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Shape Control of the Rational Quadratic Trigonometric Bézier Curve with two Shape Parameters

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Abstract: ***A rational quadratic trigonometric Bézier curve with two shape parameters, which is analogous to cubic Bézier curve, is presented in this paper. The shape of the curve can be adjusted as desired, by simply altering the value of shape parameters, without changing the control polygon. The rational quadratic trigonometric Bézier curve can be made close to the rational cubic Bézier curve or closer to the given control polygon than the rational cubic Bézier curve. The ellipses and circles can be representation exactly by using rational quadratic trigonometric Bézier curve.***

Keywords: Rational Trigonometric Bézier Basis Function, Rational Trigonometric Bézier Curve, Shape Parameter, Open curves, Close curves.

1. Introduction

Computer Aided Geometric Design (CAGD) plays a very important role in the field of engineering and technology. It is most widely used in the designing purpose engineering graphics. Bézier representation of curves and surfaces is the one of the extensively used in CAGD. It is most widely used in the field of engineering and technology such as in Robotics, Auto CAD, design of Highway and Railway, Animation, surface models, Image compression/ font designing.

Rational parametric curves are the one of the most flexible modeling tool as compared to the non-rational counterpart. Rational curves provide the more flexibility in the design of the curve than the polynomial curves. We can also make curve with different shape by selected different weight values instead of control points. However, because of fractional expression, it is difficult to calculate the derivatives and integrals of the rational curves. So, it is not easy to choose the more appropriate weight values to obtain a curve with desired shapes. By using basis function including independent parameter i.e shape parameter, we can also construct shape adjustable curves.

In recent years, trigonometric splines and polynomials play a very important role in the Computer Aided Geometric Design (CAGD), especially in curve design: see [2], [3], [4], [5], [6], [10], [11], [12], [13].The theory of the Bézier curves is key feature in CAGD. These are considered as ideal geometric standard for the representation of piecewise polynomial curves. In recent years, trigonometric polynomial curves like Bézier type are taken in discussion. A cubic trigonometric Bézier curve with two shape parameters was discussed by Han et al [9].It enjoyed all the geometric properties of the ordinary cubic Bézier curve and was used for spur gear tooth design with S-shaped transition curve Abbas et al [1].A study on class of TC-Bézier curve with shape parameters was presented by Liu, et al [11].

The paper is organized as follows. In section 2, the basis functions of the quadratic trigonometric Bézier curve with two shape parameters are established and the properties of the basis function have been described. In section 3, rational quadratic trigonometric Bézier curves and their properties are discussed. In section 4, By using shape parameter, shape control of the curves is studied and explained by using figures. In section 5, the representation of ellipse and circle are given. In section 6, the approximation of the rational quadratic trigonometric Bézier curve to the ordinary rational Cubic Bézier curve is presented. In section 7, conclusion of the paper.

2. Quadratic Trigonometric Bézier Basis Functions

In this section, definition and some properties of quadratic trigonometric Bézier basis functions with two shape parameters are given as follows:

Definition 2.1: For two arbitrarily real value of λ and μ where the following four functions of $t\,(t \in [0.1])$ are defined as quadratic trigonometric Bézier basis functions with two shape parameter λ, μ:

$$
\begin{aligned}
b_0(t) &= \left(1 - sin\frac{\pi}{2}t\right)\left[1 + (1-\lambda)sin\frac{\pi}{2}t\right] \\
b_1(t) &= \lambda\, sin\frac{\pi}{2}t\left(1 - sin\frac{\pi}{2}t\right) \\
b_2(t) &= \mu\, cos\frac{\pi}{2}t\left(1 - cos\frac{\pi}{2}t\right) \\
b_3(t) &= \left(1 - cos\frac{\pi}{2}t\right)\left[1 + (1-\mu)cos\frac{\pi}{2}t\right] \quad (2.1)
\end{aligned}
$$

For $\lambda = \mu = 0$, the basis functions are linear trigonometric polynomials. For $\lambda, \mu \neq 0$, the basis functions are quadratic trigonometric polynomials.

