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Weakly b-Continuous Functions

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Abstract: ***In 1961, Levine [7] introduced weakly continuous functions and in 1987, Noiri [12] introduced and studied weakly □-continuous functions. Later on Ekici [5], in 2008, introduced and studied BR-continuous and hence weakly BR-continuous functions in a similar fashion, by means of b-regular and b-open [4] sets. This prompted us to introduce and study weakly b-continuous by making use of b-open sets. We studied several characterizations of weakly b-continuous functions. Some basic properties including restrictions and compositions of such functions have also been studied.***

Keywords: Weakly continuous functions, weakly α-continuous functions, Weakly BR-continuous functions, Weakly b-continuous and b-open sets.

1. Introduction

Levine [7] introduced the concept of a weakly continuous function. In 2008 Ekici [5] has introduced and studied the class of functions namely BR-continuous functions and weakly BR-continuous functions by making use of b-regular sets. He obtained some characterizations of weakly BR-continuous functions and established relationships among such functions and several other existing functions. In a similar manner here our purpose is to introduce and study generalizations in form of new classes of functions namely weakly b-continuous. The author [6] has already introduced and studied b-continuous functions.

Let (X, τ) and (Y, σ) (or X and Y) denote topological spaces. For a subset A of a space X, the closure A and the interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** (resp. **regular closed)** if A=int(cl(A)) (resp. A=cl(int(A)). A subset A is said to be **preopen [9]** (resp. **semi open [8], b-open [4], α-open [11], semi preopen [3] or β-open [1])** if A⊂int(cl(A)) (resp. A⊂cl(int(A)), A⊂int(cl(A))∪cl(int(A)), A⊂int (cl(int(A))), A⊂cl(int(cl(A)))). A subset G of X is called b-neighbourhood of x∈X if there exists a b-open set B containing x such that B⊂G.

A point x∈X is said to be a **θ-cluster point** of A [14] if A∩cl(U)≠ϕ for every open set U containing x. The set of all θ-cluster points of A is called **θ-closure** of A and is denoted by θ-cl(A). A subset A is called θ-closed if -cl(A)=A [14]. The complement of a θ-closed set is called θ-open set. The complement of a b-open (resp. preopen, semi open, α-open, semi preopen) set is called **b-closed**(resp. **preclosed, semi closed, α-closed, semi prelosed**). The intersection of all b-closed (resp. preclosed, semi closed, α-closed, semi preclosed) sets of X containing A is called **b-closure** (resp. **preclosure, semi closure, α-closure, semi preclosure**) of A and denoted by b-cl(A)(resp. p-cl(A), s-cl(A), α-cl(A), sp-cl(A)). The union of all b-open (resp. preopen, semi open, α-open, semi preopen) sets of X contained in A is called **b-interior** (resp. **preinterior, semi interior, α-interior, semi preinterior)** of A and denoted by b-int(A) (resp. p-int(A), s-int(A), α-int(A), sp-int(A)). A subset A is said to be **b-regular** [4] if it is b-open as well as b-closed. The family of all b-open (resp. b-regular) sets of X is denoted by BO(X) (resp. BR(X)). A point x∈X is called **b-θ-cluster point** [13] of a subset A of X if b-cl(B)∩A≠ϕ for every b-open set B containing x. The set of all b-θ-cluster points of A is called **b-θ-closure** of A and is denoted by **b-θ-cl(A)**. A subset A of X is said to be **b-θ-closed** if A=b-θ-cl(A). The complement of a b-θ-closed set is said to be **b-θ-open**. A point x∈X is called **b-θ-interior** point of A⊂X if there exists a b-regular set U containing x such that U⊂A and is denoted by x∈b-θ-int(A).

2. Definitions and Characterizations

Definition 2.1:- A function f : X → Y is said to be **b-continuous [6]** (resp. **strongly θ-b-continuous [14]**) if for each x∈X and each open set V of Y containing f(x), there exists a b-open set U containing x such that f(U)⊂V (resp. f(b-cl(U))⊂V).

Definition 2.2:- A function f : X → Y is said to be **weakly continuous [7]** (resp. **weakly α-continuous [12]**) if for each x∈X and each open set V of Y containing f(x), there exists an open (resp. α-open) set U containing x such that f(U)⊂cl(V).

