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Numerical Solution of Second-Order Differential Equations by Differential Transformation Method (DTM)

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Abstract: ***In this paper, the differential transformation method (DTM) is used to find numerical solution of the second-order differential equations (exponential type).The method reduces the equations to recurrence relations which is further to approximate solutions. Comparing the method with the exact solution shows that the present approach is effective and powerful. Four modeling problems from mathematical physics are discussed to illustrate the effectiveness and the performance of the proposed method. The results demonstrate reliability and efficiency of this method for second order differential equations.***

Keywords: Second Order Differential Equations (SODE), Differential Transformation Method (DTM), Taylor Series, Recurrence Relation and Approximate Solution

1. Introduction

Second order differential equations are frequently encountered in applied sciences like engineering, physics and geology. These require the determination of a function of a single independent variable satisfying a given differential equation and subject to different value at the independent variables at the initial domain. Many problems in engineering and science can be formulated as linear and non linear differential equations e.g. Steady state condition, Heat transfer in a thin heated wire Electric potential inside a thin conductor, Deflection of a thin elastic thread under load and many others.

The Differential Transformation Method (DTM) is very effective and powerful for solving various kind of differential equations (see [1],[2],[3],[4],[5],[6]). DTM is an iterative procedure for obtaining analytical numerical Taylor series solution of differential equations. It is used to construct a semi-analytical numerical technique that use Taylor series for the solution of differential equation in the form of a polynomial.

The main advantage of this method is that it can be applied directly to linear and nonlinear ordinary differential equations with out requiring linearization, discretization or perturbation. Another advantage is that DTM is capable of greatly reducing the size of computational work while accurate is still providing the series solution with fast convergence [7].

The outline of the paper follows: In section 1, we briefly introduce the concepts of Second order differential equations and DTM as a numerical method using for applied mathematical problems. In section 2,some basic definition and the uses of the definition. While section 3, we applied DTM to find the approximate analytical solution of the first and second order linear boundary value problems. Comparison with the exact solutions will be performed.

2. Differential Transform Method (DTM)

The differential transformation of the function y(t) is defined as follows:

$$Y(K) = 1/K!\left[d^k/dt^k\, y(t)\right]_{t=0} \quad (1)$$

Where y(t) is the original function and Y(k) is the transformation function. Here d^k/dt^k means that K the derivate with respect to t

The differential inverse transform of Y(K) is defined as

$$y(t) = \sum_{k=0}^{\infty} Y(K)t^k \;................................(2)$$

Combining equations (1) and (2) we obtained

$$y(t) = \sum_{k=0}^{\infty} 1/K!\left[d^k/dt^k\, y(t)\right]_{t=0} t^k \;.......(3)$$

Equation (3) is called approximate solution of the functions y (t).

Base on the above definitions, the fundamental mathematical operation performed by the differential transform method is show in the table 1

Table 1: One Dimensional Differential Transformation

Functional Form	*Transformed Form*
y(t) = w(t) $\pm$ v(t)	Y(K) = W(K) $\pm$ V(K)
y(t) = η v(t)	Y(K) = η V(K) , η is a constant
y(t) = $\dfrac{d^m y(t)}{dt^m}$	Y(K) = $\dfrac{(K+m)!}{k!}Y(K+m)$
y(t) = ℓ^t	Y(K) = $\dfrac{1}{K!}$
y(t) = $\ell^{\lambda t}$	Y(K) = $\dfrac{\lambda^K}{K!}$
y(t) =sin$(ct+\beta)$	Y(K) = $\dfrac{C^K}{K!}\sin\left(\dfrac{\pi k}{2}+\beta\right)$

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y(t) = $\cos\left(ct+\beta\right)$	Y(K) = $\frac{C^K}{K!}\cos\left(\frac{\pi k}{2}+\beta\right)$
y(t) = t^n	Y(K) = $\delta(K-n)$, δ is constant delta $\begin{cases}1,k=m\\0,k\neq m\end{cases}$

3. Numerical Application

Example 1

Consider second order differential equation

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}-6y(t)=\ell^{2t}\text{..............................}(4)$$

Subject to initial value conditions are

$$y(0)=1,\ \ y^1(0)=-2\text{..............................}(5)$$

The exact solution is given as

$$y(t)=\frac{21}{20}\ell^{-2t}+\frac{1}{5}\ell^{(3t)}.-\frac{1}{4}\ell^{2t}\text{..................}(6).$$

Taking differential transformation of equation (4) using table 1 and the following recurrence relation is obtained.

$$Y(K+2)=\frac{\left[(K+1)Y(K+1)+6Y(K)+\left(2^K/K!\right)\right]}{(K+1)(K+2)}\text{..}(7)$$

Where Y(K) is the differential transformation of y(t) and the transformation of initial conditions are

Y(0) = 1 (8)

Y(1)= -2 (9)

