A Comparative Study of Parametric Models of Old-Age Mortality

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Abstract: ***Four mortality models: Gompertz, Makeham, Logistic and Coale - Kisker models have been considered for a comparative study of Old- Age Mortality for both male and female based on reliable data of ten countries. Sufficiently long run of reliable data selecting from ten countries: Australia, Hong Kong, Canada, England, Israel, Japan, New Zealand, Poland, Singapore, United Kingdom have been used for this investigation. The main objective of this paper is to select the best fit mortality model from ages 85 onwards. Using the complete life table of ten different countries as input, the parameters of these mortality models have been estimated using Levenberg – Marquardt iteration procedure. The estimated parameters of these models have been used for the testing the validity of the models which could be used for the projection of mortality rates at ages. Matlab version 7.11.0 has been used for the estimation of the parameters. It is observed from our result that among the four models the four parameter logistic model and three parameter Coale - Kisker give satisfactory results as the expected values are found to be very closed to observed data. The Gompertz model seems to be not suitable for prediction of old age mortality.***

Keywords: Force of mortality, Gompertz model, Makeham model, Logistic model, Coale and Kisker model

1. Introduction

Mortality modeling is an old subject. Mortality modelling is one of the traditional and fundamental demographic issues. The purpose of the mortality modelling is to find relations and hidden regularities and patterns in the mortality development. Knowledge of these patterns could be used among others for the forecasting of the future development of mortality. Many attempts have been made to find mathematical formulae that will summarise the way in which the probability of dying depends on age. Such formulae have many potential applications. For example, they may be useful in the projection of population numbers and as aids in actuarial work such as the construction of life tables. The first explanatory model, and the most influential parametric mortality modelling, is that proposed by Benjamin Gompertz [3]. He recognised that an exponential pattern in age captured the behaviour of human mortality for large portions of the life table [6]. The function for such survivorship curves, which is named after Gompertz, is described by the well-known Equation:

$$l(x) = n_0 \times exp\left[-\frac{a}{b}(e^{bx} - 1)\right] \qquad (1)$$

The exponential increase in the age-specific mortality rate that may be observed over time t in cohorts of different organisms (cohort analysis mostly used in experiments) or derived from survivorship curves described by Equation (1) (cross-sectional or period analysis mostly used in demography) is known as the Gompertz law and is described by Equations (2) or (3):

$$-\frac{1}{l(x)} \times \frac{d}{dx} l(x) = \mu(x) = ae^{bx} \qquad (2)$$

$$\ln \mu(x) = \ln a + bx \qquad (3)$$

Ever since Gompertz, many models have been suggested to mathematically describe survival and mortality curves [2], of which the Gompertz model and the Weibull [15]model are the most generally used at present [8, 9]. Interestingly, the Gompertz model is more commonly used to describe biological systems, whereas the Weibull model is more commonly applicable to technical devices [8, 9]. Makeham [13] suggested that a better fit to real survivorship curves might be achieved by introduction of an additive constant to Equation (2) thus leading to

$$\mu(x) = ae^{bx} + c \qquad (4)$$

The parameters of Equation (4) (and Equations (1)–(3) by inference) may be interpreted in biologically meaningful and intuitively comprehensible terms. The additive term c represents the rate of deaths resulting from causes that no organism can resist irrespective of its age, e.g., deaths caused by predators that do not discriminate prey ages or by infectious agents so virulent that the ages of their hosts make no difference. The other additive term is the product of the initial rate of deaths resulting from resistible stresses(a), which reflects frailty (the reciprocal of vitality) at x= 0, and the exponent of the parameter$b$, which reflects the rate of the decrease in vitality that occurs with increasing t, i.e., with aging.

