SOME MORE REMARKS ON COMULTIPLICATION MODULES

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Abstract. In this paper we will discuss some more results of comultiplication modules.

1. Introduction

Throughout of this paper, $R$ denotes the commutative ring with unity.

The concept of comultiplication modules was first introduced by H. Ansari-Toroghy and F. Farshadifar [3] in 2007. An $R$-module $M$ is called comultiplication module if for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = \text{ann}_M(I)$. An $R$-module $M$ is comultiplication module if and only if for any submodule $N$ of $M$, $N = (0 :_M \text{ann}_R(N))$ [3]. Also, every proper submodule of a comultiplication module is comultiplication module [3]. However, converse may not be true. For example, if $V$ is a two dimensional vector space over a field $k$ then $V$ cannot be comultiplication module but every proper subspace of $V$ is comultiplication as every one dimensional vector space is comultiplication module. If $R$ is local ring then every comultiplication module is cocyclic. Also a finitely generated second submodule of a comultiplication module is multiplication module [6]. It was also shown that every non zero comultiplication module contains a minimal submodule [6].

In 2009, Reza Ebrahimi Atani and Shahabaddin Ebrahimi Atani [14] described the indecomposable comultiplication modules and non-separated indecomposable comultiplication module over pullback of local Dedekind domains.

Proposition 2.4. $M$ is a self-cogenerator if and only if for every submodule $N$ of $M$.

In 2011, Yousef Al-Shaniafi and Patrick F. Smith [9] studied the localization of comultiplication modules over a general ring $R$. If every maximal ideal $m$ of $R$ is good for $M$ then $M$ is comultiplication $R$-module if and only if $M_m$ is a comultiplication $R_m$-module for every maximal ideal $m$ of $R$ [9]. Every comultiplication module have unique complements [10]. Also every comultiplication module have unique Goldie dimensions [10].

2. Main results

We begin with the definition of Multiplication and Comultiplication modules.

Definition 2.1. An $R$-module $M$ is called multiplication module if for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$ [9].

Definition 2.2. An $R$-module $M$ is called comultiplication module if for any submodule $N$ of $M$, $N = (0 :_M \text{ann}_R(N))$ [3]. Also, every proper submodule of a comultiplication module is comultiplication module [3]. However, converse may not be true. For example, if $V$ is a two dimensional vector space over a field $k$ then $V$ cannot be comultiplication module but every proper subspace of $V$ is comultiplication as every one dimensional vector space is comultiplication module. If $R$ is local ring then every comultiplication module is cocyclic. Also a finitely generated second submodule of a comultiplication module is multiplication module [6]. It was also shown that every non zero comultiplication module contains a minimal submodule [6].

In 2009, Reza Ebrahimi Atani and Shahabaddin Ebrahimi Atani [14] described the indecomposable comultiplication modules and non-separated indecomposable comultiplication module over pullback of local Dedekind domains.

Proposition 2.4. $M$ is a self-cogenerator if and only if for every submodule $N$ of $M$, 

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there exists an index set $I$ and an endomorphisms $\phi_i$ for all $i \in I$ on $M$ such that $N = \bigcap_{i \in I} \ker \phi_i$.

Proof. Let $M$ be a self-cogenerated module and let $N$ be any submodule of $M$. Then by definition, there exists a one-one map $\phi : M/N \to M'$ for some index set $I$. Consider the map $\phi_i : M \xrightarrow{\pi} M/N \xrightarrow{\phi} M'$, that is $\phi_i = \pi_i \phi \pi : M \to M$. Note that $x \in \bigcap_{i \in I} \ker \phi_i$ if and only if $\pi_i \phi(x) = 0$ for all $i$, that is, $\pi_i \phi(x + N) = (0)$ for all $i$, if and only if $\phi(x + N) = (0)$, that is, $x \in N$. Therefore,