Substitution equations (8) and (9) at k = 0,1,2,3, 4……….. into equation (7) we have

Y(2) = $\frac{5}{2}$

Y(3) = -$\frac{-5}{6}$

Y(4) = $\frac{29}{24}$

Y(5) = $\frac{7}{120}$

Y(6) = $\frac{197}{720}$

Y(7) = $\frac{271}{5040}$

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Using equations (7----9) and the inverse transformation in equation (2), we obtained the close form solution up to N = 6

$$y(t)=\sum_{k=0}^{\infty}Y(K)t^K\text{................................}(10)$$

$$y(t)=1(t)^0-2(t)^1+\frac{5}{2}(t)^2-\frac{5}{6}(t)^3+\frac{29}{24}(t)^4+\frac{7}{120}(t)^5-$$

$$\frac{197}{720}(t)^6+O(t)^7+\text{..}$$

$$y(t)=1-2t+\frac{5}{2!}t^2-\frac{5}{3!}t^3+\frac{29}{4!}t^4+\frac{7}{5!}t^5-\frac{197}{6!}t^6+O(t^{7^7})\ldots$$

This is approximate solution of y(t).

Table 2: Comparison of EXACTand DTM solution (Example1)

t	EXACT SOLUTION	DTM
0.0	1.00000000	1.00000000
0.1	0.82428830	0.82428830
0.2	0.69530362	0.69530353
0.3	0.61264310	0.61264047
0.4	0.57943358	0.57940614
0.5	0.60304075	0.60287233
0.6	0.69615418	0.69540681
0.7	0.87836080	0.87570681
0.8	1.17836850	1.17038398
0.9	1.63709830	1.61585886
1.0	2.31194540	2.26071429

Example 2

Consider second order linear differential equation

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}-2y(t)=\ell^{t}\text{.................................}(11)$$

Subject to initial conditions

$$y(0)=1,\ \ y^1(0)=-1\text{................................}(12)$$

Which exact solution is given as

$$y(t)=-\frac{1}{3}.\ell^{((\sqrt{3}-1)t)}.\sqrt{3}.+\frac{1}{3}\ell\left(-\left(1+\sqrt{3}\right)t\right)\sqrt{3}.+\ell.^{t}\text{....}(13)$$

Taking differential transformation of equation (11) using table 1 and the following recurrence relation is obtained.

$$Y(K+2)=\frac{\left[-2(K+1)Y(K+1)+2Y(K)+\frac{1}{K!}\right]}{(K+1)(K+2)}\text{..}(14)$$

Where Y(K) is the differential transformation of y(t) and the transformation of initial conditions are

Y(0) = 1 (15)

Y(1) = - 1 (16)

Substitution equations (15) and (16) at k = 0,1,2,3, 4………… into equation (14) we have

Y(2) = $\frac{5}{2}$

Y(3) = $-\frac{11}{6}$

Y(4) = $\frac{33}{24}$

Y(5) = $\frac{-29}{40}$

Y(6) = $\frac{241}{720}$

Y(7) = $\frac{-131}{1008}$

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Using equations (14----16) and the inverse transformation in equation (2), we obtained the close form solution up to N=6

$$y(t)=\sum_{k=0}^{\infty}Y(K)t^K \text{.................................}(17)$$

$$y(t)=1-t+\frac{5}{2}t^2-\frac{11}{6}t^3+\frac{33}{24}t^4-\frac{29}{40}t^5+\frac{241}{720}t^6+.O(t^7)\text{.........}$$

This is approximate solution of $y(t)$.

Table 3: Comparison of Exact and DTM solution (Example 2)

0.0	1.00000000	1.00000000
0.1	0.92329724	0.92329725
0.2	0.88732119	0.88754711
0.3	0.88509401	0.88511304
0.4	0.91162672	0.91176330
0.5	0.96347986	0.96410435
0.6	1.03843308	1.04057989
0.7	1.13523812	1.14130043
0.8	1.25343572	1.26826118
0.9	1.39322174	1.42570783
1.0	1.55535122	1.62063492

Example 3

Consider second order differential equation of the type

$$\frac{d^2y}{dt^2}-6y(t)=\ell^{4t} \text{....................................}(18)$$

Subject to initial value conditions

$$y(0)=0,\ \ y^1(1)=-1\text{...............................}(19)$$

Exact solution is given as

$$y(t)=\ell^{\sqrt{6t}}\left(\frac{-7\sqrt{6}}{60}-\frac{1}{20}\right).+\ell^{(-\sqrt{6t})}\left(-\frac{1}{20}+\frac{7\sqrt{6}}{60}\right)+$$

$$\frac{1}{10}\ell^{(4t)}\text{..}(20)$$

Taking differential transformation of equation (18) using table 1 and the following recurrence relation is obtained.

