International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Impact Factor (2012): 3.358

Note On q–Beta Operators

Chandra Prakash[1], Rahul Tomar[2]

Department of Mathematics, Shyam Lal College, University of Delhi, Delhi-110032, India

Abstract: ***In the present paper, we deal with King type modification of q-Beta operators. We have studied the rate of convergence and Voronovskaya-type asymptotic formula for these operators.***

Keywords: operators, q-Beta, convergence, Voronovskaya-type asymptotic formula, King type modification

1. Introduction

In 1987 Lupas [8] introduced a q–analogue of the Bernstein operator and investigated its approximation and shape-preserving properties. Subsequently, Phillips [11] proposed another generalization of the classical Bernstein polynomials based on q-integers and obtained the rate of convergence and Voronovskaya-type asymptotic formula for these operators. More results on q–Bernstein operators were obtained by Ostrovska [9,10].

In the last decade the applications of the q–calculus in the approximation is one of the main area of research. Some new generalizations of the well known positive linear operators, based on q–integers were introduced and obtained different approximation properties by several authors. Recently, approximation properties of the operators e.g. q–Durrmeyer [6, 7] and q–Szasz Mirakyan operators [1, 2] have been established.

Throughout the present papers, we consider $0 < q < 1$: Following [7] and references therein, for $q > 0$ and each nonnegative integer n we have

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_{q!} = \begin{cases} [n]_q [n-1]_q [n-1]_q \ldots\ldots [1]_q, & n = 1,2,\ldots\ldots \\ 1, & n = 0. \end{cases}$$

Further, we use the q-Pochhammer symbol, which is defined as

$$(-x,q)_n = (1+x)_q^n = \begin{cases} (1+x)(1+x)(1+qx)\ldots\ldots(1+q^{n-1}x), & n=1,2,\ldots, \\ 1, & n=0 \end{cases}$$

The q–Jackson integrals and q–improper integrals are given by (See[4] and [5])

$$\int_0^a f(x)\,d_q x = (1-q)\,a \sum_{n=0}^{\infty} f(aq^n)\, q^n$$

and $\displaystyle\int_0^{\infty/A} f(x)\,d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A},\ A>0,$

provided the sums converge absolutely.

The q–gamma function [3] is defined by

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx)\, d_q x,\ t>0,$$

where q–exponential function E_q is defined as

$$E_q(z) = \prod_{j=0}^{\infty} \left(1 + (1-q) q^j z\right) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_{q!}}, |q|<1$$

Consequently,

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \Gamma_q(1) = 1.$$

The q–beta function [3] is given by

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(x;q)_{t+s}}\, d_q x,$$

where $K(x,t) = \dfrac{1}{1+x} x^t \left(x^{-1};q\right)_t (x;q)_{1-t}$.

In particular, for any positive integer n

$$K(x,n) = q^{\frac{n(n-1)}{2}}, K(x,0) = 1$$

and

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}.$$

Very recently, a q–analogue of modified Beta operators were introduced in [12] and studied some approximation properties

$$M_{n,q}(f,x) = \frac{1}{[n]_q} \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} q^k p_{n,k}^q(t) f(t)\, d_q t, \tag{1.1}$$

where

$$p_{n,k}^q(x) = \frac{q^{k(k-1)/2}}{B_q(k+1,n)} \frac{x^k}{(x,q)_{n+k+1}},\ x \in [0,\infty)$$

and $B_q(k+1,n) = \dfrac{[k]_{q!}[n-1]_{q!}}{[n+k]_{q!}}$.