The early interpretations of the Gompertz law implied an autocatalytic nature of the age-dependent deterioration of biological functions resulting in an exponential decrease in the resistance to deadly stresses (vitality). However, the age dependent decline of most biological functions within the middle age span is close to linear. Thus, there emerged the issue of reconciling of the time trajectory of the vitality of organisms and the time trajectory of the rate of their deaths. Explanations of why the Gompertz-Makeham mortality law works usually relate the constant c to the risk of death from all causes that do not depend on age, whilst the term $ae^{bx}$is related to the risk of death because of the deterioration of the body due to ageing processes.

In more recent years other mathematical models have been suggested, in particular the logistic model,$\mu_x = c + \frac{ae^{bx}}{1+de^{bx}}$, suggested by Perks[14] and Beard [11]. Some models of mortality are purely descriptive. They give a simple formula which fits the data in a particular range of ages, but no reason why this should be so, and hence no guarantee that the formula will continue to apply in other circumstances. A descriptive model used by Coale and Kisker [1], $\mu_x =$

$ae^{bx}e^{x^2}$ in a limited range of ages. We shall describe this as the quadratic model. The paper by Doray[10] discussed that logistic type models for the force of mortality provide better fit to mortality data of people aged over 85 than Makeham's models where the force of mortality increases exponentially with age.

The following models namely Gompertz, Makeham, Logistic and Coale and Kisker models will be considered in our investigation for finding a suitable mortality model for oldest - old mortality rates.

1.1 Models for force of mortality to be compared

Gompertz's Model

The very first attempt to develop a parametric model of mortality was that of Gompertz [3]. Gompertz modelled the aging or senescent component of mortality with two parameters: a positive scale parameter a that a varies with level of mortality, and a positive shape parameter b that measures the rate of increase in mortality with age. The force of mortality in the Gompertz model is

$$\mu_x = ae^{bx} \tag{5}$$

The paper Bongaarts[5] discussed that for many purposes the Gompertz model provides a satisfactory fit to adult mortality rates, but this model underestimate of actual mortality atyoungest adult ages (under 40) and overestimate at the oldest ages (over 80).

Makeham model

The earliest modification to the Gompertz model, proposed by Makeham [13], involves addition of a constant term, so that

$$\mu_x = c + ae^{bx} \tag{6}$$

The new parameter $c$represents mortality resulting from causes, such as accidents or sexually transmitted diseases, unrelated to either maturation or senescence, which is the same for all ages. The paper Bongaarts[5] discussed that the Makeham model represents a clear improvement over the Gompertz model at younger ages, but it still overestimates mortality at the oldest ages.

Logistic model: The logistic model is known under a variety of names. It was first discovered by Perks [14], who found empirically that the values of μ_x in a life table which he was examining could be fitted by a certain curve, which was in fact a logistic function (though he did not describe it as such at the time). Here we take the logistic function in the following form:

$$\mu_x = c + \frac{ae^{bx}}{1+de^{bx}} \tag{7}$$

It is to be noted that the Makeham model ($d = 0$) is a special case of the logistic model. When $d$is small, any theories which may explain why should follow a logistic function will also help to explain why the Makeham and Gompertz laws work so well over much of the age range.

Coale and Kisker model (Quadratic model)

The idea that $\ln(\mu_x)$ can be fitted by a quadratic function of over x a limited range of ages was used by Coale & Kisker [1] for the purpose of interpolating in the range of ages from 85 to 110, between data up to age 85 and an assumed value at age 110. The relevant formula is

$$\mu_x = ae^{bx}k^{x^2} \tag{8}$$

$$\Leftrightarrow \ln \mu_x = \ln a + bx + x^2 \ln k$$

$$= A + bx + Cx^2, \text{where} C < 0.$$

This is also known as quadratic model. It is important to note that Coale and Kisker model could be used in a limited range of ages. Wilmoth[7] used the model for estimating $\mu_x$at age 110 from data which extended above age 85.

