Soft Atoms and Soft Complements of Soft Lattices

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Abstract: Soft set theory was introduced by Molodtsov in 1999 as a mathematical tool for dealing with problems that contain uncertainty. Faruk Karaaslan et al.[6] defined the concept of soft lattices, modular soft lattices and distributive soft lattices over a collection of soft sets. In this paper, we define the concept of complemented soft lattices and complemented distributive soft lattices over a collection of soft sets, study their related properties and illustrate them with some examples. We also define the concept of soft Boolean algebras, soft atoms of soft lattices and discuss the theorems related to soft atoms. In addition, we establish representation theorem for finite soft Boolean algebras.

Key words: Soft atoms, soft complements, complemented soft lattices, soft Boolean algebras.

1. Introduction

Soft set theory was introduced by Molodtsov [9] in 1999 as a mathematical tool for dealing with uncertainty. Maji et al.[8] defined some operations on soft sets and proved related properties. Irfan Ali et al.[5] studied some new operations in soft set theory. Li [7], Nagarajan et al.[10] defined the soft lattices using soft sets. Faruk Karaaslan et al.[6] defined the concept of soft lattices over a collection of soft sets by using the operations of soft sets defined by Cagman et al.[1]. Nagarajan et al. [11] proved characterization theorems for modular and distributive soft lattices. Ridvan Sahin et al. [12] applied the notion of soft set theory of Molodtsov to the theory of Boolean algebras. In this paper, we define the concept of complemented soft lattices, complemented distributive soft lattices and Soft Boolean algebras over a collection of soft sets. We study their related properties with some examples. We also define the concept of soft Boolean algebras, soft atoms of soft lattices and discuss the theorems related to soft atoms. In addition, we establish representation theorem for finite soft Boolean algebras. The readers are asked to refer Cagman et al.[1], Maji et al.[8] and Molodtsov [9] for basic definitions and results of soft set theory. Faruk Karaaslan et al.[6], Nagarajan et al.[10] and Nagarajan et al. [11] for results of soft lattices. Throughout this work, \( U \) refers to the initial universe, \( \mathcal{P}(U) \) is the power set of \( U \), \( E \) is a set of parameters, \( A \subseteq E \) and \( S(U) \) is the set of all soft sets over \( U \).

2. Complemented Soft Lattice

In this section, we give the definition of complemented soft lattices and study their related properties with some examples.

Definition 2.1 Let \( (L, \lor, \land) \) be a soft lattice and \( f_A \in L \). If \( f_A \leq f_X \) for all \( f_X \in L \), then \( f_A \) is called the least element of \( L \). If \( f_X \leq f_A \) for all \( f_X \in L \), then \( f_A \) is called the greatest element of \( L \). The least the and greatest elements of a soft lattice are the empty soft set \( f_{\emptyset} \) and the universal soft set \( f_{E} \) respectively.

Definition 2.2 A soft lattice \( (L, \lor, \land) \) is said to be a bounded soft lattice if \( L \) has both the least element \( f_{\emptyset} \) and the greatest element \( f_{E} \).

Example 2.3 Let \( A = \{e_1\}, B = \{e_2\} \), where \( A, B \subseteq E \). Assume that \( f_{\emptyset} = \emptyset \), \( f_A = \{e_1, \{u_1, u_2\}\} \), \( f_B = \{e_2, \{u_3, u_4\}\} \), \( f_E = \{(e_1, \{u_1, u_2\}), (e_2, \{u_3, u_4\})\} \).