$$\bigcap_{i \in I} \ker \phi_i = N.$$

Conversely, suppose that for every submodule $N$ of $M$, there exists endomorphisms $\phi_i : M \to M$, $i \in I$, for some index set $I$, such that

$$\bigcap_{i \in I} \ker \phi_i = N.$$

Define $\phi : M \to M'$ as $\phi(x) = (\phi_i(x))_{i \in I}$. Obviously,

$$\ker(\phi) = \bigcap_{i \in I} \ker (\phi_i) = N.$$

Therefore, $\phi$ induces a one-one map $\phi'$ from $M/N$ to $M'$ and hence $M$ is a self-cogenerated module. \hfill \square

**Definition 2.5.** Let $M$ be an $R$-module and let $p$ be a prime ideal of $R$. Define a set $S_p = \{r \in R : \frac{r}{1} \equiv 0 \text{ in } R_p\}$. Note that, this is an ideal of $R$. Define a set $T_p = \{x \in M : \frac{x}{1} \equiv 0 \text{ in } M_p\}$. Note that, this is a submodule of $M$.

A prime ideal $p$ is called good for $M$ if there exists some $a \in R \setminus p$ such that $aT_p = (0)$. For example, if $T_p$ is finitely generated for any prime ideal $p$ and for any $R$-module $M$ then $p$ is good for $M$ [9].

In the view of above terminology, the following theorem characterize the comultiplication module.

**Theorem 2.6.** Let $R$ be any ring, and let $M$ be an $R$-module such that every maximal ideal of $R$ is good for $M$. Then $M$ is a comultiplication $R$-module if and only if $M_m$ is a comultiplication $R_m$-module for every maximal ideal $m$ of $R$.

Proof. Suppose that $M$ is a comultiplication $R$-module. Let $m$ be any maximal ideal of $R$. Let $N$ be any submodule of $M_m$. Define $N' = \{x \in M : x/1 \in N\}$. Obviously, $N'$ is a submodule of $M$. Let $ann_{R_m}(N) \subseteq ann_{R_m}(y)$ for some $y = y_1/s \in M_m$.

Let $r \in ann_{R}(N')$. This implies that $r/1 \in ann_{R_m}(N) \subseteq ann_{R_m}(y)$. Hence, $(r/1)(y_1/s) = 0$ implies that $tr/y_1 = 0$ for some $t \in R \setminus m$. Therefore, $ty_1/1 \in M_m$. As $m$ is good for $M$, there exists some $s \in R \setminus m$ such that $dt/1 = 0$. This implies that $r \in ann_{R}(dy_1)$ implies $ann_{R}(N') \subseteq ann_{R_m}(dy_1)$ and hence, $dy_1 \in N'$. This implies that $dy_1/1 \in N$ and hence, $y = y_1/s = dy_1/1 \in N$. Therefore, $M_m$ is a comultiplication module.

Conversely, suppose that $M_m$ is comultiplication as an $R_m$-module for all maximal ideal $m$ of $R$. Let $L$ be a submodule of $M$ and $x \in M$ such that $ann_{R_m}(L) \subseteq ann_{R_m}(Rx)$. By Theorem [9], $(L :_Rx)$ is not a maximal ideal of $R$. Suppose that $(L :_Rx)$ is proper ideal of $R$. Then there exists a maximal ideal $m$ of $R$ such that $(L :_Rx) \subseteq m$. Then $M_m$ is a submodule of $M_m$. Let $\alpha/s \in ann_{R_m}(L_m)$ for some $\alpha \in L$ and $s \in R \setminus m$. Now, for any $y \in L$, $y/1 \in M_m$. This implies that $(\alpha/s)(y_1/1) = 0$, that is, $t\alpha y = 0$ for some $t \in R \setminus m$. Therefore, $\alpha y \in T_m$. Now, $m$ is good for $M$, implies that $\alpha \alpha y = 0$. Since $y$ is arbitrary element in $L$, we conclude that $\alpha \alpha L = (0)$, that is, $\alpha \alpha \in ann_{R_m}(L) \subseteq ann_{R_m}(Rx)$. Hence, $\alpha \alpha x = 0$. Therefore, $\alpha/s \equiv \alpha/\alpha \in ann_{R_m}(x/1)$. Hence, we have $ann_{R_m}(L_m) \subseteq ann_{R_m}(x/1)$. Therefore, $x/1 \in L_m$. This implies that $x \in L$. Therefore, we conclude that $M$ is a comultiplication module. \hfill \square
We know that every submodule of $M$ is a comultiplication module [9]. The converse of above lemma need not be true. Consider $V$ be a two dimensional vector space over a field $F$. Then obviously, it is a module over $F$. But it is not a comultiplication module, as for any subspace $V_1$ of dimension one is proper submodule of $V$ and neither $V_1 = ann_V(0)$ nor $V_1 = ann_V(F)$. However $V_1$ is a comultiplication module, as $V_1 = ann_{V_1}(0)$ and $(0) = ann_{V_1}(0)$.