Theorem 2.1: The basis functions (2.1) have the following properties:

(a) Nonnegativity: $b_i(t) \geq 0, i = 0,1,2,3.$
(b) Partition of unity: $\sum_{i=0}^{3} b_i(t) = 1.$
(c) Monotonicity: For a given parameter t, $b_0(t)$ and $b_3(t)$ are monotonically decreasing λ *and* μ respectively; $b_1(t)$ and $b_2(t)$ are monotonically λ *and* μ respectively.
(d) Symmetry: $b_i(t;\lambda,\mu) = b_{3-i}(1-t;\lambda,\mu)$, for $i = 0,1,2,3.$

Proof: (a) For $t \in [0,1]$ and $\lambda,\mu \in [0,2]$, then

$$\left(1-sin\frac{\pi}{2}t\right) \geq 0, (1-\lambda)sin\frac{\pi}{2}t \geq 0,$$
$$sin\frac{\pi}{2}t \geq 0, \left(1-cos\frac{\pi}{2}t\right) \geq 0, (1-\mu)cos\frac{\pi}{2}t \geq 0,$$
$$cos\frac{\pi}{2}t \geq 0, \lambda \geq 0, \mu \geq 0.$$

It is obvious that
$b_i(t) \geq 0, i = 0,1,2,3.$

(b) $\sum_{i=0}^{3} b_i(t) = \left(1-sin\frac{\pi}{2}t\right)\left[1+(1-\lambda)sin\frac{\pi}{2}t\right] + \lambda sin\frac{\pi}{2}t\left(1-sin\frac{\pi}{2}t\right) + \mu cos\frac{\pi}{2}t\left(1-cos\frac{\pi}{2}t\right)\left[1+(1-\mu)cos\frac{\pi}{2}t\right] = 1.$

The remaining cases following obviously.

Fig.1. Shows the curves of the rational quadratic trigonometric basis function for $\lambda = \mu = 2$ (red solid), $\lambda = \mu = 1$ (blue dashed), $\lambda = \mu = 0$ (green solid).

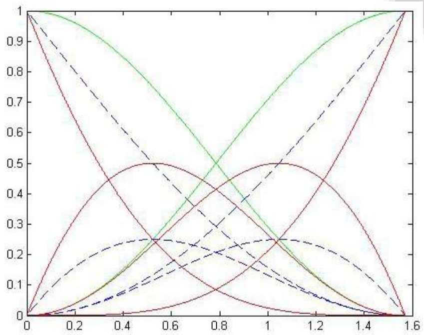

Figure 1: The quadratic trigonometric basis function

3. Rational Quadratic trigonometric Bézier curve

We construct the rational quadratic trigonometric Bézier curve with two shape parameters as follows:

Definition 3.1: The Rational Quadratic Trigonometric Bézier curve with two shape parameters is defined as:

$$C(t) = \frac{\sum_{i=0}^{3} w_i P_i b_i(t)}{\sum_{i=0}^{3} w_i b_i(t)} \quad (3.1)$$

$t \in [0,1], \lambda,\mu \in [0,2]$, where $P_i(i = 0,1,2,3)$ in R^2 or R^3 the control points $P_i(i = 0,1,2,3)$ are the basis functions defined in (2.1) and w_i is scalar, called the weight of function. We assume that $w_i \geq 0$. If $w_i = 1$, we get non-rational trigonometric Bézier cure, since the denominator is identically equal to one.

The curve defined by (3.1) possesses some properties which can obtain easily from the properties of the basis function.

Theorem 3.1: The Rational Quadratic trigonometric Bézier curves (3.1) have the following properties:
(a) End point properties:
$C(0) = P_0, C(1) = P_3$

$$C'(0) = \frac{w_1\lambda}{w_0}[P_1 - P_0],$$
$$C'(1) = \frac{w_2\mu}{w_3}[P_3 - P_2]$$

If $w_0 = w_3 = 1$ and $w_1 = w_2 = 2$, then
$C'(0) = 2\lambda[P_1 - P_0]$
$C'(1) = 2\mu[P_3 - P_2]$
(b) Symmetry: P_0, P_1, P_2, P_3 and P_3, P_2, P_1, P_0 define the same curve in different parametrizations, that is
$C(t;\lambda,\mu;\ P_0, P_1, P_2, P_3) =$
$C(1-t;\mu,\lambda; P_3, P_2, P_1, P_0),$
$t \in [0,1], \lambda,\mu \in [0,2]$
(c) Geometric invariance: The shape of the curve (3.1) is independent of the choice of coordinates, i.e., it satisfies the following two equations:
$C(t;\lambda,\mu;\ P_0+q, P_1+q, P_2+q, P_3+q) =$

$C(t;\lambda,\mu;\ P_0, P_1, P_2, P_3) + T$

$C(t;\lambda,\mu;\ P_0 * T, P_1 * T, P_2 * T, P_3 * T) =$

$C(t;\lambda,\mu;\ P_0, P_1, P_2, P_3) * T$

where q is an arbitrary vector in R^2 or R^3 and T is an arbitrary $d \times d$ matrix, $d = 2$ *or* 3.