Then \( L = \{f_{\emptyset}, f_A, f_B, f_E\} \subseteq S(U) \) is a soft lattice with the operations \( \lor \) and \( \land \). In this soft lattice, the least element is \( f_{\emptyset} \) and the greatest element is \( f_E \). Thus \((L, \lor, \land)\) is a bounded soft lattice. The Hasse diagram of it is given in figure 1.
Example 2.5 Let 
\( U = \{u_1, u_2, u_3, u_4, u_5, u_6\}, E = \{e_1, e_2, e_3\} \), 
\( P = \{e_1\}, Q = \{e_2\}, R = \{e_3\}, S = \{e_1, e_2\} \), 
\( T = \{e_1, e_3\}, V = \{e_2, e_3\} \),
where \( P, Q, R, S, T, V \subseteq E \). Assume that \( f_\varnothing = \varnothing \), 
\( f_P = \{(e_1, \{u_1, u_2\}\} \), \( f_Q = \{(e_2, \{u_1, u_2\}\} \), 
\( f_R = \{(e_3, \{u_3, u_5, u_6\}\} \), 
\( f_S = \{(e_1, \{u_1, u_2\}, (e_2, \{u_3, u_4\}\} \), 
\( f_T = \{(e_1, \{u_1, u_2\}, (e_3, \{u_3, u_5, u_6\}\} \), 
\( f_V = \{(e_2, \{u_3, u_4\}, (e_3, \{u_3, u_5, u_6\}\} \), 
\( f_E = \{(e_1, \{u_1, u_2\}, (e_2, \{u_3, u_4\}, (e_3, \{u_5, u_6\}\}) \). Then 
\( L = \{f_\varnothing, f_P, f_Q, f_R, f_S, f_T, f_V, f_E\} \subseteq S(U) \) is a soft 
lattice with the operations \( \cup \) and \( \cap \). Here the soft 
complement of \( f_\varnothing \) is \( f_P \), the soft complement of \( f_P \) is 
\( f_Q \), and the soft complement of \( f_R \) is \( f_S \). Thus 
\( (L, \cup, \cap) \) is a complemented soft lattice. The Hasse 
diagram of it is given in figure 2.

![Figure 2]

Remark 2.6 If \( f_A \) is a soft complement of \( f_B \), then \( f_B \) is a soft complement of \( f_A \).

Remark 2.7 If \( L \) has \( f_\varnothing \) and \( f_E \), then 
\( f_\varnothing \cap f_E = f_\varnothing \) and \( f_\varnothing \cup f_E = f_E \) and so \( f_\varnothing \) is a soft 
complement of \( f_E \) and \( f_E \) is a soft complement of \( f_\varnothing \).

Lemma 2.8 If \( (L, \cup, \cap) \) is a finite soft chain with more 
than two elements then \( L \) is not soft complemented.
Proof. Let \( f_A \in L \) such that \( f_A \neq f_\varnothing \) and \( f_A \neq f_E \).
Suppose \( f_B \in L \). Since \( L \) is a chain, either \( f_A \leq f_B \) or 
\( f_B \leq f_A \). Case(i) Let \( f_A \leq f_B \). Then \( f_A \cap f_B = f_A \) and 
\( f_A \cup f_B = f_B \). Therefore \( f_B \) can not be a soft complement of 
\( f_A \). Case(ii) Let \( f_B \leq f_A \). Then \( f_A \cap f_B = f_B \) and 
\( f_A \cup f_B = f_A \). Therefore \( f_B \) can not be a soft complement of 
\( f_A \). Thus no element \( f_B \in L \) be a soft complement of 
\( f_A \). Hence \( L \) in not a complemented soft lattice.

Theorem 2.9 If \( (L, \cup, \cap) \) is a complemented 
distributive soft lattice, then every element of \( L \) has a 
unique soft complement.
Proof. Let us suppose that \( f_A \in L \) has two soft 
complements \( f_{B_1} \) and \( f_{B_2} \). Then by the definition of the 
soft complement 
and \( f_A \land f_{B_2} = f_\varnothing \), \( f_A \lor f_{B_2} = f_E \), 
\( f_{B_1} = f_{B_1} \land f_E = f_{B_1} \land (f_A \lor f_{B_2}) \) 
\( = (f_{B_1} \land f_A) \lor (f_{B_1} \land f_{B_2}) = f_\varnothing \lor (f_{B_1} \land f_{B_2}) \) 
Now \( = (f_{B_2} \lor f_{B_1}) \lor (f_{B_2} \lor f_{B_1}) = f_{B_2} \lor (f_{B_2} \lor f_{B_1}) \) 
\( = f_{B_2} \lor f_{B_2} = f_{B_2} \).

Hence \( f_{B_1} = f_{B_2} \) which contradicts the assumption that 
\( f_A \in L \) has two different soft complements \( f_{B_1} \) and \( f_{B_2} \). 
Thus the soft complement of any \( f_A \in L \) is unique.