2. Material and Methods

The four mortality models i.e. Gompertz, Makeham, Logistic, Coale and Kisker model (Quadratic model**)** have been considered for this study. These nonlinear models can be written in the form as

$$\mu_i = f(x_i, \mathbf{B}) + \varepsilon_i, \tag{9}$$

$i = 1,2,\cdots,n,$ where μ is the response variables, x is the independent variable, **B**is the vector of parameters $\beta_j$to be estimated $(\beta_1, \beta_2, \cdots, \beta_p)$, ε_i is a random error term,p is the number of unknown parameters, and n is the number of observations[4].The estimators ofβ_j's are found by minimizing the sum of squares residual(SS_{Res}) function

$$(SS_{Res}) = \sum_{i=1}^{n}[\mu_i - f(x_i, \mathbf{B})]^2 \tag{10}$$

under the assumption that the ε_i are normal and independent with mean zero and common variance σ^2. Since μ_i and x_i are fixed observations, the sum of squares residual is a function of**B**. Least squares estimates of **B**are values which when substitutedinto equations (9) will make the(SS_{Res}) a minimum and are found by differentiating equations (10) with respect to each parameter and setting the result to zero. This provides the p normal equations that must be solved for$\hat{\mathbf{B}}$**,** where p denotes the number of unknown parameters. These normal equations take the form

$$\sum_{i=1}^{n}\{\mu_i - f(x_i, \mathbf{B})\}\left[\frac{\partial f(x_i, \mathbf{B})}{\partial \beta_j}\right] = 0 \tag{11}$$

for$j = 1,2,\cdots,p$.

3. Results

3.1 Estimation of Parameters

The parameters of the models are estimated using the Levenverg - Marquardt iteration method based on empirical data sets of complete life table for both males and females. The Levenberg–Marquardt algorithm also known as the damped least-squaresmethod provides a numerical solution to the problem of minimizing a function, generally nonlinear, over a space of parameters of the function. The Levenberg–Marquardt algorithm is more robust than the Gauss–Newton, which means that in many cases it finds a solution even if it starts very far off the final minimum. Matlab version 7.11.0 has been used for the estimation of the parameters. The **Table 1** contains the values of the estimated parametersfor male and female people estimated with the Levenverg - Marquardt iteration method.

3.2 Selection Criteria

After fitting the models using ten different countries separately for male and female, we evaluate the goodness of fit. We consider here two goodness-of-fit measures. The first measure is the root mean square error (RMSE) and the other is the sum squared error (SSE).

Sum Squared Error

This measures the total deviation of the response values from the fit to the response values. It is also called the summed square of residuals and is usually labelled as SSE.

$$SSE = \sum_{x=1}^{n} (\mu_x - \overline{\mu_x})(w_x)$$

A value closer to 0 indicates that the model has a smaller random error component, and that the fit will be more useful for prediction.

Root Mean Squared Error

This measure is also known as the fit standard error and the standard error of the regression. It is an estimate of the standard deviation of the random component in the data, and is defined as

$$RMSE = \sqrt{MSE}$$

where MSE is the mean square error or the residual mean square

$$MSE = \frac{SSE}{v}$$

where v is the residual degrees of freedom and is defined as the number of response values n minus the number of fitted coefficients m estimated from the response values

$$v = n - m$$

v indicates the number of independent pieces of information involving the n data points that are required to calculate the sum of squares. Note that if parameters are bounded and one or more of the estimates are at their bounds, then those estimates are regarded as fixed. The degree of freedom is increased by the number of such parameters. Just as with SSE, an $RMSE$ value closer to 0 indicates a fit that is more useful for prediction

Table 1: Estimated parameters of the Models along with RMSE and SSE for both male and female population