(d) Convex hull property: From the non-negativity and partition of unity of basis functions, it follows that the whole curve is located in the convex hull generated by its control points.

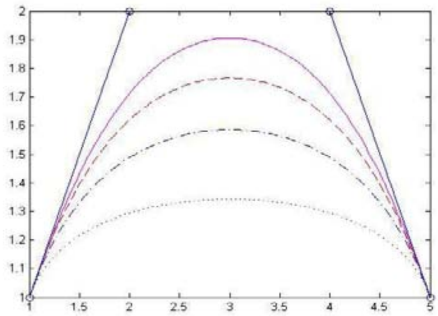

Figure 2: The effect on the shape of rational quadratic trigonometric Bézier curves with altering the values of λ *and* μ simultaneously

4. Shape control of the rational quadratic Trigonometric Bézier curve

The parameters λ and μ controls the shape of the curve (3.1). In figure 2, The rational quadratic trigonometric Bézier curve *C (t)* gets closer to the control polygon as the values of the parameters λ and μ increases with $w_0 = w_3 = 1$ and $w_1 = w_2 = 2$. In figure 2, the curves are generated by setting the values of λ,μ as $\lambda = \mu = 0.5$ (black dotted lines), $\lambda = \mu = 1$ (blue dashed-dotted lines), $\lambda = \mu = 1.5$ (red dashed lines), $\lambda = \mu = 2$ (pink solid lines). In figure 3 (a), the curves

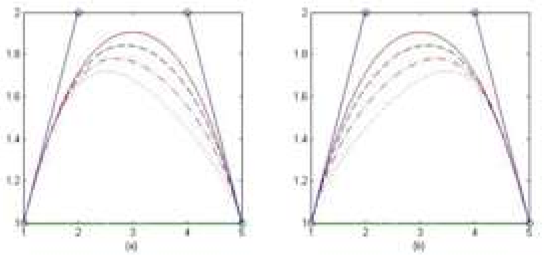

Figure 3: The effect on the shape of rational quadratic trigonometric Bézier curves with altering the values of λ *and* μ.

are generated by setting the values of μ as $\mu = 0$ (green solid lines), $\mu = 1$ (blue dashed-dotted lines), $\mu = 1.5$ (red dashed lines), $\mu = 2$ (pink solid lines) and setting $\lambda = 2$, with $w_0 = w_3 = 1, w_1 = w_2 = 2$. In figure 3(b) changing λ to $\lambda = 0$ (green solid lines), $\lambda = 0.5$ (black dotted lines), $\lambda = 1$ (blue dashed-dotted lines), $\lambda = 1$ (blue dashed-dotted lines), $\lambda = 1.5$ (red dashed lines), $\lambda = 2$ (pink solid lines) and setting $\mu = 2$.

In order to construct a closed rational quadratic trigonometric Bézier curves, we can set $P_n = P_0$ and $w_0 = w_3 = 1, w_1 = w_2 = 2$. In figure 4 and 5, The closed rational quadratic trigonometric Bézier curves of altering the values of the shape parameters λ and μ at the same time. The rational quadratic trigonometric Bézier curves are generated by setting $\lambda = 0.5, \mu = 0.5$ (red solid lines), $\lambda = 1, \mu = 1$ (blue solid lines), $\lambda = 1.5, \mu = 1.5$ (green solid lines) and $\lambda = 2, \mu = 2$ (pink solid lines).

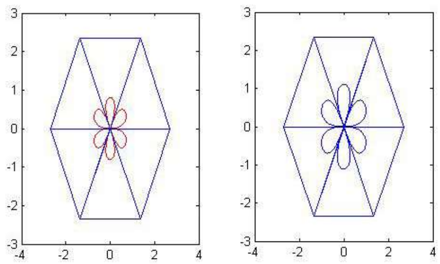

Figure 4: The close rational quadratic trigonometric Bézier curves with different values of shape parameter λ and μ (flower)