Theorem 2.10 Let \( (L, \cup, \cap) \) be a complemented 
distributive soft lattice. For \( f_A, f_B \in L \), the following are 
equivalent. (i) \( f_A \leq f_B \), 
(ii) \( f_A \land f^c_B = f_\varnothing \), (iii) \( f_A \lor f_B = f_E \), 
(iv) \( f^c_B \leq f^c_A \).
Proof. (i) \( \Rightarrow \) (ii) : Let \( f_A \leq f_B \). 
\( f_A \lor f_B = f_B \) 
\( \Rightarrow (f_A \lor f_B) \land f^c_B = f_\varnothing \) 
\( \Rightarrow (f_A \land f^c_B) \lor (f_B \land f^c_B) = f_\varnothing \) 
\( \Rightarrow f_A \land f^c_B = f_\varnothing \) as \( f_B \land f^c_B = f_\varnothing \).

(ii) \( \Rightarrow \) (iii) : \( f_A \land f^c_B = f_\varnothing \) 
\( \Rightarrow (f_A \land f^c_B) \lor (f_B \land f^c_B) = f_\varnothing \) 
\( \Rightarrow f_A \land f^c_B = f_\varnothing \) as \( f_B \land f^c_B = f_\varnothing \).

(iii) \( \Rightarrow \) (iv) : \( f_A \land f^c_B = f_\varnothing \) 
\( \Rightarrow (f_A \land f^c_B) \lor (f_B \land f^c_B) = f^c_B \) 
\( \Rightarrow (f_A \land f^c_B) \lor f^c_B = f^c_B \) 
\( \Rightarrow f_B \leq f^c_A \).

(iv) \( \Rightarrow \) (i) : \( f_B \leq f_A \) 
\( \Rightarrow f_B \land f^c_A = f^c_B \) 
\( \Rightarrow (f_B \land f^c_A) \lor (f_B \land f^c_B) = f_B \) 
\( \Rightarrow f_A \land f_B = f_B \) 
\( \Rightarrow f_A \leq f_B \).
3. Soft Boolean Algebra

Soft Boolean Algebras are special type of soft lattices. In this section, we define soft Boolean algebras with some of its properties.

Definition 3.1 A complemented distributive soft lattice is said to be a soft Boolean algebra or a soft Boolean lattice with least element \( f_\emptyset \) and greatest element \( f_E \).

Example 3.2

Let \( U = \{u_1, u_2, u_3, u_4, u_5, u_6\} \), \( E = \{e_1, e_2, e_3\} \),

where \( A, B, C \subseteq E \). Assume that \( f_\emptyset = \emptyset \), \( f_E = \{e_1, e_2, e_3\} \), \( f_C = \{e_1, u_5, u_6\} \), \( f_B = \{e_2, \{u_1, u_2\}\} \), \( f_A = \{e_3, \{u_3, u_4\}\} \).

Then \( A = \{f_A, f_B, f_C\} \) and \( P(A) = \{f_A, f_B, f_C, \{f_A, f_B, f_C\}\} \subseteq S(U) \). Hence \( P(A) \subseteq S(U) \) is a soft lattice with the operations \( \cap \) and \( \sim \). The Hasse diagram of the soft lattice \( (P(A), \cap, \sim) \) is given in figure 3.

This soft lattice \( (P(A), \cap, \sim) \) is a complemented distributive soft lattice with least element \( f_\emptyset \) and a greatest element \( A \). Hence the soft lattice \( (P(A), \cap, \sim) \) is a soft Boolean algebra.

Remark 3.3 Every element \( f_A \) in a soft Boolean algebra has a unique soft complement. The unique soft complement of \( f_A \) is denoted by \( f_A^C \).

Theorem 3.4 Let \( (B, \lor, \land, \bar{\cdot}, f_\emptyset, f_E) \) be a soft Boolean algebra. Then for all \( f_A \in B \),

(i) \( f_A \lor f_A^C = f_E \) and \( f_A \land f_A^C = f_\emptyset \).

Proof. (i) Let \( f_A^C \) be the soft complement of \( f_A \). Then \( f_A \lor f_A^C = f_E \) and \( f_A \land f_A^C = f_\emptyset \). By commutativity, \( f_A \lor f_A^C = f_E \) and \( f_A \land f_A^C = f_\emptyset \). This implies that \( f_A \) is the soft complement of \( f_A^C \). That is, \( f_A \land f_A^C = f_\emptyset \).