Lemma 1. *If we define the central moments as*

$$T_{n,m}(x) := M_{n,q}(t^m;x) = \frac{1}{[n]_q} \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} q^k p_{n,k}^q(t)\, t^m\, d_q t,$$

then $T_{n,0}(x)=1$ *and for n > m + 2, we have the following recurrence relation:*

$$\left([n+1]_q-[m+2]_q\right)T_{n,m+1}(qx)$$

$$=qx(1+x)D_q\left[T_{n,m}(x)\right]+q\left([m+1]_q+[n+1]_q x\right)T_{n,m}(qx)$$

consequently, we have

$$T_{n,1}(x)=\frac{[n+1]_q}{q^2[n-1]_q}x+\frac{1}{q[n-1]_q},$$

$$T_{n,2}(x)=\frac{[n+1]_q[n+2]_q}{q^6[n-1]_q[n-2]_q}x^2+\frac{[n+1]_q\left(q+[3]_q\right)}{q^5[n-1]_q[n-2]_q}x+\frac{[2]_q}{q^3[n-1]_q[n-2]_q}$$

2. Construction of the Operators

Many well-known operators preserve the linear as well as constant functions for example Bernstein, Baskakov, Szász-Mirakyan and Szász-Beta operators possess these properties i.e. $L_n(e_i; x) = e_i(x)$ where $e_i(x) = x^i$ $(i = 0; 1)$. To make the convergence faster King [13] proposed an approach to modify the classical Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2. As the operators $M_{n;q}(f\,;\,x)$ introduced in (1.1) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions, for this purpose the modification of $M_{n;q}(f\,; x)$ as follows:

$$\tilde{M}_{n,q}(f,x)=M_{n,q}\left(f,r_{n,q}(x)\right)=\frac{1}{[n]_q}\sum_{k=0}^{\infty}p_{n,k}^q\left(r_{n,q}(x)\right)\int_0^{\infty/A}q^k p_{n,k}^q(t)f(t)d_qt, \tag{2.1}$$

where $r_{n,q}(x)=\dfrac{q^2[n-1]_q x-q}{[n+1]_q}$, $r_{n,q}(x)\in[0,\infty)$.

Thus $x\in I_{n,q}=\left[1/q[n-1]_q,\infty\right)$.

Lemma 2. *For each* $x\in[0,\infty)$, *we have*

(i) $\tilde{M}_{n,q}(1,x)=1$

(ii) $\tilde{M}_{n,q}(t,x)=x$

(iii)

$$\tilde{M}_{n,q}(t^2,x)=\frac{[n+2]_q[n-1]_q}{q^2[n-2]_q[n+1]_q}x^2+\frac{1}{q^3[n-2]_q}\left((1+q)^2-2\frac{[n+2]_q}{[n+1]_q}\right)x$$
$$+\frac{1}{q^4[n-2]_q[n-1]_q}\left(\frac{[n+2]_q}{[n+1]_q}-[2]_q\right).$$

Lemma 3. *For each* $x\in[0,\infty)$, *be Lemma* [2], *we have*

(i) $\tilde{M}_{n,q}(t-x,x)=0$

(ii)

$$\alpha_n(x)=\tilde{M}_{n,q}\left((t-x)^2,x\right)=\left(\frac{[n+2]_q[n-1]_q}{q^2[n-2]_q[n+1]_q}-1\right)x^2+\frac{1}{q^3[n-2]_q}$$
$$\left((1+q)^2-2\frac{[n+2]_q}{[n+1]_q}\right)x$$
$$+\frac{1}{q^4[n-2]_q[n-1]_q}\left(\frac{[n+2]_q}{[n+1]_q}-[2]_q\right).$$

We establish the following direct approximation theorem for the operators (2.1).

Theorem 1. *Let* $f\in C_B\left(I_{n,q}\right)$ *and* $0<q<1$. *Then for all* $x\in[0,\infty)$ *and for n > 2, there exist a positive constant C such that*

$$\left\|\tilde{M}_{n,q}(f,x)-f(x)\right\|\le Cw_2\left(f,\alpha_n(x)\right).$$

Proof. Let $g\in C^2\left(I_{n,q}\right)$ and $x,t\in I_{n,q}$. By Taylor's expansion we have