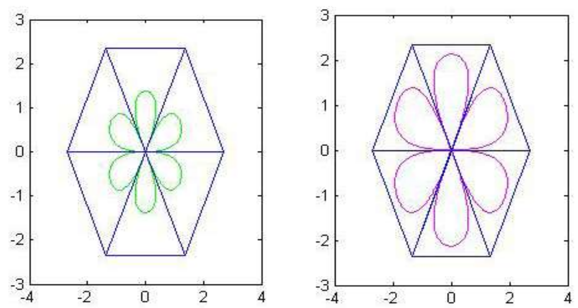

Figure 5: The close rational quadratic trigonometric Bézier curves with different values of shape parameter λ and μ (flower)

5. The representation of Ellipse

Theorem 5.1: Let P_0, P_1, P_2 and P_3 be four control points on an ellipse with semi axes $(2\sqrt{2})a$ and $(4\sqrt{2})b$, by the proper selection of coordinates, their coordinates can be written in the form

$$P_0 = \binom{2a}{0}, P_1 = \binom{a}{2b},$$
$$P_2 = \binom{-a}{2b}, P_3 = \binom{-2a}{0}$$

Then the corresponding rational quadratic trigonometric Bézier curve with the shape parameters $\lambda = \mu = 2$ with $w_0 = w_1 = w_2 = w_3 = \frac{1}{2}$ and local domain $t \in [0,4]$ represents arc of an ellipse with

$$\begin{cases} x(t) = 2a\left(cos\frac{\pi}{2}(t) - sin\frac{\pi}{2}(t)\right) \\ y(t) = 4b\left(cos\frac{\pi}{2}(t) + sin\frac{\pi}{2}(t) - 1\right) \end{cases} \quad (5.1)$$

Proof: If we take $\lambda = \mu = 2$ and $w_0 = w_1 = w_2 = w_3 = \frac{1}{2}$

$$P_0 = \binom{2a}{0}, P_1 = \binom{a}{2b},$$
$$P_2 = \binom{-a}{2b}, P_3 = \binom{-2a}{0}$$

into (3.1), then the coordinates of rational quadratic trigonometric Bézier curve are

$$\begin{cases} x(t) = 2a\left(cos\frac{\pi}{2}(t) - sin\frac{\pi}{2}(t)\right) \\ y(t) = 4b\left(cos\frac{\pi}{2}(t) + sin\frac{\pi}{2}(t) - 1\right) \end{cases}$$

This gives the intrinsic equation

$$\left(\frac{x(t)}{(2\sqrt{2})a}\right)^2 + \left(\frac{y(t) + 4b}{(4\sqrt{2})b}\right)^2 = 1.$$

It is an equation of an ellipse.
Fig.6 shows the Ellipse.

Corollary 5.2: According to theorem (5.1), if, $a = \frac{1}{2\sqrt{2}}, b = \frac{1}{4\sqrt{2}}$ then the corresponding Rational Quadratic trigonometric Bézier curve with the shape parameter $\lambda = \mu = 2$ and local domain $t \in [0,4]$ represents arc of an circle. Fig. 7 shows the Circle.

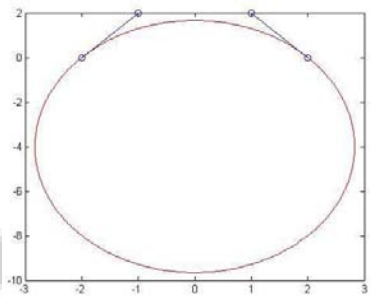

Figure 6: The representation of ellipse

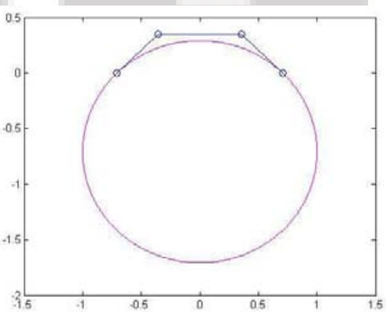

Figure 7: The representation of ellipse

6. Approximability

Control polygons play an important role in geometric modeling. It is an advantage if the curve begins modeled tends to preserve the shape of its control polygon. Now we show the relation of the rational quadratic trigonometric Bézier curves and rational cubic Bézier curves with same control points.