Country		Male				Female			
		Gompertz (a,b)	Makeham (a,b,c)	Logistic (a,b,c,d)	Coale and Kisker (a,b,e)	Gompertz (a,b)	Makeham (a,b,c)	Logistic (a,b,c,d)	Coale and Kisker (a,b,e)
Australia	$\hat{a}$	0.2157	0.3189	0.2774	0.2202	0.01035	0.00362	0.002379	0.002341
	$\hat{b}$	0.4314	0.3053	0.5989	0.4516	0.001501	0.0001704	0.000068	0.000071
	$\hat{c}$	--	-0.0984	0.0041	--	--	-0.2044	-0.002291	--
	$\hat{d}$	--	--	0.2858	--	--	--	0.4808	--
	$\hat{e}$	--	--	--	0.9699	--	--	--	0.934
	RMSE	0.00637	0.00312	0.00281	0.00280	0.01035	0.00362	0.002379	0.002341
	SSE	0.00057	0.00013	0.00010	0.00010	0.001501	0.000170	0.000068	0.000071
Hongkong	$\hat{a}$	0.1967	0.2213	0.2195	0.1981	0.1155	0.1203	0.1200	0.1158
	$\hat{b}$	0.434	0.392	0.432	0.4404	0.48	0.464	0.4767	0.4830
	$\hat{c}$	--	-0.02314	-0.01227	--	--	-0.00448	-0.002809	--
	$\hat{d}$	--	--	0.0434	--	--	--	0.01175	--
	$\hat{e}$	--	--	--	0.9903	--	--	--	0.9959
	RMSE	0.001684	0.000185	0.000046	0.000069	0.000421	0.000053	0.000035	0.000039
	SSE	0.000037	0.0000004	0.00000002	0.00000006	0.000002	0.00000003	0.00000001	0.00000002
Canada	$\hat{a}$	0.2087	0.3309	0.2902	0.2137	0.1643	0.2222	0.1983	0.1676
	$\hat{b}$	0.4275	0.2836	0.5561	0.4501	0.505	0.3918	0.6878	0.5313
	$\hat{c}$	--	-0.1168	-0.01089	--	--	-0.05429	0.00578	--
	$\hat{d}$	--	--	0.293	--	--	--	0.2297	--
	$\hat{e}$	--	--	--	0.966	--	--	--	0.9667
	RMSE	0.006416	0.001886	0.001391	0.00132	0.004877	0.001741	0.000461	0.000854
	SSE	0.000576	0.000046	0.000023	0.00002	0.000333	0.000039	0.000003	0.000010
England	$\hat{a}$	0.2309	0.3234	0.2485	0.2356	0.1873	0.2718	0.2368	0.1916
	$\hat{b}$	0.446	0.3323	0.8533	0.4671	0.4872	0.3541	0.6773	0.5156
	$\hat{c}$	--	-0.08779	0.05057	--	--	-0.07982	0.00408	--
	$\hat{d}$	--	--	0.3581	--	--	--	0.2667	--
	$\hat{e}$	--	--	--	0.9699	--	--	--	0.9628
	RMSE	0.008216	0.006171	0.005070	0.005641	0.006924	0.00374	0.00334	0.003306
	SSE	0.000945	0.000495	0.000309	0.000414	0.000671	0.00018	0.00013	0.000142
Israel	$\hat{a}$	0.2034	0.6924	0.2911	0.2117	0.2005	0.2537	0.238	0.2036
	$\hat{b}$	0.3675	0.1156	0.8936	0.3987	0.5054	0.4151	0.6083	0.5255
	$\hat{c}$	--	-0.4807	0.04149	--	--	-0.04973	-0.001928	--
	$\hat{d}$	--	--	0.7168	--	--	--	0.1601	--
	$\hat{e}$	--	--	--	0.9464	--	--	--	0.9742
	RMSE	0.009849	0.003632	0.000572	0.001789	0.00458	0.0012790	0.00009142	0.0004434
	SSE	0.001261	0.000158	0.000004	0.000038	0.00027	0.0000196	0.000000091	0.0000024
Japan	$\hat{a}$	0.3368	0.3832	0.3831	0.3395	0.2527	0.3246	0.2837	0.2547
	$\hat{b}$	0.6832	0.6225	0.623	0.7086	0.8038	0.6799	1.023	0.8994
	$\hat{c}$	--	-0.04219	-0.04202	--	--	-0.06599	-0.00003574	--
	$\hat{d}$	--	--	0.0003203	--	--	--	0.126	--
	$\hat{e}$	--	--	--	0.9766	--	--	--	0.928600
	RMSE	0.007096	0.000417	0.000427	0.001708	0.01721	0.0103	0.007411	0.007869
	SSE	0.001208	0.000004	0.000004	0.000067	0.00711	0.0024	0.001208	0.001424