**Remark 2.7.** In general an $R$-module $M$ may not contains a simple submodule. Consider $Z$ as a $Z$-module. Suppose $N$ is a simple submodule of $Z$. Then $N = Zx$ for any non-zero $x \in Z$. Hence $Zx \cong \frac{Z}{ann_Z(x)} \cong Z$, which is a contradiction. Thus $Z$ does not have any simple submodule. However, if $M$ is comultiplication module then every non-zero submodule of $M$ contains a simple submodule [6, 3.2].

If $M$ comultiplication module then $\theta(N) \subseteq N$, for any submodule $N$ of $M$ and for any $R$-endomorphism $\theta$ on $M$ [3]. However, if the map $\theta$ is monomorphism, then $\theta(N) = N$.

**Definition 2.8.** A non-zero submodule $S$ of an $R$-module $M$ is called second submodule if for any $a \in R$, the map $\phi : S \rightarrow S$ given by $x \rightarrow ax$ is either surjective or zero [4].

**Proposition 2.9.** If $N$ is a second submodule of an $R$-module $M$ then $ann_R(N)$ is a prime ideal of $R$.

**Proof.** Let $N$ be a second submodule of an $R$-module $M$. If possible, suppose $abN = (0)$ and $aN \neq (0)$. Define the homomorphisms $f_a : N \rightarrow N$ and $g_b : N \rightarrow N$ respectively by $f_a(x) = ax$, $g_b(x) = bx$ for all $x \in N$. Then $\text{Im}(f_a) = aN \neq (0)$ and $\text{Im}(g_b) = bN \neq (0)$. Since $N$ is a second submodule, we conclude that $aN = N$ and $bN = N$. This implies that $abN = N = (0)$, which contradicts that $N$ is a second submodule. Therefore, $ann_R(N)$ is a prime ideal.

**Example 2.10.** Let $n$ be a fixed positive integer. Then

1. $\mathbb{Z}_n$ is a comultiplication $\mathbb{Z}$-module.
2. $\mathbb{Z}_n$ is a comultiplication $\mathbb{Z}_n$-module

**Proof.** We prove only (1). The proof of (2) is same as that of (1).

Let $N$ be a submodule of $\mathbb{Z}_n$. Let $o(N) = d$. Then $n = md$ for some positive integer $m$. This implies that $N = m\mathbb{Z}_n$. Put $I = d\mathbb{Z}$. Then $d\mathbb{Z}$ is an ideal in $\mathbb{Z}$ such that $N = ann_{\mathbb{Z}_n}(d\mathbb{Z})$. Therefore, $\mathbb{Z}_n$ is a comultiplication $\mathbb{Z}$-module.

**Example 2.11.** Let $p$ be any prime number. Let $M = \mathbb{Z}_{(p^{\infty})}$. Then $M$ is a comultiplication $\mathbb{Z}$-module [3, 3.2].

