# HARMONIC ANALYSIS IN FUZZY SYSTEMS

TIMUR KARAÇAY
BASKENT UNIVERSITY, ANKARA,TR

Abstract. This work introduces methods of harmonic analysis into Fuzzy Systems via the complexification of the system. Let $X$ be a universe, $F$ be a family of fuzzy sets in $X$. It is shown that $X$ and $F$ generate topological groups $G$ and $\Gamma$, respectively, that are dual to each other. These structures then provide us with powerful mathematical tools of harmonic analysis, enabling us to prove the postulates $0 \Rightarrow 0, 0 \Rightarrow 1, 1 \Rightarrow 1$ of *boolean* logic as well as the introduction of the notions *logical dependencies*, and the *comparisons* of propositions.

## 1. Introduction

*Two-valued* logic basisd on *Boolean Algebra* has very firm foundations and is the basis of 20th century technology. No mathematician will even doubt the perfection of this system.

However scientists dealing with the mathematical formulations of natural phenomenons realized that the boolean logic lacks something, and consequently, they set forth to open a new page in the history of logics, the so called *many-valued* logics.

*Plato* was believed to be the first man who indicated that there should be a third region between *false* and *true*. In the early 1900's *Jan Lukasiewicz* first described a *three-valued* logic, and then *four-valued* and *five-valued* logics. Finally he declared that in principle there was nothing prevent the derivation of an *infinite-valued* logic. In 1965 *Lotfi Zadeh* described the mathematics of *fuzzy set theory*, and by extension *fuzzy logic*.

Each technology has its pros and cons, and so is fuzzy systems ( [1], [2], [3], [5], [6], [9], [10], [12], [13], [14]). Fuzzy systems has already extensions to a wide and diverse area of study such as fuzzy sets, logics, numbers, probability, information, linguistics, possibility, psychometry, etc. Each area has its own approachment to represent the system. This work concerns a representation of fuzzy systems via topological groups.

Throughout the paper $\mathbf{N}, \mathbf{Z}, \mathbf{R}, \mathbf{C}$ will denote the sets of *natural numbers, integers, real numbers* and *complex* numbers, in their respective order.

Let us remind that the *boolean* and *fuzzy* logics are furnished on the sole lattice structures of the range of propositions; mostly, on the set $\{0, 1\}$ and on the interval $[0, 1]$, respectively. Ofcourse, the outcomes of the systems depend on and bound by the structure of the range of propositions. To enlarge the system, we shall use the circle group $T$ as the range of fuzzy functions, where $T$ is the unit circle

$$T = \{z : \ z \in \mathbf{C}, \ |z| = 1\}. \tag{1}$$

The unit circle $T$ forms a subgroup of $C^*$, the multiplicative group of all nonzero complex numbers. We call $T$ the *circle group* in order to emphasis its group structure. Since $C^*$ is Abelian, it follows that $T$ is as well. The notation $T$ for the circle group stems from the fact that $T^n$ (the direct product of $T$ with itself n times) is geometrically an n-*torus*. The circle group is then a 1-torus.

The *circle group* is not a lattice, therefore one should not expect that the outcomes would be direct generalisations of classical fuzzy systems. Instead, the circle group has a natural topology when regarded as a subspace of the complex plane. Since *multiplication* and *inversion* are continuous functions on $C^*$, the circle group has the structre of a topological group. In addition, since the unit circle is a *closed* subset of the complex plane, the circle group is a *closed subgroup* of $C^*$ (itself regarded as a topological group). It is this structure that our system will depend on.

## 2. Extending Fuzzyness

Let $X$ be a universe on which our fuzzy sets are defined. As mentioned above, we extend the notion of a fuzzy system as follows.