New Zealand	$\hat{a}$	0.2268	0.2605	0.2571	0.2287	0.1945	0.2433	0.2242	0.1976
	$\hat{b}$	0.4751	0.4228	0.4779	0.4848	0.5464	0.4555	0.6944	0.5712
	$\hat{c}$	--	-0.03148	-0.01527	--	--	-0.04531	0.005046	--
	$\hat{d}$	--	--	0.05351	--	--	--	0.1685	--
	$\hat{e}$	--	--	--	0.9867	--	--	--	0.9709
	RMSE	0.002964	0.001341	0.00135	0.001332	0.005075	0.00191	0.0005587	0.001051
	SSE	0.000123	0.000023	0.00002	0.000023	0.000361	0.00005	0.0000038	0.000014
Poland	$\hat{a}$	0.2340	0.2680	0.2605	0.2355	0.1962	0.2461	0.2376	0.1988
	$\hat{b}$	0.3776	0.3344	0.4325	0.3825	0.4488	0.3688	0.488	0.4622
	$\hat{c}$	--	-0.03242	0.001995	--	--	-0.04714	-0.01278	0.9805
	$\hat{d}$	--	--	0.1163	--	--	--	0.1235	--
	$\hat{e}$	--	--	--	0.9913	--	--	--	--
	RMSE	0.001814	0.000428	0.000058	0.000206	0.0033790	0.0005837	0.00007340	0.00006919
	SSE	0.000046	0.0000024	0.00000004	0.000001	0.0001599	0.0000044	0.00000006	0.00000006
Singapore	$\hat{a}$	0.2024	0.2226	0.2212	0.2035	0.1499	0.1729	0.1704	0.1512
	$\hat{b}$	0.4327	0.3985	0.4304	0.4379	0.4647	0.4116	0.4733	0.4741
	$\hat{c}$	--	-0.01904	-0.01036	--	--	-0.02155	-0.009357	--
	$\hat{d}$	--	--	0.03438	--	--	--	0.06139	--
	$\hat{e}$	--	--	--	0.9922	--	--	--	0.9867
	RMSE	0.001403	0.000151	0.000034	0.000053	0.001761	0.0002468	0.00004331	0.00004270
	SSE	0.000026	0.00000027	0.00000001	0.00000003	0.000040	0.0000007	0.00000002	0.00000002
UK	$\hat{a}$	0.2214	0.2624	0.1939	0.2238	0.192	0.2879	0.2299	0.1969
	$\hat{b}$	0.4173	0.3595	1.012	0.4276	0.4792	0.3379	0.816	0.5098
	$\hat{c}$	--	-0.03889	0.08014	--	--	-0.09078	0.02484	--
	$\hat{d}$	--	--	0.3753	--	--	--	0.346	--
	$\hat{e}$	--	--	--	0.984	--	--	--	0.5098
	RMSE	0.007535	0.007355	0.006077	0.007121	0.007570	0.004238	0.00299	0.003382
	SSE	0.000795	0.000703	0.000443	0.000659	0.000802	0.000234	0.00011	0.000149

4. Discussion

Based on RMSE and SSE values, the four models have been rank in the following order, starting with the best. However, between the first and second positions is difficult to establish, as the respective values of RMSE and SSE of these models are roughly equal.

1. **Logistic model**: This model is the most successful of the four in describing the trajectory of old age mortality for ten different countries. The fit is good in individual countries, the deviations are small.
2. **Coale and Kisker model**: The quadratic model was designed by Coale and Kisker (1990) to start at age 85. This model gives a very good fit at ages 85, but it fails to describe even approximately the values below 85 years.
3. **Makeham model**: It is observed from the result that the fitted value of the parameter c in the Makeham model is not so small hence itcan't be neglected. So at high ages the difference between the Gompertz model and Makeham model is significant in our case. But Thatcher et al. [2] concluded that at high ages the difference between the Gompertz model and Makeham model is negligible.
4. **Gompertz model**: This classical model overestimates the rise of mortality with age.