Definition 2.3:- A function f : X → Y is said to be **weakly b-continuous** if for each x∈X and each open set V of Y containing f(x), there exists a b-open set U containing x such that f(U)⊂cl(V).

Theorem 2.4:- For a function f : X → Y, the following are equivalent :

(a) f is weakly b-continuous at x∈X.

(b) for each neighbourhood V of f(x), there exists an b-open set U containing x (or b-neighbourhood U of x) such that f(U)⊂cl(V).

(c) b-cl(f^{-1}(int(cl(V))))⊂f^{-1}(cl(V)) for every subset V of Y.

(d) b-cl(f^{-1}(int(F)))⊂f^{-1}(F) for every regular closed subset F of Y.

(e) b-cl(f^{-1}(V))⊂f^{-1}(cl(V)) for every open set V of Y.

(f) f^{-1}(V)⊂b-int(f^{-1}(cl(V))) for every open set V of Y.

(g) b-cl(f^{-1}(V))⊂f^{-1}(cl(V)) for each preopen set of Y.

(h) f^{-1}(V)⊂b-int(f^{-1}(cl(V))) for each preopen set V of Y.

Proof :- (a)⟺(b) obvious by definition.

(a)⟹(c) Let V⊂Y and x∈X–f^{-1}(cl(V)). Then f(x)∈Y–cl(V) and there exists an open set U containing f(x), such that U∩V=ϕ. We have cl(U)∩int(cl(V))=ϕ. Since f is weakly b-continuous, so, there exists a b-open set W containing x such that $f(W)\subset cl(U)$. Then $W\cap f^{-1}(int(cl(V)))=\phi$ and $x\in X-b\text{-}cl(f^{-1}(int(cl(V))))$. Hence $b\text{-}cl(f^{-1}(int(cl(V))))\subset f^{-1}(cl(V))$.

(c)$\Longrightarrow$(d) Let F be any regular closed set in Y. Then $b\text{-}cl(f^{-1}(int(F)))=b\text{-}cl(f^{-1}(int(cl(int(F)))))\subset f^{-1}(cl(int(F)))=f^{-1}(F)$.

(d)$\Longrightarrow$(e) Let V be an open subset of Y. Since cl(V) is regular closed in Y, then $b\text{-}cl(f^{-1}(V))\subset b\text{-}cl(int(cl(V)))\subset f^{-1}(cl(V))$.

(e)$\Longrightarrow$(f) Let V be any open set in Y. Since $Y-cl(V)$ is open in Y, then $X-b\text{-}int(f^{-1}(cl(V)))\subset b\text{-}cl(f^{-1}(Y-cl(V)))\subset f^{-1}(cl(Y-cl(V)))\subset X-f^{-1}(V)$. Hence $f^{-1}(V)\subset b\text{-}int(f^{-1}(cl(V)))$.

(f)$\Longrightarrow$(a) Let $x\in X$ and V be any open subset of Y containing f(x), then $x\in f^{-1}(V)\subset b\text{-}int(f^{-1}(cl(V)))$. Take $W=b\text{-}int(f^{-1}(cl(V)))\subset f^{-1}(cl(V))$. Thus $f(W)\subset cl(V)$ and hence f is weakly b-continuous at $x\in X$.

(a)$\Longrightarrow$(g) Let V be any preopen set in Y and $x\in X-f^{-1}(cl(V))$. There exists an open set G containing f(x), such that $G\cap V=\phi$. We have, $cl(G\cap V)=\phi$. Since V is preopen, then $V\cap cl(G)\subset int(cl(V))\cap cl(G)\subset cl(int(cl(V))\cap G)\subset cl(int(cl(V)))\cap G\subset cl(int(cl(V\cap G)))\subset cl(V\cap G)=\phi$. Since f is weakly b-continuous and G is an open set containing f(x), there exists a b-open set W in X containing x such that $f(W)\subset cl(G)$. Then $f(W)\cap V=\phi$ and $W\cap f^{-1}(V)=\phi$. This implies that $x\in X-b\text{-}cl(f^{-1}(V))$ and thus $b\text{-}cl(f^{-1}(V))\subset f^{-1}(cl(V))$.

