SOME MORE REMARKS ON COMULTIPLICATION MODULES

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Abstract. In this paper we will discuss some more results of comultiplication modules.

1. Introduction

Throughout of this paper, R denotes the commutative ring with unity.

The concept of comultiplication modules was first introduced by H. Ansari-Toroghy and F. Farshadifar [3] in 2007. An R-module M is called comultiplication module if for every submodule N of M, there exists an ideal I of R such that $N = ann_M(I)$. An R-module M is comultiplication module if and only if for any submodule N of M, $N = (0 :_M ann_R(N))$[3]. Also, every proper submodule of a comultiplication module is comultiplication module[3]. However, converse may not be true. For example, if V is a two dimensional vector space over a field k then V cannot be comultiplication module but every proper subspace of V is comultiplication as every one dimensional vector space is comultiplication module. If R is local ring then every comultiplication module is cocyclic. Also a finitely generated second submodule of a comultiplication module is multiplication module [6]. It was also shown that every non zero comultiplication module contains a minimal submodule[6].

In 2009, Reza Ebrahimi Atani and Shahabaddin Ebrahimi Atani [14] described the indecomposable comultiplication modules and non-separated indecomposable comultiplication module over pullback of local Dedekind domains.

In 2011, Yousef Al-Shaniafi and Patrick F. Smith [9] studied the localization of comultiplication modules over a general ring R. If every maximal ideal $\mathfrak{m}$ of R is good for M then M is comultiplication R-module if and only if $M_{\mathfrak{m}}$ is a comultiplication $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R [9]. Every comultiplication module have unique complements [10]. Also every comultiplication module have unique Goldie dimensions [10].

2. Main results

We begin with the definition of Multiplication and Comultiplication modules.

Definition 2.1. An R-module M is called multiplication module if for every submodule N of M, there exists an ideal I of R such that $N = IM$ [9] .

Definition 2.2. An R-module M is called comultiplication module if for any submodule N of M, there exists an ideal I of R such that $N = ann_M(I)$ [9] .

Definition 2.3. An R-module M is a self-cogenerator if for every submodule N of M, the factor module M/N embeds in direct product M^I of copies of M, for some index set I, that is, there exists a one-one map $\phi : M/N \to M^I$ [9].

Proposition 2.4. *M is a self-cogenerator if and only if for every submodule N of M,*

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there exists an index set I and an endomorphisms ϕ_i for all $i \in I$ on M such that $N = \bigcap_{i\in I} \ker \phi_i$.

Proof. Let M be a self-cogenerated module and let N be any submodule of M. Then by definition, there exists a one-one map $\phi : M/N \to M^I$ for some index set I. Consider the map $\phi_i : M \xrightarrow{\pi} M/N \xrightarrow{\phi} M^I \xrightarrow{\pi_i} M$, that is $\phi_i = \pi_i\phi\pi : M \to M$. Note that $x \in \cap_{i\in I}\ker\phi_i$ if and only if $\pi_i\phi\pi(x) = 0$ for all i, that is, $\pi_i\phi(x+N) = (0)$ for all i, if and only if $\phi(x+N) = (0)$, that is, $x \in N$. Therefore,

$$\cap_{i\in I}\ker\phi_i = N.$$

Conversely, suppose that for every submodule N of M, there exists endomorphisms $\phi_i : M \to M$, $i \in I$, for some index set I, such that

$$\cap_{i\in I}\ker\phi_i = N.$$

Define $\phi : M \to M^I$ as $\phi(x) = (\phi_i(x))_{i\in I}$. Obviously,

$$ker(\phi) = \cap_{i\in I} ker(\phi_i) = N.$$

Therefore, ϕ induces a one-one map ϕ' from M/N to M^I and hence M is a self-cogenerator module.

□

Definition 2.5. Let M be an R-module and let $\mathfrak{p}$ be a prime ideal of R. Define a set $S_\mathfrak{p} = \{r \in R : \frac{r}{1} \equiv 0 \text{ in } R_\mathfrak{p}\}$. Note that, this is an ideal of R. Define a set $T_\mathfrak{p} = \{x \in M : \frac{x}{1} \equiv 0 \text{ in } M_\mathfrak{p}\}$. Note that, this is a submodule of M.

A prime ideal $\mathfrak{p}$ is called good for M if there exists some $a \in R \setminus \mathfrak{p}$ such that $aT_\mathfrak{p} = (0)$. For example, if $T_\mathfrak{p}$ is finitely generated for any prime ideal $\mathfrak{p}$ and for any R-module M then $\mathfrak{p}$ is good for M [9].

In the view of above terminology, the following theorem characterize the comultiplication module.