**Definition 1.** *A function from the universal set $X$ into the circle group $T$ is called a fundamental fuzzy function.*

The absolute value of the real part of a fundamental fuzzy function is a fuzzy set in the sense of Zadeh. But such obstructions are not necessary for our purpose, since we need no lattice structures on the range.

fundamental fuzzy functions are nothing but the *characters* in topological groups ([4], [7], [11]). Let $F$ be a set of fundamental fuzzy functions on the universal set $X$. Definition 1 then reads as

$$\gamma \in F \Rightarrow \gamma : X \to T. \tag{2}$$

Since, the set $F$ of fundamental fuzzy functions on $X$ is a well-defined crisp set, one may also consider another set (family) $\tilde{F}$ of fundamental fuzzy functions on $F$. As in (2) we write

$$x \in \tilde{F} \Rightarrow x : F \to T \tag{3}$$

These mappings establish a duality between $F$ and $\tilde{F}$ as follows. For a fixed $\gamma \in F$, while $x$ varies in $\tilde{F}$, the symbol $(x, \gamma)$ will be interpreted as "*$\gamma$ is a function from $\tilde{F}$ into $T$.*" Likewise, for a fixed $x \in \tilde{F}$, while $\gamma$ varies in $F$, the symbol $(x, \gamma)$ will also be interpreted as "*$x$ is a function from $F$ into $T$.*"

In view of this duality between $F$ and $\tilde{F}$, we may interchangebly use the symbols

$$x(\gamma) = (x, \gamma) = \gamma(x), \quad \gamma \in F,\ x \in \tilde{F} \tag{4}$$

to make it adequate to its context, and consistent with the notations used in harmonic analysis on topological groups.

We define the addition operations on the sets $F$ and $\tilde{F}$, respectively, by the relations

$$(\gamma_1 + \gamma_2)(x) = (\gamma_1(x)\gamma_2(x) \qquad = (x, \gamma_1)(x, \gamma_2), \quad \gamma_1, \gamma_2 \in F,\ x \in \tilde{F} \tag{5}$$

$$(x_1 + x_2)(\gamma) = (\gamma(x_1)\gamma(x_2)) \qquad = (x_1, \gamma)(x_2, \gamma), \quad \gamma \in F,\ x_1, x_2 \in \tilde{F} \tag{6}$$

The meanings of the two additions will always be clear from their context. It is easy to see that the two addition operations satisfy the *commutativity* and *associativity* and hence $(F, +)$ and $(\tilde{F}; +)$ are *semigroups.* Adding the unit elements (zeroes 0) defined by

$$0(\gamma) = (0, \gamma) = 1, \quad 0 \in F, \gamma \in F,\ 1 + 0i = 1 \in T \tag{7}$$

$$\gamma(0) = (x, 0) = 1, \quad x \in F, 0 \in \tilde{F},\ 1 + 0i = 1 \in T \tag{8}$$

to $F$ and $\tilde{F}$ makes them *monoids.*

Using the duality notations (4) and the inverse mapping in the circle group $T$, the inverses for all $\gamma \in F$ and all $x \in \tilde{F}$, may be defined by the relations

$$(-\gamma)(x) = (x, -\gamma) = \overline{(x, \gamma)} \tag{9}$$

$$(-x)(\gamma) = (-x, \gamma) = \overline{(x, \gamma)}. \tag{10}$$

Adding these inverses to the related monoids generate the Abelian groups $(\Gamma, +)$ and $(G, +)$, in their respective order.

It is now legitimate to replace the sets $F$ and $\tilde{F}$ with the groups $\Gamma$ and $G$, in their respective order, in the relations (2) to (6).

We note that, by the duality quoted above, the groups $(\Gamma, +)$ and $(G, +)$ may be regarded as the *dual group* to each other (see, [4], [11]).

## 3. Fuzzy Sets Characterized as LCA Group

Starting from a universal set $X$, we have now obtained two Abelian groups $(G, +)$ and $(\Gamma, +)$ dual to each other. In order to introduce methods of analysis into fuzzy systems we shall endow both groups with suitable topologies to make them Locally Compact Abelian Topological Groups (LCA). Thence we will be able to employ the results of harmonic analysis on these LCA groups ( [4], [7], [8], [11]).