(g)$\Longrightarrow$(h) Let V be any preopen set. Since $Y-cl(V)$ is open in Y, then $X-b\text{-}int(f^{-1}(cl(V)))=b\text{-}cl(f^{-1}(Y-cl(V)))\subset f^{-1}(cl(Y-cl(V)))\subset X-f^{-1}(V)$. This shows that $f^{-1}(V)\subset b\text{-}int(f^{-1}(cl(V)))$.

(h)$\Longrightarrow$(a) Let $x\in X$ and V be any open set in Y containing f(x). We have $x\in f^{-1}(V)\subset b\text{-}int(f^{-1}(cl(V)))$. Take $W=b\text{-}int(f^{-1}(cl(V)))$. Then $f(W)\subset cl(V)$ and hence f is weakly b-continuous at x in X.

Theorem 2.5:- For a function $f : X \to Y$ the following are equivalent :

(1) f is weakly b-continuous at $x\in X$.

(2) $x\in b\text{-}int(f^{-1}(cl(U)))$ for each neighbourhood U of f(x).

Proof:- (1)$\Longrightarrow$(2) Let U be any neighbourhood of f(x). Then there exists a b-open G containing x such that $f(G)\subset cl(U)$. Since $G\subset f^{-1}(cl(U))$ and G is b-open then $x\in G\subset b\text{-}int\ G\subset b\text{-}int(f^{-1}(cl(U)))$.

(2)$\Longrightarrow$(1) Let $x\in b\text{-}int(f^{-1}(cl(U)))$ for each neighbourhood U of f(x). Then $V=b\text{-}int(f^{-1}(cl(U)))$. This implies that $f(V)\subset cl(U)$ and V is b-open. Hence f is weakly b-continuous at $x\in X$.

Theorem 2.6:- If $f : X\to Y$ is a weakly b-continuous function and Y is Hausdorff, then f has b-closed point inverses.

Proof:- Let $y\in Y$ and $x\in X$ such that $f(x)\neq y$. Since Y is Hausdorff, there exist disjoint open sets G and H such that $f(x)\in G$ and $y\in H$. Also, $G\cap H=\phi$, implies $cl(G)\cap H=\phi$. We have $y\notin cl(G)$. Since f is weakly b-continuous, so, there exists a b-open set U containing x such that $f(U)\subset cl(G)$. Assume that U is not contained in $\{x\in X : f(x)=y\}$. If possible for some $u\in U$, $f(u)=y$, then $y= f(u) \in cl(G)$. This contradicts $cl(G)\cap H=\phi$. Hence $U\subset\{x\in X : f(x)\neq y\}$ and U is b-open in X. Thus, set $\{x\in X : f(x)\neq y\}$ is b-open in X, equivalently, $f^{-1}\{(y)\}= \{x\in X : f(x)=y\}$ is b-closed in X.

Theorem 2.7:- For a function $f : X \to Y$, the following are equivalent :

(a)f is weakly b-continuous.

(b) $f(b\text{-}cl(G))\subset\theta\text{-}cl(f(G))$ for each subset G of X.

(c) $b\text{-}cl(f^{-1}(H))\subset f^{-1}(\theta\text{-}cl(H))$ for each subset H of Y.

(d) $b\text{-}cl(f^{-1}(int(\theta\text{-}cl(H))))\subset f^{-1}(\theta\text{-}cl(H))$ for every subset H of Y.

Proof:- (a)$\Longrightarrow$(b) Let $G\subset X$, $x\in b\text{-}cl(G)$ and U be any open set in Y containing f(x). There exists a b-open W containing x such that $f(W)\subset cl(U)$. Since, $x\in b\text{-}cl(G)$, then $W\cap G=\phi$. This implies that $\phi\neq f(W)\cap f(G)\subset cl(U)\cap f(G)$ and $f(x)\in\theta\text{-}cl(f(G))$. Thus, $f(b\text{-}cl(G))\subset\theta\text{-}cl(f(G))$.

(b)$\Longrightarrow$(c) Let $H\subset Y$. Then $f(b\text{-}cl(f^{-1}(H)))\subset\theta\text{-}cl(H)$ and hence $b\text{-}cl(f^{-1}(H))\subset f^{-1}(\theta\text{-}cl(H))$.

