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Characterization of 2-Inner Product Using Euler-Lagrange Type of Equality

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Abstract: ***Problem of necessary and sufficient conditions for 2-normed space be 2-pre-Hilbert space is focus of interest of many mathematicians. Some of the characterizations of a 2-inner product may be found in [2], [4] and [5]. In this paper, it's given necessary and sufficient condition for the existing of 2-inner product into 2-normed space $(L,\|\cdot,\cdot\|)$ using the Euler-Lagrange type of equality and also is given generalization of this characteristic.***

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1. Introduction

Let L be a real vector space with dimension greater than 1 and, $\|\cdot,\cdot\|$ be a real function defined on $L\times L$ which satisfies the following:

a) $\|x,y\|\geq 0$, for each $x,y\in L$ и $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent;

b) $\|x,y\|=\|y,x\|$, for each $x,y\in L$;

c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for each $x,y\in L$ and for each $\alpha\in\mathbf{R}$,

d) $\|x+y,z\|\leq\|x,z\|+\|y,z\|$, for each $x,y,z\in L$.

Function $\|\cdot,\cdot\|$ is called a 2-*norm of* L, and $(L,\|\cdot,\cdot\|)$ is called a 2-normed vector space ([6]).

Let $n>1$ be a natural number, L be a real vector, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be real function on $L\times L\times L$ such that:

i) $(x,x|y)\geq 0$, for each $x,y\in L$ и $(x,x|y)=0$ if and only if a and b are linearly dependent;

ii) $(x,y|z)=(y,x|z)$, for each $x,y,z\in L$;

iii) $(x,x|y)=(y,y|x)$, for each $x,y\in L$;

iv) $(\alpha x,y|z)=\alpha(x,y|z)$, for each $x,y,z\in L$ and for each $\alpha\in\mathbf{R}$; and

v) $(x+x_1,y|z)=(x,y|z)+(x_1,y|z)$, for each $x_1,x,y,z\in L$.

Function $(\cdot,\cdot|\cdot)$ is called a 2-*inner product*, and $(L,(\cdot,\cdot|\cdot))$ is called a 2-*pre-Hilbert space* ([2]).

Concepts of 2-norm and 2-inner product are two dimensional analogies of concepts of norm and inner product. R. Ehret proved ([5]) that, if $(L,(\cdot,\cdot|\cdot))$ be 2-pre-Hilbert space, than

$$\|x,y\|=(x,x|y)^{1/2},\text{ for each } x,y\in L \tag{1}$$

defines 2-norm. So, we get 2-normed vector space $(L,\|\cdot,\cdot\|)$ and for each $x,y,z\in L$ the following equalities are true:

$$(x,y|z)=\frac{\|x+y,z\|^2-\|x-y,z\|^2}{4}, \tag{2}$$

$$\|x+y,z\|^2+\|x-y,z\|^2=2(\|x,z\|^2+\|y,z\|^2) \tag{3}$$

In fact, equality (3) is analogy of parallelogram equality, and we'll named *parallelepiped equality.* The reason for that is the following situation. Let $(L,(\cdot,\cdot))$ be a real pre-Hilbert space. Then,

$$\|x,z\|=\begin{vmatrix}(x,x) & (x,z)\\ (x,z) & (z,z)\end{vmatrix}^{1/2},\text{ for each } x,z\in L$$

define standard 2-norm. If $L=\mathbf{R}^3$ is used with the usual scalar product and the sets $\{x,y\},\{x,z\}$ and $\{y,z\}$ are linearly independent, then $\|x,z\|$, $\|y,z\|$, $\|x+y,z\|$ and $\|x-y,z\|$ ce equal to the areas of the parallelograms constructed above the vectors x and z, y and z, $x+y$ and z, $x-y$ and z, respectively. It's easy to find that

$$\begin{aligned}\|x+y,z\|^2+\|x-y,z\|^2&=[(x,x)+2(x,y)+(y,y)](z,z)\\&\quad-[(x,z)+(y,z)]^2\\&\quad+[(x,x)-2(x,y)+(y,y)](z,z)\\&\quad-[(x,z)-(y,z)]^2\\&=2[(x,x)(z,z)-(x,z)^2]\\&\quad+2[(y,y)(z,z)-(y,z)^2]\\&=2(\|x,z\|^2+\|y,z\|^2).\end{aligned}$$

It means that the sum of the squares of diagonal intersection areas of the parallelepiped is double the sum of the square of appropriate side's areas of the same parallelepiped.

Furthermore, if $(L,\|\cdot,\cdot\|)$ be as 2-norm vector space as such that for each $x,y,z\in L$, (3) is true, then (2) defines 2-inner product on L, and (1) is satisfied.