We give $G$ the weak topology $\mathscr{G}$ induced by $\Gamma$. That is the coarsest topology for which all the functions of (2) are continuous, when the sets $X$, and $F$ are replaced by $G$ and $\Gamma$, respectively.

We will show that $(G, \mathscr{G})$ is a locally compact abelian (LCA) topological group. This will be achieved if it is shown that the topology on $G$ is *Hausdorff*, *locally compact*, and that the group operations are *continuous.* To make the paper self contained, what follows employ standard techniques of topological groups to accomplish this task.

Let $\mathscr{V}$ be the set of all subsets of $G$ of the form $\gamma^{-1}(V)$ ($\gamma \in \Gamma$ and $V$ is open in $T$), and let $\mathscr{B}$ be the set of all finite intersections of sets of $\mathscr{V}$. Then, $\mathscr{B}$ is a basis of the topology $\mathscr{G}$ on $G$, which is called the *weak (initial, projective)* topology on $G$ for the family $\gamma \in \Gamma$. $\mathscr{G}$ is the coarsest topology on $G$ for which all the mappings $\gamma \in \Gamma$ are continuous.

**Lemma 1.** *Let $V$ be a symmetric neighborhood of 1 in $T$ and let $p$ be an element of $\Gamma$. Then the set*

$$U_{V,p} = \{x \in G : \ (x,p) \in V\} \tag{11}$$

*is a symmetric neighborhood of 0, in $G$.*

*Proof.* $V$ is symmetric in $T$ means that if $z \in V$ then $\bar{z} \in V$. It follows from (9) that

$$x \in U_{V,p} \ \Rightarrow \ (x,p) \in V \ \Rightarrow \ \overline{(x,p)} \in V \ \Rightarrow \ (-x,p) \in V \ \Rightarrow \ -x \in U_{V,p}$$

.

**Lemma 2.** *Let $V_1$ and $V_2$ be two symmetric neighborhoods of 1 in $T$ with the property that $V_1 \subset V_2$, and let $p \in \Gamma$. Then we have $U_{V_1,p} \subset U_{V_2,p}$.*

*Proof.* This is a straightforward result of Lemma 1.

**Lemma 3.** *The family of all the sets $U_{V,p}$, where $V$ runs through a basis of symmetric neighborhoods of the unit 1 in $T$ and where $p \in \Gamma$ forms a basis of symmetric neighborhoods of 0 in $G$.*

*Proof.* Let $W$ be a neighborhood of 0 in $G$. By the definition of the topology $\mathscr{G}$, there exists an element $p \in \Gamma$ and a symmetric neighborhood $V$ of 1 in $T$ such that $p^{-1}(V) \subset W$. But, then (9) shows that $U_{V,p} \subset W$, concluding the proof.

**Lemma 4.** *If $V$ runs through a basis of symmetric neighborhood of 1 in $T$, and if $p$ runs through $\Gamma$, then the family of the sets $\{x + U_{V,p}\}$ forms a basis of neighborhoods of $x$ for any $x \in G$.*

*Proof.* Let $W$ be a neighborhood of $x \in G$. We must show that there exists a symmetric neighborhood $V$ of 1 in $T$ so that $x + U_{V,p} \subset W$.

Since 1 in $T$ has a basis of symmetric neighborhoods, the definition of the topology $\mathscr{G}$ on $G$ guarantees the existence of an element $p \in \Gamma$ and a symmetric neighborhood $V$ of 1 in $T$ with the property that $p^{-1}((x,p)V) \subset W$. But then, we have

$$u \in U_{V,p} \quad \Rightarrow \quad (u,p) \in V \quad \Rightarrow \quad x + U_{V,p} \subset W,$$

which is the required result.