(ii) We know that \( f_\emptyset \lor f_E = f_E \) and \( f_\emptyset \land f_E = f_\emptyset \). This implies that \( f_E \) is the soft complement of \( f_\emptyset \). That is, \( f_E \lor f_\emptyset = f_E \) and \( f_E \land f_\emptyset = f_\emptyset \). By the principle of duality, \( f_A \lor f_A^C = f_E \) and \( f_A \land f_A^C = f_\emptyset \).

Remark 3.5 A soft Boolean algebra will generally be denoted by \( (B, \lor, \land, \bar{\cdot}, f_\emptyset, f_E) \), where \( B \subseteq S(U) \). The two operations \( \lor \) (join) and \( \land \) (meet) are binary operations on \( B \) and soft complementation is a unary operations on \( B \). The corresponding soft poset will be denoted by \( (B, \leq) \).

The bounds of the soft lattice are \( f_\emptyset \) and \( f_E \) where \( f_\emptyset \) is the least element and \( f_E \) is the greatest element.

Note 3.6 As a soft Boolean algebra should contain \( f_\emptyset \) and \( f_E \), every soft Boolean algebra has at least two elements.

Theorem 3.7 De Morgan’s Law: Let \( (B, \lor, \land, \bar{\cdot}, f_\emptyset, f_E) \) be a soft Boolean algebra. Then for any \( f_A, f_B \in B \),

(i) \( f_A \lor f_B = f_A \lor f_B \)

(ii) \( f_A \land f_B = f_A \land f_B \)

Proof. Let \( (B, \lor, \land, \bar{\cdot}, f_\emptyset, f_E) \) be a soft Boolean algebra and \( f_A, f_B \in B \). Then

\[ (f_A \lor f_B) \lor (f_A \lor f_B) = f_A \lor f_B \]

\[ (f_A \lor f_B) \land (f_A \lor f_B) = f_A \lor f_B \]

Thus \( f_A \lor f_B \) is the soft complement of \( f_A \lor f_B \). That is \( (f_A \lor f_B) \lor f_A \lor f_B \).

By the principle of duality, we have \( f_A \lor f_B = f_A \lor f_B \).

4. Soft Atoms of Soft Lattices

In this section, we define the soft atoms of soft lattices. We show that a finite soft Boolean algebra has exactly \( 2^n \) elements for some positive integer \( n \). Moreover, any two soft Boolean algebras of order \( 2^n \) are isomorphic to each other.

Definition 4.1 Let \( f_A \) and \( f_B \) be two elements in a soft lattice. The element \( f_B \) is said to be a cover for \( f_A \) if \( f_A \land f_B = f_A \land f_B \) and there is no element \( f_C \) in the soft lattice such that \( f_A < f_C < f_B \). If \( f_B \) covers an element

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\(f_A\), we denote it by \(f_A \prec f_B\).

**Definition 4.2** An element which covers the least element \(f_\emptyset\) is said to be a soft atom of the soft lattice.

**Remark 4.3** Let \(f_A, f_B \in L\) and \(f_A\) be a soft atom in \(L\). If \(f_A \neq f_B\), then \(f_B \wedge f_A = f_\emptyset\). As \(f_A\) is a soft atom \(f_B \wedge f_A \leq f_A \Rightarrow f_B \wedge f_A = f_\emptyset\) if \(f_B\) and \(f_A\) are two distinct soft atoms, then \(f_B \wedge f_A = f_\emptyset\).

**Theorem 4.4** Let \(B\) be a finite soft Boolean algebra. If \(f_B \neq f_\emptyset\) is an element in \(B\), then there exists a soft atom \(f_A\) such that \(f_A \leq f_B\).

**Proof.** If \(f_B\) itself is a soft atom, then we take \(f_A = f_B\). If \(f_B\) is not a soft atom, as \(B\) is finite, we can find a soft chain \(f_B \prec f_B_1 \prec \ldots \prec f_B_n \prec f_B\) satisfying \(f_B \prec f_B_1 \prec f_B_2 \prec \ldots \prec f_B_{n-1} \prec f_B\) for all \(i = 1, 2, \ldots, n-1\) and \(f_B_{n-1} \prec f_B\). So \(f_B_1\) is a soft atom such that \(f_B_1 \leq f_B\) and we take \(f_A = f_B_1\).