$$Y(K+2)=\frac{\left[6Y(K)+(\frac{4^K}{K!})\right]}{(K+1)(K+2)}\text{......................}(21)$$

Where Y(K) is the differential transformation of y(t) and the transformation of initial conditions are

Y(0) = 0 (22)

Y(1) = -1 (23)

Substitution equations (22) and (23) at k = 0,1,2,3, 4………… into equation (21) we have

$Y(2)=\frac{1}{2}$

$Y(3)=-\frac{1}{3}$

$Y(4)=\frac{11}{12}$

$Y(5)=\frac{13}{30}$

$Y(6)=\frac{119}{60}$

$Y(7)=\frac{337}{210}$

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Using equations (21----23) and the inverse transformation in equation (2), we obtained the close form solution up to N=6

$$y(t)=\sum_{k=0}^{\infty}Y(K)t^K \text{.................................}(24)$$

$$y(t)=0-t+\frac{1}{2}t^2-\frac{1}{3}t^3+\frac{11}{12}t^4+\frac{13}{30}t^5+\frac{119}{60}t^6+.O(t^7)\text{.........}$$

$$y(t)=-t+\frac{1}{2}t^2-\frac{1}{3}t^3+\frac{11}{12}t^4+\frac{13}{30}t^5+\frac{119}{60}t^6+.O(t^7)\text{.........}$$

This is approximate solution of $y(t)$

Table 4: Comparison of DTM and EXACT solution (Example 3)

t	EXACT SOLUTION	DTM
0.0	0.00000000	0.00000000
0.1	-0.09523677	-0.09523519
0.2	-0.18102300	-0.18091386
0.3	-0.25505921	-0.25372519
0.4	-0.31066219	-0.30267636
0.5	-0.33455456	-0.30230655
0.6	-0.30336756	-0.30004654
0.7	-0.17820920	-0.17811009
0.8	0.10368212	0.10321170
0.9	0.64167499	0.64087114
1.0	1.59109283	1.59003211

Example 4.

Consider second order differential equation

$$\frac{d^2y}{dt^2}+y(t)=\ell^{2t} \qquad 25)$$

Subject to initial value conditions

$$y(0)=1,\ \ y^1(1)=-2\text{..................................}(25)$$

Exact solution is given as

$$y(t)=-\frac{1}{5}\left(12\sin t-4\cos t-\ell^{2t}\right)\text{....................}(26)$$

Taking differential transformation of equation (25) using table 1 and the following recurrence relation is obtained.

$$Y(K+2)=\frac{\left[-Y(K)+(\frac{2^K}{K!})\right]}{(K+1)(K+2)}\text{......................}(27)$$

Where Y(K) is the differential transformation of y(t) and the transformation of initial conditions are

Y(0) = 1 (28)

Y(1) = -2 (29)

Substitution equations (28) and (29) at k = 0,1,2,3, 4………… into equation (27), we have

$Y(2) = 0$

$Y(3) = \frac{2}{3}$

$Y(4) = \frac{1}{6}$

$Y(5) = \frac{1}{30}$

$Y(6) = \frac{1}{60}$

$Y(7) = \frac{7}{1260}$.

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Using equations (27----29) and the inverse transformation in equation (2), we obtained the close form solution using up to N=6

$$y(t) = \sum_{k=0}^{\infty} Y(K)t^K \text{..................................(24)}$$

$$y(t) = 1 - 2t + 0t^2 + \frac{2}{3}t^3 + \frac{1}{6}t^4 + \frac{1}{30}t^5 + \frac{1}{60}t^6 + .O(t^7)\text{.........}$$

$$y(t) = 1 - 2t + \frac{2}{3}t^3 + \frac{1}{6}t^4 + \frac{1}{30}t^5 + \frac{1}{60}t^6 + .O(t^7)\text{.........}$$

This is approximate solution of $y(t)$

Table 5 : Comparison of Exact and DTM solution (Example 4)

t	EXACT SOLUTION	DTM
0.0	1.00000000	1.00000000
0.1	0.80068368	0.80068368
0.2	0.60561181	0.60561180
0.3	0.41944446	0.41944444
0.4	0.24735296	0.24735204
0.5	0.09510011	0.09509549
0.6	-0.03085006	-0.03087488
0.7	-0.12308711	-0.12329599
0.8	-0.17368277	-0.17394318
0.9	-0.17276712	-0.17345245
1.0	-0.10947729	-0.11111111

4. Conclusion

In this paper, we have used the differential transformation method (DTM) to find approximate solutions of the second order differential equations (exponential types).The method was used in direct way without using linearization, perturbation or discretization. The approximate analytical solutions of the equations were calculated in the form of a series with easily computable component by DTM. Comparing the method with the exact solution shows that the present approach is effective and powerful. Four modeling problems from mathematical physics are discussed to illustrate the effectiveness and the performance of the proposed method

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