In [4] C. Diminnie and A. White characterize 2-pre-Hilbert space using 2-functional partial derivatives i.e. they prove that if $(L,(\cdot,\cdot|\cdot))$ be a 2-pre-Hilbert space in which the norm is defined by (1), then for each $x,y,z\in L$. the following equality is true

$$(x,y|z)=\lim_{t\to 0}\frac{\|x+ty,z\|-\|x,z\|}{2t}.$$

The sequence $\{x_n\}_{n=1}^{\infty}$ in 2-norm real space is named *convergent* if there exists $x \in L$ such that

$$\lim_{n\to\infty} \| x_n - x, y \| = 0 \text{, for each } y \in L.$$

The vector $x \in L$ is named *bound of the sequence* $\{x_n\}_{n=1}^{\infty}$ and denote

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x,\ n \to \infty, \text{ ([12]).}$$

Lemma 1 ([12]). Let L be 2-normed real space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in L. If

$$\lim_{n\to\infty} \| x_n - x, y \| = 0,$$

then

$$\lim_{n\to\infty} \| x_n, y \| = \| x, y \|. \ \blacksquare$$

2. Characterization of 2-inner product

Theorem 1. Let $(L, \|\cdot,\cdot\|)$ be a 2-normed real space. 2-norm is get from 2-inner product if and only if

$$\frac{\|ax+by,z\|^2}{\gamma} + \frac{\|\beta bx - \alpha ay, z\|^2}{\gamma\alpha\beta} = \frac{\|x,z\|^2}{\alpha} + \frac{\|y,z\|^2}{\beta} \qquad (4)$$

for each $x,y,z \in L$ and for each $a,b \in \mathbf{R}$, $\alpha,\beta > 0$, $\gamma = \alpha a^2 + \beta b^2$.

Proof. Let for each $x,y,z \in L$ and $a,b \in \mathbf{R}$, $\alpha,\beta > 0$, $\gamma = \alpha a^2 + \beta b^2$, equality (4) holds. If $\alpha = \beta = a = b = 1$ in (4) we get the equality (3). It means that (2) defines 2-inner product on L, from which is obtained the 2-norm i.e. the equality (1) is satisfied.
Conversely, let exist 2-inner product from which is obtained the 2-norm, i.e. the equality (1) is satisfied. Then, for each $x,y,z \in L$ and $a,b \in \mathbf{R}$, $\alpha,\beta > 0$, $\gamma = \alpha a^2 + \beta b^2$ is satisfied

$$\frac{\|ax+by,z\|^2}{\gamma} + \frac{\|\beta bx - \alpha ay,z\|^2}{\gamma\alpha\beta} =$$

$$= \frac{1}{\gamma}(ax+by, ax+by \mid z) + \frac{1}{\gamma\alpha\beta}(\beta bx - \alpha ay, \beta bx - \alpha ay \mid z)$$

$$= \frac{1}{\gamma}[a^2(x,x\mid z) + 2ab(x,y\mid z) + b^2(y,y\mid z)]$$

$$+ \frac{1}{\gamma\alpha\beta}[\beta^2 b^2(x,x\mid z) - 2ab\alpha\beta(x,y\mid z) + \alpha^2 a^2(y,y\mid z)]$$

$$= \frac{1}{\gamma}(\alpha a^2 + \beta b^2)\left[\frac{\|x,z\|^2}{\alpha} + \frac{\|y,z\|^2}{\beta}\right] = \frac{\|x,z\|^2}{\alpha} + \frac{\|y,z\|^2}{\beta},$$

i.e. the equality (4) holds ■

Corollary 1. Let $(L, \|\cdot,\cdot\|)$ be a 2-normed real space. 2-norm is obtained by 2-inner product if and only if

$$\| ax+by, z \|^2 + \| bx - ay, z \|^2 = (a^2+b^2)(\| x,z \|^2 + \| y,z \|^2), \quad (5)$$

for each $x,y,z \in L$ and $a,b \in \mathbf{R}$.

Proof. Directly is implied by theorem 1 when $\alpha = \beta = 1$. ■

The equality (4) is Euler-Lagrange type, and it characterizes the 2-inner product. In fact, the following corollary is an analogy of corollary 1, i.e. its trigonometric form.

Corollary 2. Let $(L, \|\cdot,\cdot\|)$ be a real 2-normed space. 2-norm is obtained by 2-inner product if and only if

$$\| x\cos\alpha + y\sin\alpha, z \|^2 + \| y\cos\alpha - x\sin\alpha, z \|^2 = \| x,z \|^2 + \| y,z \|^2, \quad (6)$$

for each $x,y,z \in L$ and for each $\alpha \in \mathbf{R}$.