**Theorem 4.5** Let \(B\) be a finite soft Boolean algebra and \(f_B \neq f_\emptyset\) in \(B\). Let \(f_A_1, f_A_2, \ldots, f_A_k\) be all the soft atoms of \(B\) such that \(f_A_i \leq f_B\) for all \(i = 1, 2, \ldots, k\). Then \(f_B = f_A_1 \lor f_A_2 \lor \ldots \lor f_A_k\).

**Proof.** Let \(f_B \neq f_\emptyset\) in \(B\). Define \(A(f_B) = \{f_A \in B : f_\emptyset \prec f_A \text{ and } f_A \leq f_B\}\). By theorem 4.4, \(A(f_B) \neq f_\emptyset\). As \(B\) is finite, \(A(f_B)\) is a finite soft set. Let \(A(f_B) = \{f_A_1, f_A_2, \ldots, f_A_k\}\) and \(f_C = f_A_1 \lor f_A_2 \lor \ldots \lor f_A_k\). As each \(f_A_i \leq f_B\), we have \(f_C \leq f_B\). We claim that \(f_B \leq f_C\). It is enough to show that \(f_B \wedge f_C = f_\emptyset\).

If \(f_B \wedge f_C \neq f_\emptyset\), then \(A(f_B) \cap f_C \neq f_\emptyset\). Consider a soft atom \(f_A\) of \(B\) such that \(f_A \leq f_B \wedge f_C\) (by theorem 4.4). Then \(f_A \leq f_C\) and \(f_A \leq f_B\). As \(f_A \leq f_B\) and \(f_A\) is a soft atom, \(f_A \in A(f_B)\). So \(f_A = f_A_i\) for some \(i\) and \(f_A \leq f_C\). As \(f_A \leq f_C\) and \(f_A \leq f_C\) we have \(f_A \leq f_C \lor f_C = f_\emptyset\) which is a contradiction to \(f_A\) is a soft atom. Thus \(f_B \wedge f_C = f_\emptyset\) and hence \(f_B \leq f_C\). So \(f_B = f_A_1 \lor f_A_2 \lor \ldots \lor f_A_k\).

**Theorem 4.6** Let \(B\) be a finite soft Boolean algebra and \(f_B \neq f_\emptyset\) in \(B\). If \(f_A_1, f_A_2, \ldots, f_A_k\) and \(f_B_1, f_B_2, \ldots, f_B_m\) are soft atoms of \(B\) such that (i) \(f_A_1, f_A_2, \ldots, f_A_k\) are distinct

(ii) \(f_B_1, f_B_2, \ldots, f_B_m\) are distinct and (iii) \(f_B = f_A_1 \lor f_A_2 \lor \ldots \lor f_A_k = f_B_1 \lor f_B_2 \lor \ldots \lor f_B_m\) then \(k = m\) and \(\{f_A_1, f_A_2, \ldots, f_A_k\} = \{f_B_1, f_B_2, \ldots, f_B_m\}\).

**Proof:** By (i), (ii) and (iii) \(f_A_1, f_A_2, \ldots, f_A_k\) are distinct soft atoms of \(B\) and \(f_B_1, f_B_2, \ldots, f_B_m\) are distinct soft atoms of \(B\) such that \(f_B = f_A_1 \lor f_A_2 \lor \ldots \lor f_A_k = f_B_1 \lor f_B_2 \lor \ldots \lor f_B_m\). Then each \(f_A_i \leq f_B\) and each \(f_B_j \leq f_B\). So \(f_A_i = f_A_j \lor (f_B_1 \lor f_B_2 \lor \ldots \lor f_B_m)\) = \((f_A_i \lor f_B_1) \lor (f_A_i \lor f_B_2) \lor \ldots \lor (f_A_i \lor f_B_m)\). As \(f_A_i \neq f_\emptyset\), we can find \(j\) such that \(f_A_i \lor f_B_j \neq f_\emptyset\). As both \(f_A_i\) and \(f_B_j\) are soft atoms and \(f_A_i \lor f_B_j \neq f_\emptyset\), \(f_A_i = f_B_j\). Thus each \(f_A_i\) is same as some \(f_B_j\). Hence \(k \leq m\). Similarly \(f_B_j = f_A_i \lor f_B = f_A_j \lor (f_B_1 \lor f_B_2 \lor \ldots \lor f_B_k)\) = \((f_B_j \lor f_A_1) \lor (f_B_j \lor f_A_2) \lor \ldots \lor (f_B_j \lor f_A_k)\). As \(f_B \neq f_\emptyset\), \(f_B_1 \lor f_B_2 \lor \ldots \lor f_B_k\) are soft atoms, \(f_B_1 \neq f_B_2\). Thus for each \(f_B_j\) there exists \(f_A_i\) such that \(f_B_j = f_A_i\). Therefore, \(m \leq k\).