**Proof.** Fix a prime integer $p$. Define a set $Q_p = \{ \frac{r}{p^t} : r, t \in \mathbb{Z} \}$.

Then $Q_p$ is additive abelian group containing $\mathbb{Z}$. Define a set $M = \mathbb{Z}_{(p^{\infty})} = \frac{Q_p}{p\mathbb{Z}}$. Then $M$ is a $\mathbb{Z}$-module. Let $N$ be any submodule of $M$. Then $N = \mathbb{Z}(1/p^t + Z)$ for some integer $i$. Set $I = p^i\mathbb{Z}$. Now, $ann_M(p^i\mathbb{Z}) = N$. Hence $M$ is a comultiplication module. □

Now we have an example which shows that not every $R$-module is comultiplication module.

**Example 2.12.** [3, 3.9] Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Now, $2\mathbb{Z}$ is a submodule of $\mathbb{Z}$, we have

$$ann_\mathbb{Z}(2\mathbb{Z}) = \{ m \in \mathbb{Z} : 2Zm = 0 \} = (0).$$

Now, if $Z$ is a comultiplication module then by [3], we have

$$2\mathbb{Z} = (0 \circ_\mathbb{Z} ann_\mathbb{Z}(2\mathbb{Z})).$$
But
\[(0 : \mathbb{Z} \text{ann}_\mathbb{Z}(2\mathbb{Z})) = (0 : \mathbb{Z} 0) = \mathbb{Z} \neq 2\mathbb{Z}.
\]
Therefore, \(\mathbb{Z}\) is not a comultiplication module.

**Definition 2.13.** A non empty family of submodules \(\{N_i\}_{i \in I}\) of an \(R\)-module \(M\) is called coindependent provided for every nonempty finite subset \(J\) of \(I\) and element \(i \in I \setminus J\), \(N_i + \bigcap_{j \in J} N_j = M\) [10].

**Definition 2.14.** An \(R\) module \(M\) has finite dual Goldie dimension provided \(M\) does not contain an infinite coindependent family of proper submodules. In this case the supremum of cardinalities of family of coindependent submodules is called dual Goldie dimension [10].

**Theorem 2.15.** A ring \(R\) is semi-local if and only if the \(R\)-module \(R\) has finite dual Goldie dimension.

**Proof.** First, suppose that \(R\) is a semi local ring. If possible, suppose that \(\{I_i\}_{i \in \Delta}\) for all \(i\) be an infinite family of coindependent ideals of \(R\). Then, \(I_i \subseteq m_i\) for some maximal ideals \(m_i\) of \(R\). Since the family \(\{I_i\}_{i \in \Delta}\) is coindependent, \(I_i + I_j = R\) for all \(i \neq j\) and hence any maximal ideal can contain \(I_j\) for atmost one \(j \in \Delta\). But this contradicts that \(R\) is semi-local. Therefore, \(R\) as an \(R\)-module has finite coindependent family of submodules and hence has finite dual Goldie dimension.

Conversely, suppose that \(R\)-module \(R\) has finite dual Goldie dimension. If possible, suppose that \(R\) is not semi-local. But then the family of all maximal ideals in \(R\) will form an infinite family of coindependent ideals of \(R\), which contradicts that \(R\) has finite dual Goldie dimension. Therefore, \(R\) is semi-local.

\[\square\]

**Remark 2.16.** [10] The dual Goldie dimension of a semi-local ring \(R\) is the number of distinct maximal ideals of \(R\). If possible, suppose that \(R\) has \(m + 1\) dual Goldie dimension, where \(m\) is the number of distinct maximal ideals \(m_i\) \((1 \leq i \leq m)\) of \(R\). Then there exists a coindependent family \(\{I_i\}_{i=1}^{m+1}\) of ideals of \(R\) such that every maximal ideal can contain \(I_j\) for atmost one \(j\). But this contradicts that \(R\) has only \(m\) maximal ideals.

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