Proof. Let (6) is true for each $x,y,z \in L$ and for each $\alpha \in \mathbf{R}$. By letting $\alpha = \frac{\pi}{4}$ in (6), we obtain the following equality:

$$\| x\cos\tfrac{\pi}{4} + y\sin\tfrac{\pi}{4}, z \|^2 + \| y\cos\tfrac{\pi}{4} - x\sin\tfrac{\pi}{4}, z \|^2 = \| x,z \|^2 + \| y,z \|^2,$$

i.e. the equality

$$\frac{\|x+y,z\|^2}{2} + \frac{\|y-x,z\|^2}{2} = \| x,z \|^2 + \| y,z \|^2,$$

which is equivalent to (3). It means that (2) defines 2-inner product from which is obtained the 2-norm, i.e. the equality (1) is satisfied.
Conversely, let there exist 2-inner product from which the 2-norm is obtained and let $x,y,z \in L$ and $\alpha \in \mathbf{R}$. By Corollary 1, for every $x,y,z \in L$ and every $a,b \in \mathbf{R}$ the equality (5) is satisfied. Letting $a = \cos\alpha$ and $b = \sin\alpha$ into (5), we get that for every $x,y,z \in L$ and every $\alpha \in \mathbf{R}$:

$$\| x\cos\alpha + y\sin\alpha, z \|^2 + \| x\sin\alpha - y\cos\alpha, z \|^2 =$$

$$= (\cos^2\alpha + \sin^2\alpha)(\| x,z \|^2 + \| y,z \|^2)$$

$$= \| x,z \|^2 + \| y,z \|^2,$$

i.e. the equality (6) is correct. ■

Remark 1. In the second part of the proof of the corollary 2, the equality (6) derives from the equality (5). But, the equalities (5) and (6) are actually equivalent. Let $x,y,z \in L$ and $a,b \in \mathbf{R}$. Then exist $\alpha \in \mathbf{R}$ so that $\frac{a}{\sqrt{a^2+b^2}} = \cos\alpha$ and $\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$. Finally, from the equality (6), we get the equality:

$$\| x\frac{a}{\sqrt{a^2+b^2}} + y\frac{b}{\sqrt{a^2+b^2}}, z \|^2 + \| x\frac{b}{\sqrt{a^2+b^2}} - y\frac{a}{\sqrt{a^2+b^2}}, z \|^2 =$$

$$= \| x,z \|^2 + \| y,z \|^2$$

which is equivalent to the equality (5).

Lemma 2. Let $(L, \|\cdot,\cdot\|)$ be a real 2-normed space. If there exists any $\alpha \in \mathbf{R}$ such that for every $x,y,z \in L$, equality (6) is satisfied, then for every $x,y,z \in L$ and for every $n \in \mathbf{N}$,

$$\begin{aligned} &\| x\cos n\alpha + y\sin n\alpha, z \|^2 + \| y\cos n\alpha - x\sin n\alpha, z \|^2 = \\ &= \| x,z \|^2 + \| y,z \|^2. \end{aligned} \qquad (7)$$

Proof. Letting $n = 1$, the equality (7) transforms as (6), i.e. it's satisfied. Let suppose that $n = k \geq 1$ exists such that the equality (7) is satisfied i.e. for each $x,y,z \in L$

$$\begin{aligned} &\| x\cos k\alpha + y\sin k\alpha, z \|^2 + \| y\cos k\alpha - x\sin k\alpha, z \|^2 = \\ &= \| x,z \|^2 + \| y,z \|^2. \end{aligned} \qquad (8)$$

If we take $x\cos\alpha + y\sin\alpha$ and $y\cos\alpha - x\sin\alpha$ for x and y respectively, then by (6) and (8) we get

$$\|x,z\|^2+\|y,z\|^2=$$
$$=\|x\cos\alpha+y\sin\alpha,z\|^2+\|y\cos\alpha-x\sin\alpha,z\|^2$$
$$=\|(x\cos\alpha+y\sin\alpha)\cos k\alpha+(y\cos\alpha-x\sin\alpha)\sin k\alpha,z\|^2+$$
$$+\|(y\cos\alpha-x\sin\alpha)\cos k\alpha-(x\cos\alpha+y\sin\alpha)\sin k\alpha,z\|^2$$
$$=\|x(\cos\alpha\cos k\alpha-\sin\alpha\sin k\alpha)+y(\sin\alpha\cos k\alpha+\cos\alpha\sin k\alpha)\|^2+$$
$$+\|y(\cos\alpha\cos k\alpha-\sin\alpha\sin k\alpha)-x(\sin\alpha\cos k\alpha+\cos\alpha\sin k\alpha)\|^2$$
$$=\|x\cos(k+1)\alpha+y\sin(k+1)\alpha,z\|^2$$
$$+\|y\cos(k+1)\alpha-x\sin(k+1)\alpha,z\|^2.$$

It means that (7) is true for $n=k+1$. So, from the principle of mathematical induction is true for all $n\in\mathbf{N}$. ■

The following result of mathematical analysis is necessary for the other characterization of the 2-inner product. Let $B(0;1)$ denote the unit circle in $\mathbf{R}^2$ and let

$$\pi\mathbf{Q}=\{k\pi \mid k\in\mathbf{Q}\}.$$

Then, the validity of the following lemma is obvious.