Theorem 2.6. *Let R be any ring, and let M be an R-module such that every maximal ideal of R is good for M. Then M is a comultiplication R-module if and only if $M_\mathfrak{m}$ is a comultiplication $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Suppose that M is a comultiplication R-module. Let $\mathfrak{m}$ be any maximal ideal of R. Let N be any submodule of $M_\mathfrak{m}$. Define $N' = \{x \in M : x/1 \in N\}$. Obviously, N' is a submodule of M. Let $ann_{R_\mathfrak{m}}(N) \subseteq ann_{R_\mathfrak{m}}(y)$ for some $y = y_1/s \in M_\mathfrak{m}$.

Let $r \in ann_R(N')$. This implies that $r/1 \in ann_{R_\mathfrak{m}}(N) \subseteq ann_{R_\mathfrak{m}}(y)$. Hence, $(r/1)(y_1/s) = 0$ implies that $try_1 = 0$ for some $t \in R \setminus \mathfrak{m}$. Therefore, $ry_1 \in T_\mathfrak{m}$. As $\mathfrak{m}$ is good for M, there exists some $d \in R \setminus \mathfrak{m}$ such that $dT_\mathfrak{m} = (0)$. This implies that $r \in ann_R(dy_1)$ implies $ann_R(N') \subseteq ann_R(dy_1)$ and hence, $dy_1 \in N'$. This implies that $dy_1/1 \in N$ and hence, $y = y_1/s = dy_1/ds \in N$. Therefore, $M_\mathfrak{m}$ is a comultiplication module.

Conversely, suppose that $M_\mathfrak{m}$ is comultiplication as an $R_\mathfrak{m}$-module for all maximal ideal $\mathfrak{m}$ of R. Let L be a submodule of M and $x \in M$ such that $ann_R(L) \subseteq ann_R(Rx)$. By Theorem [?], $(L :_R x)$ is not a maximal ideal of R. Suppose that $(L :_R x)$ is proper ideal of R. Then there exists a maximal ideal $\mathfrak{m}$ of R such that $(L :_R x) \subseteq \mathfrak{m}$. Then $L_\mathfrak{m}$ is a submodule of $M_\mathfrak{m}$. Let $\alpha/s \in ann_{R_\mathfrak{m}}(L_\mathfrak{m})$ for some $\alpha \in L$ $s \in R \setminus \mathfrak{m}$. Now, for any $y \in L$, $y/1 \in L_\mathfrak{m}$. This implies that $(\alpha/s)(y/1) = 0$, that is, $t\alpha y = 0$ for some $t \in R \setminus \mathfrak{m}$. Therefore, $\alpha y \in T_\mathfrak{m}$. Now, $\mathfrak{m}$ is good for M, implies that $a\alpha y = 0$. Since y is arbitrary element in L, we conclude that $a\alpha L = (0)$, that is, $a\alpha \in ann_R(L) \subseteq ann_R(Rx)$. Hence, $a\alpha x = 0$. Therefore, $\alpha/s \equiv a\alpha/as \in ann_{R_\mathfrak{m}}(x/1)$. Hence, we have $ann_{R_\mathfrak{m}}(L_\mathfrak{m}) \subseteq ann_{R_\mathfrak{m}}(x/1)$. Therefore, $x/1 \in L_\mathfrak{m}$. This implies that $x \in L$. Therefore, we conclude that M is a comultiplication module. □

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We know that every submodule of M is a comultiplication module [9]. The converse of above lemma need not be true. Consider V be a two dimensional vector space over a field F. Then obviously, it is a module over F. But it is not a comultiplication module, as for any subspace V_1 of dimension one is proper submodule of V and neither $V_1 = ann_V(0)$ nor $V_1 = ann_V(F)$. However V_1 is a comultiplication module, as $V_1 = ann_{V_1}(0)$ and $(0) = ann_{V_1}(F)$.

Remark 2.7. In general an R- module M may not contains a simple submodule. Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Suppose N is a simple submodule of $\mathbb{Z}$. Then $N = \mathbb{Z}x$ for any non zero $x \in \mathbb{Z}$. Hence

$$\mathbb{Z}x \cong \frac{\mathbb{Z}}{ann_{\mathbb{Z}}(x)} \cong \mathbb{Z},$$

which is a contradiction. Thus $\mathbb{Z}$ does not have any simple submodule. However, if M is comultiplication module then every nonzero submodule of M contains a simple submodule [6, 3.2].

If M comultiplication module then $\theta(N) \subseteq N$, for any submodule N of M and for any R-endomorphism θ on M [3]. However, if the map θ is monomorphism, then $\theta(N) = N$.

Definition 2.8. A non zero submodule S of an R-module M is called second submodule if for any $a \in R$, the map $\phi : S \to S$ given by $x \to ax$ is either surjective or zero [4].

Proposition 2.9. *If N is a second submodule of an R- module M then $ann_R(N)$ is a prime ideal of R.*

Proof. Let N be a second submodule of an R- module M. If possible, suppose $abN = (0)$ and $aN \neq (0), bN \neq (0)$.