**Lemma 5.** *Given any neighborhood $W$ of 0 in $G$, there exist an element $p$ in $\Gamma$ and a symmetric neighborhood $V$ of 1 in $T$ such that $U_{V,p} + U_{V,p} \subset W$*

*Proof.* Lemma 3 shows the existence of an element $p \in \Gamma$ and a symmetric neighborhood $V_1$ of 1 in $T$ such that $U_{V_1,p} \subset W$ Since $T$ is a LCA group, there exists a neighborhood $V$ of 1 in $T$ with the property that $V + V \subset V_1$. It follows from Lemma 1 that $U_{V,p} + U_{V,p} \subset U_{V_1,p}$, concluding the proof.

**Lemma 6.** *$(G, \mathscr{G})$ is a Hausdorff topological space.*

*Proof.* Let $x \in G$ with $x \neq 0$. By the construction, $F$ separates points on $X$, so does the generated group $\Gamma$. Hence, there is a $p \in \Gamma$ such that $(x,p) \neq (x,0) = 1$ in $T$. Since $T$ is a Hausdorff space, there is a neighborhood $V$ of 1 in $T$ with the property that

$$V \bigcap (x,p)V = \phi$$

in $T$. Now Lemma 1 gives

$$U_{V,p} \bigcap (x + U_{V,p}) = \phi$$

concluding the proof.

**Lemma 7.** *$p$ is an open mapping from $G$ into $T$ for each $p \in \Gamma$.*

*Proof.* By Lemmas 3 and 4, it will be sufficient to show that $p(U_{V,p})$ is open in $T$ for all $p \in \Gamma$ and for all symmetric, open neighborhood $V$ of 1 in $T$. But that is so since $p(U_{V,p}) = U_{V,p}$, by Lemma 1.

**Lemma 8.** *If $V$ is a compact neighborhood of 1 in $T$, and if $p \in \Gamma$, then $U_{V,p}$ is compact in $G$.*

*Proof.* Let $\{W_\alpha \mid \alpha \in I\}$ be an open covering of $U_{V,p}$. We then have

$$U_{V,p} \subset \bigcup \{W_\alpha \mid \alpha \in I\} \Rightarrow V \subset \bigcup \{p(W_\alpha) \mid \alpha \in I\}$$

Lemma 7 and the compactness of $V$ in $T$ together imply the existence of a finite family of open sets of the form $\{p(W_{\alpha_k}) \mid k = 1, 2, ..., n\}$ with the property

$$V \subset \bigcup \{p(W_{\alpha_k}) \mid k = 1, 2, ..., n\}$$

which in turn implies

$$U_{V,p} \subset \bigcup \{W_{\alpha_k} \mid k = 1, 2, ..., n\}$$

**Lemma 9.** *$(G,\mathscr{G})$ is locally compact.*

*Proof.* Since $T$ is locally compact, the required result is a conclusion of Lemma 6.

Combining all the results of the lemmas stated above gives

**Theorem 1.** *$(G,+,\mathscr{G})$ is a LCA topological group.*

*Proof.* It only remains to show that the group operations addition and inversion of $G$ are continuous, i.e. the mappings

$$(x,y) \to x+y \; of \; G\times G \;\; into \; G, \;\; and$$
$$x \to -x \;\; of \; G \; into \;\; G$$

are continuous.

Let $W$ be a neighborhood of $(x+y)$ in $G$. Lemmas 3 and 4 imply the existence of symmetric neighborhoods $V_1$ and $V$ of 1 in $T$ such that $(x+y)+U_{V_1,p} \subset W$, and $U_{V,p}+U_{V,p} \subset U_{V_1,p}$. It follows that

$$(x+U_{V,p})+(y+U_{V,p}) \subset (x+y)+U_{V_1,p},$$

proving $(i)$. To show the continuity of the mapping $(ii)$, let $W$ be a neighborhood of $-x$. Again,by Lemma 3, we have a symmetric neighborhood $V$ of 1 in $T$ such that $-x+U_{V,p} \subset W$. Since $U_{V,p}$ is symmetric, $-x-U_{V,p} \subset W$ follows, which ends the proof.