5. Conclusion

In this paper, four mortality models namely Gompertz, Makeham, Logistic and Coale -Kisker have been fitted for ten different countries by using Levenberg Marquardt iteration method. The comparison has been made on the basis of RMSE and SSE. The Gompertz model fits well only for Hong Kong for female population. Makeham model give satisfactory fit for Hong Kong female and Singapore male population. Moreover, the fitting of the Gompertz model seems to be not satisfactory. The Logistic and Coale - Kisker models were far closer to the observed values and it is not easy to rank them. The Coale - Kisker model which is also known as quadratic model is pragmatic, but it has little theoretical support. (On the contrary: if extended indefinitely it would imply that the force of mortality will eventually reach zero, and this can only happen if immortality is possible). Essentially, the quadratic model uses a parabola as an approximation to a more general curve. Finally, we may conclude that the logistic model approximation is the best of the four models for projection of old-age mortality.

References

[1] J. Coale and E. E. Kisker, "Defects in data on old-age mortality in the United States: New procedures for calculating mortality schedules and life tables at the highest ages," In Asian and Pacific Population Forum, IV **(**1), pp. 1-31, 1990.
[2] R. Thatcher and V. Kannisto and J. W. Vaupel, The force of mortality at ages 80 to 120, Odense: Odense University Press, 22, 1998.
[3] Gompertz, "On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies,"Philosophical transactions of the Royal Society of London, CXV, pp. 513-583, 1825.
[4] M. Borah and D. J. Mahanta, "Rapid parameter estimation of three parameter nonlinear growth models," International Journal of Mathematical Archive (IJMA), IV (2), pp. 274-282, 2013.
[5] J. Bongaarts, "Long-range trends in adult mortality: Models and projection methods,"Demography, XLII (1) pp. 23-49, 2005.

[6] J. P. Higgins and S. G. Thompson, J. J. Deeks and D. G. Altman, "Measuring inconsistency in meta-analyses,"BMJ: British Medical Journal, CCCXXVII (7414), pp. 557, 2003.
[7] J. R. Wilmoth, "Are mortality rates falling at extremely high ages? An investigation based on a model proposed by Coale and Kisker,"Population Studies XLIX, pp. 281-295, 1995.
[8] L. A. Gavrilov and N. S. Gavrilova, "The reliability theory of aging and longevity,"Journal of theoretical Biology, CCXIII(4), pp. 527-545, 2001.
[9] L. A. Gavrilov and N. S. Gavrilova,"Reliability theory explains human aging and longevity. Science's SAGE KE(Science of Aging Knowledge Environment),"V, pp. 1-10, 2003.
[10] L. G. Doray, "Inference for logistic-type models for the force of mortality", Paper presented at the living to 100 and beyond symposium,Orlando, Fla. (January 7-January 9) 2008.
[11] R. E. Beard, Appendix: Note on some mathematical mortality models, In Ciba Foundation Symposium-The Lifespan of Animals (Colloquia on Ageing),John Wiley & Sons, Ltd., V, pp. 302-311, 1959.
[12] S. J Olshansky and B. A. Carnes, "Ever since Gompertz,"Demography, XXXIV (1), pp. 1-15, 1997.
[13] W. M. Makeham, "On the law of mortality and the construction of annuity tables,"The Assurance Magazine, and Journal of the Institute of Actuaries, VIII (6), pp. 301-310, 1860.
[14] W. Perks, "On some experiments in the graduation of mortality statistics,"Journal of the Institute of Actuaries, LXIII (1),pp. 12-57, 1932.
[15] W. Weibull, "A statistical distribution function of wide applicability,"Journal of applied mechanics, XVIII(3), pp. 293-297, 1951.

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