Lemma 3. For every $\alpha\in\mathbf{R}\setminus\pi\mathbf{Q}$, the set

$$\{(\cos n\alpha,\sin n\alpha)\mid n\in\mathbf{N}\}$$

is dense into $B(0;1)$. ■

Theorem 2. Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. 2-norm is get by 2-inner product if and only if there exists $\alpha\in\mathbf{R}\setminus\pi\mathbf{Q}$ such that for every $x,y,z\in L$ the equality (6) is correct.

Proof. Let exists 2-inner product, and 2-norm is get by that product. Then, by corollary 1, the equality (6) is satisfied for every $\alpha\in\mathbf{R}$ and for every $x,y,z\in L$. Thus, the equality (6) is satisfied for some $\alpha\in\mathbf{R}\setminus\pi\mathbf{Q}$ and for every $x,y,z\in L$.

Conversely, let exist $\alpha\in\mathbf{R}\setminus\pi\mathbf{Q}$ such that for every $x,y,$ $z\in L$ the equality (6) is correct. But, $(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})\in B(0;1)$, and by lemma 3 follows existence of sequence of natural numbers $\{m_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} m_n=\infty$ and

$$\lim_{n\to\infty}(\cos m_n\alpha,\sin m_n\alpha)=(\tfrac{\sqrt{2}}{2},\tfrac{\sqrt{2}}{2}).\quad(9)$$

Moreover, for every $x,y,z\in L$

$$0\le\|x\cos m_n\alpha+y\sin m_n\alpha-\tfrac{x\sqrt{2}+y\sqrt{2}}{2},z\|$$
$$\le|\cos m_n\alpha-\tfrac{\sqrt{2}}{2}|\cdot\|x,z\|+|\sin m_n\alpha-\tfrac{\sqrt{2}}{2}|\cdot\|y,z\|$$

and

$$0\le\|y\cos m_n\alpha-x\sin m_n\alpha-\tfrac{y\sqrt{2}-x\sqrt{2}}{2},z\|$$
$$\le|\cos m_n\alpha-\tfrac{\sqrt{2}}{2}|\cdot\|y,z\|+|\sin m_n\alpha-\tfrac{\sqrt{2}}{2}|\cdot\|x,z\|.$$

Letting $n\to\infty$ into the last two inequalities, the equality (9) implies

$$\lim_{n\to\infty}\|x\cos m_n\alpha+y\sin m_n\alpha-\frac{x\sqrt{2}+y\sqrt{2}}{2},z\|=0,$$

for every $x,y,z\in L$ and

$$\lim_{n\to\infty}\|y\cos m_n\alpha-x\sin m_n\alpha-\frac{y\sqrt{2}-x\sqrt{2}}{2},z\|=0,$$

for every $x,y,z\in L$. Now, by the last two equalities and lemma 1,

$$\lim_{n\to\infty}\|x\cos m_n\alpha+y\sin m_n\alpha,z\|=\|\frac{x\sqrt{2}+y\sqrt{2}}{2},z\|,\quad(10)$$

for every $x,y,z\in L$ and

$$\lim_{n\to\infty}\|y\cos m_n\alpha-x\sin m_n\alpha,z\|=\|\frac{y\sqrt{2}-x\sqrt{2}}{2},z\|,\quad(11)$$

for every $x,y,z\in L$. Furthermore, $m_n\in\mathbf{N}$, for $n\in\mathbf{N}$. Thus, by lemma 2, for every $n\in\mathbf{N}$

$$\|x\cos m_n\alpha+y\sin m_n\alpha,z\|^2+\|y\cos m_n\alpha-x\sin m_n\alpha,z\|^2=$$
$$=\|x,z\|^2+\|y,z\|^2.$$

Finally, letting $n\to\infty$ into the equality above, we get that the equalities (10) and (11) imply

$$\frac{\|x+y,z\|^2}{2}+\frac{\|y-x,z\|^2}{2}=\|x,z\|^2+\|y,z\|^2.$$

This equality and equality (3) are equivalent. It means that equality (2) defines 2-inner product and 2-norm is get by that, i.e. (1) is satisfied. ■

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