We now give $\Gamma$ the open-compact topology, namely the topology whose basis consists of the sets of the form

$$V(K,U) = \{\gamma \in \Gamma | f(K) \subset U\} \tag{12}$$

where $K$ is compact in $G$ and $U$ is open in $T$. This makes $(\Gamma,+,\mathscr{S})$ a LCA topological group which is LCA dual group of $(G,+,\mathscr{G})$. It is well known that if $G$ is compact then $\Gamma$ is discrete and vice-versa.

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## 4. Logical Dependence and Logical Basis

The inner product, orthogonality, (complete) orthonormal systems, and Fourier coefficients of fuzzy functions can be defined as usual, in order to employ results of Hilbert spaces. From now on we assume that $G$ is compact and hence $\Gamma$ is discrete. Therefore, a complete orthonormal system of complex fuzzy functions is at most countably infinite.

**Definition 2.** *Let $S$ be a finite family of fundamental fuzzy functions in $\Gamma$. Any sum of the form*

$$f(x) = \sum c_\gamma(-x,\gamma), \; (c_\gamma \in \mathbf{C}, \; \gamma \in S)$$

*is said to be a finite combination of fundamental fuzzy functions of $S$.*

**Definition 3.** *Let $A$ be a family of fundamental fuzzy functions in $\Gamma$, not necesserily finite. Then the set $L(A)$ of all combinations of all finite family of $A$ is called the span of $A$.*

$L(A)$ is the minimal linear space which contains $A$. We now extend the notion of fuzzyness as follows.

**Definition 4.** *Let $A$ be any subset of $\Gamma$. Any element $f \in L(A)$ is said to be a fuzzy function.*

The following two lemmas follow from the definition.

**Lemma 10.** *We have*

$$A \subset B \Rightarrow L(A) \subset L(B)$$

*for all $A,B \subset \Gamma$.*

**Lemma 11.** *The relation*

$$L(A\cap B) = L(A)\cap L(B)$$

*holds for all $A,B \subset \Gamma$.*

**Definition 5.** *The basis of a fuzzy function $f$, denoted by $B_f$, is defined to be the set*

$$B_f = \cap\{A \mid f \in L(A)\} \tag{13}$$

Clearly, the relation

$$< \gamma_1, \gamma_2 >= \begin{cases} 1 & if \quad \gamma_1 = \gamma_2 \\ 0 & if \quad \gamma_1 \neq \gamma_2 \end{cases} \tag{14}$$

holds for each pair $\gamma_1, \gamma_2 \in \Gamma$, implying that fundamental fuzzy functions are orthonormal, and that $\Gamma$ is at most countably infinite.

Note that, the basis $B_f$ of a fuzzy function $f$ is the set of fundamental fuzzy functions $\gamma \in \Gamma$ with the property that

$$\widehat{f}(\gamma) = \begin{cases} < f, \gamma > \neq 0 & if \quad \gamma \in B_f \\ 0 & if \quad \gamma \notin B_f \end{cases} \tag{15}$$

where $\widehat{f}(\gamma)$ is the Fourier coefficient of $f$ corresponding to the fundamental fuzzy function $\gamma \in \Gamma$.

The basis of a fuzzy function $f$ is the minimal set $A$ in $\Gamma$ for which $f \in L(A)$.

**Definition 6.** *Two fuzzy functions $f$ and $g$ are independent iff $B_f \cap B_g = \emptyset$.*

**Lemma 12.** *Different fundamental fuzzy functions are independent.*

*Proof.* This is a consequence of (14).

An equivalence relation, denoted by $\approx$, on the set of fuzzy functions can be defined by

$$f \approx g \quad \Leftrightarrow \quad B_f = B_g. \tag{16}$$

For the sake of simplicity of notations, we shall use $f$ instead of its equivalent class.

