longrightarrow [0,\infty]$ given by $d(x,y) = ||x - y||$. Clearly d defines metric on X**.** Hence it satisfies $(2) - (4)$ of definition 3.2. So, we will verify only definition $d3.2(1)$. Since $\mu(|x - y|) \geq T(x - y)$ by definition 3.14(1), and $T(x - y) \geq\; min\{T(x),T(-y)\}$ by(definition 2.5(1)), we have

$$\mu(d(x,y)) = \mu(||x - y||) \geq T(x - y)$$
$$\geq min\{T(x),T(-y)\}$$
$$= \; min\,\{T(x),T(y)\} = T_{Ux\,U}(x,y)$$

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