Thus \(k = m, m \leq k \Rightarrow k = m\).

Hence \(\{f_A_1, f_A_2, \ldots, f_A_k\} = \{f_B_1, f_B_2, \ldots, f_B_m\}\).

**Corollary 4.7** Let \(B\) be a finite soft Boolean algebra and every \(f_B \neq f_\emptyset\) in \(B\) can be expressed as a join of soft atoms and this expression is unique.

**Proof.** By theorems 4.4, 4.5 and 4.6 the proof follows.

**Definition 4.8** Let \(B_1\) and \(B_2\) be soft Boolean algebras. A mapping \(\eta: B_1 \rightarrow B_2\) is said to be a soft Boolean homomorphism from \(B_1\) into \(B_2\) if \(\eta\) is a soft lattice homomorphism and for all \(f_x \in B_1\), we have \(\eta(f_x^\emptyset) = (\eta(f_x))^\emptyset\).

**Definition 4.9** A soft Boolean homomorphism is said to be a soft Boolean isomorphism if it is bijective.
Theorem 4.10 Representation theorem for finite soft Boolean algebras: Let B be a finite soft Boolean algebra and let A be the set of all soft atoms of B. Then B is soft isomorphic to \( \mathcal{P}(A) \).

Proof. To each \( f_B \neq f_\emptyset \) in B, let

\[ A(f_B) = \{ f \in A : f_\emptyset < f_A \text{ and } f_A \leq f_B \} \]

and let

\[ A(f_\emptyset) = \emptyset. \]

Then for all \( f_B \neq f_\emptyset \), \( A(f_B) \neq f_\emptyset \) and \( A(f_B) \subseteq A \) where \( A = A(f_\emptyset) \) is the set of all soft atoms in B. Now we define a map \( \eta : B \to \mathcal{P}(A) \) given by

\[ \eta(f_B) = A(f_B) \]

for all \( f_B \in B \). We now show that the map is a soft Boolean isomorphism.

i) \( \eta \) is one-to-one:

If \( \eta(f_B) = \eta(f_{B_1}) \), then for \( f_{B_1}, f_{B_2} \in B \),

then \( \eta(f_{B_1}) = \{ f_A, f_{A_2}, \ldots, f_{A_k} \} \).

Then by theorem 4.6, \( f_{B_1} = f_A \lor f_{A_2} \lor \ldots \lor f_{A_k} = f_{B_2} \).

So \( \eta(f_{B_1}) = \eta(f_{B_2}) \Rightarrow f_{B_1} = f_{B_2} \).

ii) \( \eta \) is onto: Let \( C \in \mathcal{P}(A) \). If \( C = f_{\emptyset} \), then \( \eta(f_{\emptyset}) = A(f_{\emptyset}) = f_\emptyset = C \).

If \( C \neq f_{\emptyset} \), let \( C = \{ f_{C_1}, f_{C_2}, \ldots, f_{C_m} \} \). Let

\[ f_B = f_{C_1} \lor f_{C_2} \lor \ldots \lor f_{C_m}. \]

Then each \( f_{C_i} \leq f_B \) and so \( f_{C_i} \in A(f_B) \) for all \( i = 1, 2, \ldots, m \). If \( f_A \in A(f_B) \), then \( f_A \leq f_B \) and \( f_B \neq f_A = f_A \land f_B = f_A \land (f_{C_1} \lor f_{C_2} \lor \ldots \lor f_{C_m}) = (f_A \land f_{C_1}) \lor (f_A \land f_{C_2}) \lor \ldots \lor (f_A \land f_{C_m}) \).