Define the homomorphisms $f_a : N \to N$ and $g_b : N \to N$ respectively by $f_a(x) = ax$, $g_b(x) = bx$ for all $x \in N$. Then $\mathrm{Im(f_a)} = aN \neq (0)$ and $\mathrm{Im(g_b)} = bN \neq (0)$. Since N is a second submodule, we conclude that $aN = N$ and $bN = N$. This implies that $abN = N = (0)$, which contradicts that N is a second submodule. Therefore, $ann_R(N)$ is a prime ideal. □

Example 2.10. Let n be a fixed positive integer. Then

(1.) $\mathbb{Z}_n$ is a comultiplication $\mathbb{Z}$-module.
(2.) $\mathbb{Z}_n$ is a comultiplication $\mathbb{Z}_n$-module [6, 3.8].

Proof. We prove only (1). The proof of (2) is same as that of (1).

Let N be a submodule of $\mathbb{Z}_n$. Let $o(N) = d$. Then $n = md$ for some positive integer m. This implies that $N = m\mathbb{Z}_n$. Put $I = d\mathbb{Z}$. Then $d\mathbb{Z}$ is an ideal in $\mathbb{Z}$ such that $N = ann_{\mathbb{Z}_n}(d\mathbb{Z})$. Therefore, $\mathbb{Z}_n$ is a comultiplication $\mathbb{Z}$-module.

□

Example 2.11. Let p be any prime number. Let $M = \mathbb{Z}_{(p^\infty)}$. Then M is a comultiplication $\mathbb{Z}$-module [3, 3.2].

Proof. Fix a prime integer p. Define a set

$$\mathbb{Q}_p = \{\frac{r}{p^t} : r,\ t \in \mathbb{Z}\}.$$

Then $\mathbb{Q}_p$ is additive abelian group containing $\mathbb{Z}$. Define a set $M = \mathbb{Z}_{(p^\infty)} = \frac{\mathbb{Q}_p}{\mathbb{Z}}$. Then M is a $\mathbb{Z}$-module. Let N be any submodule of M. Then $N = \mathbb{Z}(1/p^i + \mathbb{Z})$ for some integer i. Set $I = p^i\mathbb{Z}$. Now, $ann_M(p^i\mathbb{Z}) = N$. Hence M is a comultiplication module. □

Now we have an example which shows that not every R-module is comultiplication module.

Example 2.12. [3, 3.9] Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Now, $2\mathbb{Z}$ is a submodule of $\mathbb{Z}$, we have

$$ann_{\mathbb{Z}}(2\mathbb{Z}) = \{m \in \mathbb{Z} : 2\mathbb{Z}m = 0\} = (0).$$

Now, if $\mathbb{Z}$ is a comultiplication module then by [3], we have

$$2\mathbb{Z} = (0 :_{\mathbb{Z}} ann_{\mathbb{Z}}(2\mathbb{Z})).$$

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But

$$(0 :_{\mathbb{Z}} ann_{\mathbb{Z}}(2\mathbb{Z})) = (0 :_{\mathbb{Z}} 0) = \mathbb{Z} \neq 2\mathbb{Z}.$$

Therefore, $\mathbb{Z}$ is not a comultiplication module.

Definition 2.13. A non empty family of submodules $\{N_i\}_{i\in I}$ of an R-module M is called coindependent provided for every nonempty finite subset J of I and element $i \in I \setminus J$, $N_i + \cap_{j\in J} N_j = M$ [10].

Definition 2.14. An R module M has finite dual Goldie dimension provided M does not contain an infinite coindependent family of proper submodules. In this case the supremum of cardinalities of family of coindependent submodules is called dual Goldie dimension [10] .

Theorem 2.15. *A ring R is semi-local if and only if the R- module R has finite dual Goldie dimension.*

Proof. First, suppose that R is a semi local ring. If possible, suppose that $\{I_i\}_{i\in\Delta}$ for all i be an infinite family of coindependent ideals of R. Then, $I_i \subseteq \mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$ of R. Since the family $\{I_i\}_{i\in\Delta}$ is coindependent, $I_i + I_j = R$ for all $i \neq j$ and hence any maximal ideal can contain I_j for atmost one $j \in \Delta$. But this contradict that R is semi-local. Therefore, R as an R-module has finite coindependent family of submodules and hence has finite dual Goldie dimension.

Conversely, suppose that R-module R has finite dual Goldie dimension. If possible, suppose that R is not semi-local. But then the family of all maximal ideals in R will form an infinite family of coindependent ideals of R, which contradicts that R has finite dual Goldie dimension. Therefore, R is semi-local.

□

Remark 2.16. [10] The dual Goldie dimension of a semi-local ring R is the number of distinct maximal ideals of R. If possible, suppose that R has $m + 1$ dual Goldie dimension, where m is the number of distinct maximal ideals $\mathfrak{m}_i$ $(1 \leq i \leq m)$ of R. Then there exists a coindependent family $\{I_i\}_{i=1}^{m+1}$ of ideals of R such that every maximal ideal can contain I_j for atmost one j. But this contradicts that R has only m maximal ideals.

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