Theorem 6.1: Suppose P_0, P_1, P_2 and P_3 are not collinear; the relationship between rational quadratic trigonometric Bézier curve C(t) (3.1) and rational cubic Bézier curve

$$B(t) = \frac{\sum_{i=0}^{3} P_i B_i(t) w_i}{\sum_{i=0}^{3} B_i(t) w_i} \quad (6.1)$$

where $B_i(t) = \sum_{i=0}^{3} P_i \binom{3}{i} (1-t)^{3-i} t^i, (i = 0,1,2,3)$ are Bernstein polynomials and the scalar w_i are the weights, with the same control points $P_i (i = 0,1,2,3)$ are as follows:

$C(0) = B(0), C(1) = B(1)$

$$C\left(\frac{1}{2}\right) - P^* = \frac{7[\sqrt{2}+(1-\lambda)]}{[\sqrt{2}+(1+\lambda)]}\left(B\left(\frac{1}{2}\right) - P^*\right) \quad (6.2)$$

where $P^* = \frac{(P_1+P_2)}{2}$ with the assumption that $\lambda = \mu$.

Proof: We assume that $w_0 = w_3 = 1$ and $w_1 = w_2 = 2$ then the ordinary rational cubic Bézier curve (6.1) takes the form

$$B(t) = \frac{(1-t)^3 P_0 + 6(1-t)^2 t\, P_1 + 6(1-t)t^2 P_2 + t^3 P_3}{(1-t)^3 + 6(1-t)^2 t + 6(1-t)t^2 + t^3}$$

By simple computation, then

$C(0) = P_0 = B(0), C(1) = P_3 = B(1) and$

$$B\left(\frac{1}{2}\right) = \frac{1}{14}(P_0 + 6P_1 + 6P_2 + P_3)$$

$$B\left(\frac{1}{2}\right) = P^* = \frac{1}{14}(P_0 - P_1 - P_2 + P_3) \quad (6.3)$$

For $\lambda = \mu$ and $w_0 = w_3 = 1, w_1 = w_2 = 2$

$$C\left(\frac{1}{2}\right) = \frac{[\sqrt{2}+(1-\lambda)](P_0+P_3) + 2\lambda(P_1+P_2)}{2[\sqrt{2}+(1+\lambda)]}$$

$$C\left(\frac{1}{2}\right) - P^* = \frac{[\sqrt{2}+(1-\lambda)]}{2[\sqrt{2}+(1+\lambda)]}(P_0 - P_1 - P_2 + P_3)$$

$$C\left(\frac{1}{2}\right) - P^* = \frac{7[\sqrt{2}+(1-\lambda)]}{[\sqrt{2}+(1+\lambda)]}\left(B\left(\frac{1}{2}\right) - P^*\right)$$

Then (6.2) holds.

Corollary 6.1: The rational quadratic trigonometric Bézier curve is closer to the control polygon that the rational cubic Bézier curve if and only if $\frac{3[\sqrt{2}+1]}{4} \le \lambda, \mu \le 2$.

Corollary 6.2: When $\lambda = \mu = \frac{3[\sqrt{2}+1]}{4}$, the rational quadratic trigonometric Bézier curve is closer to the rational cubic Bézier curve, i.e. $C\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right)$.

Fig. 8 shows the relationship between the rational quadratic trigonometric Bézier curve and rational cubic Bézier curve. The quadratic cubic Bézier curve. The quadratic trigonometric Bézier curve is closer to rational cubic Bézier curve. The rational quadratic trigonometric Bézier curve (blue dashed) with parameter $\lambda = \mu = \frac{3[\sqrt{2}+1]}{4}$ is analogous to ordinary cubic Bézier curve (green solid).

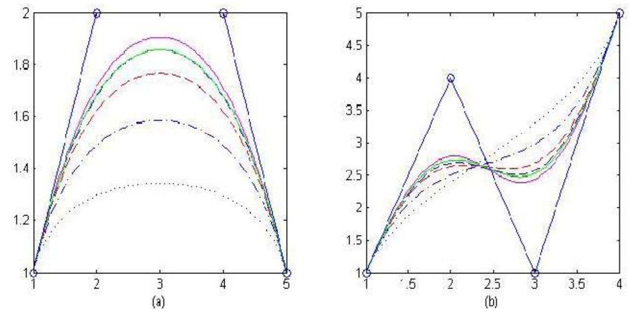

Figure 8: The relationship between the rational quadratic trigonometric Bézier curve and rational cubic Bézier curve.

7. Conclusion

In this paper, we have presented the rational quadratic trigonometric Bézier curve with two shape parameters. Each section of the curve only refers to the four control points. We can design different shape curves by changing parameters. The curve represent ellipse and circle when adjusting the control points and parameter value.

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