So \( f_A \land f_{C_j} \neq f_\emptyset \) for atleast one \( j \). As \( f_A \) and \( f_{C_j} \) are soft atoms and \( f_A \land f_{C_j} \neq f_\emptyset \), \( f_A = f_{C_j} \). Thus if \( f_A \in A(f_B) \), then \( f_A \in C \Rightarrow A(f_A) \subseteq C \).

As \( f_B = f_{C_1} \lor f_{C_2} \lor \ldots \lor f_{C_m} \subseteq C \subseteq A(f_B) \). Thus we have \( C = A(f_B) \) and \( \eta(f_B) = C \). Hence \( \eta \) is onto. iii) \( \eta(f_P \land f_Q) = \eta(f_P) \cap \eta(f_Q) \): for all \( f_P, f_Q \in B \). If \( f_P = f_\emptyset \), \( f_Q = f_\emptyset \), we have

\[ \eta(f_P \land f_Q) = f_\emptyset = \eta(f_P) \cap \eta(f_Q). \]

So assume that both \( f_P \neq f_\emptyset \) and \( f_Q \neq f_\emptyset \). Then there exists a soft atom satisfying \( f_A \leq f_P \land f_Q \). Then \( f_A \leq f_P \) and \( f_A \leq f_Q \) that is, \( f_A \in \eta(f_P) \) and \( f_A \in \eta(f_Q) \).

So, \( f_A \in \eta(f_P \land f_Q) \Rightarrow f_A \in \eta(f_P) \cap \eta(f_Q) \).

If \( \eta(f_P) \cap \eta(f_Q) = f_\emptyset \), then \( \eta(f_P \land f_Q) = f_\emptyset \) and the equation is satisfied. Assume that \( \eta(f_P) \cap \eta(f_Q) \neq f_\emptyset \). If \( f_A \in \eta(f_P) \cap \eta(f_Q) \), then \( f_A \in A(f_P) \) and \( f_A \in A(f_Q) \) and so \( f_A \leq f_P \) and \( f_A \leq f_Q \).

As \( f_A \leq f_P \) is soft atom, \( f_A \subseteq f_P \land f_Q, f_A \in A(f_P \land f_Q) = \eta(f_P \land f_Q) \).

Similarly we can prove \( \eta(f_P \land f_Q) = \eta(f_P) \land \eta(f_Q) \). This means that \( \eta(f_B) = \eta(f_Q) \) for (i), (ii), (iii) and (iv), \( \eta \) is a soft Boolean isomorphism.

Corollary 4.11 Every finite soft Boolean algebra has \( 2^n \) elements for some positive integer \( n \).

Proof. Let \( B \) be a finite soft Boolean algebra. Then if \( A \) is the set of all soft atoms of \( B \) and let \( O(A) = n \). Then there are exactly \( 2^n \) elements in \( \mathcal{P}(A) \) and by theorem 4.10, \( B \) is soft isomorphic to soft Boolean algebra \( \mathcal{P}(A) \).

So \( O(B) = O(\mathcal{P}(A)) = 2^n \) where \( O(A) = n \). Hence \( B \) has exactly \( 2^n \) elements where \( n \) is the number of elements in \( A \).

Corollary 4.12 Any two soft Boolean algebras of order \( 2^n \) are isomorphic to each other.

Proof. By theorem 4.10, every soft Boolean algebra of order \( 2^n \) is soft isomorphic to \( \mathcal{P}(A) \) where \( A \) is the set of all soft atoms and \( O(A) = n \). Hence any two soft Boolean algebras of order \( 2^n \) are soft isomorphic to each other.

5. Conclusion

In this paper, we defined the concept of complemented soft lattices, complemented distributive soft lattices and soft Boolean algebras over a collection of soft sets and discussed their related properties and illustrated them with some examples. We also defined the soft atoms of soft lattices and discussed the theorems related to soft atoms. Further, we have given representation theorem for finite soft Boolean algebras. We are studying about these soft lattices and are expected to give some more results.

References