## 5. The Implications

**Definition 7.** *Let $f$ and $g$ be two fuzzy functions. We say that $f$ implies $g$, if and only if, the basis of $f$ is contained in the basis of $g$. In symbols, we write*

$$f \Rightarrow g \Longleftrightarrow B_f \subseteq B_g \tag{17}$$

**Theorem 2.** *The implications*

$$0 \Rightarrow 0, \ 0 \Rightarrow f, \ f \Rightarrow f \tag{18}$$

*hold for any fuzzy function $f$.*

*Proof.* If $f = 0$, there is nothing to prove. So, let us assume that $f \neq 0$ in which case the basis $B_f$ of $f$ cannot be empty. But, then the relation $B_0 = \emptyset \subset B_f$ holds proving the required result.

The important assumptions $0 \Rightarrow 0$, $0 \Rightarrow 1$, and $1 \Rightarrow 1$ of boolean logic are simple consequences of Theorem 2, whenever fuzzy functions are replaced with propositions.

Clearly, the class of all orthonormal fundamental fuzzy functions are partially ordered by the inclusionn relation $\subseteq$.

If $f$ is a fuzzy function, then we have its formal Fourier expansion

$$f(x) = \sum \widehat{f}(\gamma)(-x, \gamma), \quad \gamma \in B_f, \ x \in G. \tag{19}$$

Here *"formal"* means we do not consider convergence of the series in any means.

## 6. Comparisons of Propositions

The basis of fuzzy functions put a partial ordering on $\Gamma$. Using this ordering we may define an order relation on the set of fuzzy functions as follows

**Definition 8.**

$$f \preceq g \Leftrightarrow B_f \subseteq B_g \tag{20}$$

Using this order, we may define the operations $\wedge$ and $\vee$ on propositions by the relations.

$$f \wedge g = h \Leftrightarrow B_h = B_f \cap B_g \tag{21}$$

$$f \vee g = h \Leftrightarrow B_h = B_f \cup B_g \tag{22}$$

**Definition 9.** *If the basis $B_f$ of a fuzzy function $f$ is equal to $\Gamma$, then $f$ is said to be of maximal.*

We remind that $\Gamma$ is a complete orthonormal system. *Maximal truthness* correspond to the *true* values of boolean and fuzzy logics, when fuzzy functions are regarded as propositions. Ofcourse, then the intermediate values of fuzzy logics may be regarded as fuzzy functions with basis not maximal.

Another way of comparison of proposions is the lexicographic priority. For instance, among the services that a state's citizens expects the government to provide include *employment, health services, social security, education, food security, environmental protection, etc.*. We have to agree that a government must prioritize on which ones to provide first before the others in order to maximize benefits and at the same time boost development. Let $S = \{\gamma_{i1}, \gamma_{i2}, \dots, \gamma_{in}\}$ denote the services mentioned, in a given priority order. A decision function (proposition) might then be of the form

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} \hat{f}(\gamma_{ik})(-x, \gamma_{ik}) \\ &= \hat{f}(\gamma_{i1})(-x, \gamma_{i1}) + \hat{f}(\gamma_{i2})(-x, \gamma_{i2}) + \dots + \hat{f}(\gamma_{in})(-x, \gamma_{in}) \end{aligned} \tag{23}$$

Changing the order of the set $S$ gives another permutation , but the sum (23) and the logical basis $B_f$ do not change in algebraic manner. However, if the goverment gives the priority of the services in the order of $S$, then the sum (23) determines the priority in its lexicographic order.

**Future Prospects**

If the notion *fuzzy function* is replaced by *propositions* or by *probability functions*, it will be possible to extend those notions to some wider classes.

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**Author Profile**

FIGURE 1. Timur Karaçay

Timur Karaçay received his B.S. degree in Mathematics from Ankara University in 1963, and his Ph.D. degree from Ege University in 1967. He attended the UNESCO Post Graduate School in Functional Analysis held in Aarhus University in 1967-1968. He also studied at Liverpool University (1976-77) and Tubingen University (1990) as researh fellow. He worked for Ege, Hacettepe and Akdeniz Universities. Presently he is a professor at Başkent University in Ankara.

*E-mail address*: `tkaracay@baskent.edu.tr`

*URL*: `www.baskent.edu.tr/~tkaracay`