(c)$\Longrightarrow$(d) Let $H\subset Y$. Since θ-cl(H) is closed in Y, then $b\text{-}cl(f^{-1}(int(\theta\text{-}cl(H))))\subset f^{-1}(\theta\text{-}cl(int(\theta\text{-}cl(H))))=f^{-1}(cl(int(\theta\text{-}cl(H))))\subset f^{-1}(\theta\text{-}cl(H))$.

(d)$\Longrightarrow$(a) Let H be any open set of Y. We have $H\subset int(cl(H))=int(\theta\text{-}cl(H))$.

Thus, $b\text{-}cl(f^{-1}(H))\subset b\text{-}cl(f^{-1}(int(\theta\text{-}cl(H))))\subset f^{-1}(\theta\text{-}cl(H))\subset f^{-1}(cl(H))$. This implies from Theorem 2.4(e) that f is weakly b-continuous.

Theorem 2.8:- If $f^{-1}(\theta\text{-}cl(V))$ is b-closed in X for every subset V of Y, then f is weakly b-continuous.

Proof:- Let $V\subset Y$. Since $f^{-1}(\theta\text{-}cl(V))$ is b-closed in X, then $b\text{-}cl(f^{-1}(V))\subset b\text{-}cl(f^{-1}(\theta\text{-}cl(V)))=f^{-1}(\theta\text{-}cl(V))$. This implies from above Theorem 2.7 that f is weakly b-continuous.

Theorem 2.9:- If $f : X \to Y$ is a function which is weakly b-continuous, then $f^{-1}(V)$ is b-closed in X for every θ-closed subset V of Y.

Proof:- Follows directly from Theorem 2.7. Since f is weakly b-continuous, so, $b\text{-}cl(f^{-1}(V))\subset f^{-1}(\theta\text{-}cl(V))=f^{1}(V)$ for a θ-closed set V in Y. This implies that $b\text{-}cl(f^{-1}(V))= f^{-1}(V)$. Thus, $f^{-1}(V)$ is b-closed if V is θ-closed.

Corollary 2.10:- Let $f : X \to Y$ be a weakly b-continuous function, then $f^{-1}(V)$ is b-open in X for every θ-open subset V of Y.

Theorem 2.11:- Let $f : X \to Y$ be a function. If Y is regular then following are equivalent :

(a) f is weakly b-continuous.

(b) f is b-continuous.

(c) f is strongly θ-b-continuous if and only if f is continuous [13].

Proof:- Let $x\in X$ and V be an open set of Y containing f(x). Since Y is regular,
then there exists an open set H of Y containing f(x) such that $H\subset cl(H)\subset V$. Since f is weakly b-continuous, there exists a b-open set U of X containing x such that $f(U)\subset cl(H)\subset V$. Thus f is b-continuous. Converse is obvious.

Lemma 2.12 [4]:-The intersection of an α-open set and a b-open set is a b-open set.

Lemma 2.13 [10]:- If A is α-open in X, then $BO(A)=BO(X)\cap A$.

Lemma 2.14 [2]:- If $A\subset B\subset X$, $B\in BO(X)$ and $A\in BO(B)$, then $A\in BO(X)$.

Theorem 2.15:- Let $\{ A_i : i\in I\}$ be an α-open cover of a space X and $f : X \to Y$ be
a function, then following are equivalent :

(a) f is weakly b-continuous.

(b) the restriction f/A i : A i → Y is weakly b-continuous for each i∈I.

Proof:- (a)⟹(b) Let i∈I and A_i be an α-open set in X. Let x∈A_i and V be an open set in Y containing $f/A_i(x)=f(x)$. Since f is weakly b-continuous, so, there exists a b-open set G containing x such that f(G)⊂cl(V). Moreover G∩A_i is b-open in A_i containing x and $f/A_i(G\cap A_i)=f(G\cap A_i)\subset f(G)\subset cl(V)$. Hence f/A_i is weakly b-continuous.

(b)⟹(a) Let x∈X and V be an open set in Y containing f(x). There exists i∈I, such that x∈A_i. Since $f/A_i : A_i \to Y$ is weakly b-continuous, there exists a b-open set G in A_i containing x such that $f/A_i(G)\subset cl(V)$. Since each A_i is α-open in X then G is b-open in X containing x and f(G)⊂cl(V). Hence f is weakly b-continuous.

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