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)du,$$

Applying $\tilde{M}_{n,q}$, we get

$$\tilde{M}_{n,q}(g,x)-g(x)=\tilde{M}_{n,q}\left(\int_x^t(t-u)g''(u)du\right)$$

Obviously, we have

$$\left|\int_x^t(t-u)g''(u)du\right|\le(t-x)^2\|g''\|.$$

Therefore,

$$\left\|\tilde{M}_{n,q}(g,x)\right\|\le\tilde{M}_{n,q}\left((t-x)^2,x\right)\|g''\|=\alpha_n(x)\|g''\|$$

using the Remark, we have

$$\left|\tilde{M}_{n,q}(f,x)\right|\le\|f\|$$

Thus,

$$\left|\tilde{M}_{n,q}(f,x)-f(x)\right|\le\left|\tilde{M}_{n,q}(f-g,x)-(f-g)(x)\right|+\left|\tilde{M}_{n,q}(g,x)-g(x)\right|$$

$$\le\|f-g\|+\alpha_n(x)\|g''\|.$$

Now taking the infimum over all $C_B[0,\infty)$ and using (a) we obtain the required result. This proves the theorem.

3. Voronovskaja Type Theorem

In this section we obtain a voronovskaja type result for the operators $\tilde{M}_{n,q}$.

Lemma 4. *Assume that* $q_n \in (0,1), q_n \to 1$ *as* $n \to \infty$. Then, for every $x \in [0,\infty)$ *there hold*

$$\lim_{n\to\infty}[n+1]_{qn}\tilde{M}_{n,qn}(t-x,x)=0$$

and

$$\lim_{n\to\infty}[n+1]_{qn}\tilde{M}_{n,qn}\left((t-x)^2,x\right)=2x(1+x).$$

Theorem 2. *Assume that*

$q_n \in (0,1)$, $q_n \to 1$ *as* $n \to \infty$.

Then for any $f \in C_2^*[0,\infty)$ *such that*

$f', f'' \in C_2^*[0,\infty)$ *and* $x \in [0,\infty)$, *we have*

$$\lim_{n\to\infty}[n+1]_{qn}\left(M_{n,qn}(f,x)-f(x)\right)=x(1+x)f''(x)$$

for every $x \ge 0$.

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$ be fixed. By taylor expansion we can write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \quad (3.1)$$

where $r(t,x)$ is Peano form of the remainder, $r(.,x) \in C_2^*[0,\infty)$ and $\lim_{t\to x} r(t,x)=0$.

Applying $M_{n,qn}$ to above we obtain

$$[n+1]_{qn}\left(M_{n,qn}(f,x)-f(x)\right)=f'(x)[n+1]_{qn}M_{n,qn}(t-x,x)$$

$$+\frac{1}{2}f''(x)[n+1]_{qn}M_{n,qn}\left((t-x)^2,x\right)$$

$$+[n+1]_{qn}M_{n,qn}\left(r(t,x)(t-x)^2,x\right)$$

By Cauchy-Schwarz inequality, we have

$$M_{n,qn}\left(r(t,x)(t-x)^2,x\right)\le\sqrt{M_{n,qn}\left(r(t,x)^2,x\right)}\sqrt{M_{n,qn}\left((t-x)^4,x\right)} \quad (3.2)$$

Observe that $r^2(x,x)=0$ and $r^2(.,x) \in C_2^*[0,\infty)$.

Then it follows Theorem that

$$\lim_{n\to\infty}[n+1]_{qn}M_{n,qn}\left(r(t,x)^2,x\right)=r^2(x,x)=0 \quad (3.3)$$

uniformly with respect to $x \in [0,A]$.

Now from (3.2), (3.3) and Lemma 4 we obtain

$$\lim_{n\to\infty}[n+1]_{qn}M_{n,qn}\left(r(t,x)(t-x)^2,x\right)=0$$

$$\lim_{n\to\infty}[n+1]_{qn}\left(M_{n,qn}(f,x)-f(x)\right)$$

$$=\lim_{n\to\infty}\Big(f'(x)[n+1]_{qn}M_{n,qn}(t-x,x)+\frac{1}{2}f''(x)[n+1]_{qn}M_{n,qn}\left((t-x)^2,x\right)$$

$$+[n+1]_{qn}M_{n,qn}\left(r(t,x)(t-x)^2,x\right)\Big)$$

$$=x(1+x)f''(x).$$

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