

Free terminal Time Optimal Control Problem of an SIR Epidemic Model with Vaccination

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Abstract: *This paper presents an approach that investigates a free terminal optimal time control of an SIR (Susceptible-Infected-Removed) epidemic model. In order to reduce the infected group and increase the removed individuals, we present a control simulating vaccination program considering also the minimum duration of a vaccination campaign. The optimal control and the optimal final time are found using Pontryagin's maximum principle and the additional transversality condition for the terminal time. We solved the optimality system by an iterative method, and then we confirm the performance of the optimization strategy by numerical simulations.*

Keywords: SIR epidemic models, optimal control, Pontryagin's maximum principle, numerical simulations.

1. Introduction

Mathematical modeling in epidemiology was pioneered by D. Bernoulli in 1760 in his work demonstrating the effectiveness of the technique of variolation against smallpox. Since that time, theoretical epidemiology has witnessed numerous developments. In recent years, mathematical models have become a powerful tool for predicting the developing tendency of the infectious disease, determining the key factors of the spread of infectious disease and seeking the optimum strategies of preventing and controlling the spread of infectious diseases.

Among the effective methods to prevent and control the spread of infection, the public-health authorities usually use vaccination. It's a powerful tool that allows for the mass prevention of infection rather than treating the symptoms of infection. Vaccination saves million of lives each year around the world. A tremendous number of models with vaccination have been formulated, analyzed and applied to a variety of infectious diseases, as in references [3], [5], [8]. Currently, the guidelines for diseases immunization are based on the conventional concept of time constant, while in practice, it's both difficult and expensive to implement vaccination for large population coverage in large time, especially while considering financial and logistical constraints. That's why we have a whole interest to research for an optimal final time which allows us to attempt the aim of those strategies with an optimal cost. In this context, we set a free terminal time optimal control problem in the case of an SIR (Susceptible-Infected-Removed) epidemic model with vaccination. A control representing the percentage of susceptible individuals being vaccinated per time unit is considered in order to minimize the number of infected individuals and maximize the removed individuals during the course of an epidemic. The minimum duration of the vaccination program is also considered. This paper is organized as follows. In section 2, we will describe the mathematical model with control term. The analysis of optimization problem is presented in section 3. In section 4, we will give a numerical appropriate method and the corresponding simulation results. Finally, the conclusions are summarized in section 5.

2. Model Formulation

We consider an SIR epidemic model with constant total population size. The population is divided into three disease-state compartments: susceptible individuals (S), people who can catch the disease; infectious (infective) individuals (I), people who have the disease and can transmit the disease; recovered individuals (R), people who have recovered from the disease. We assume that an individual can be infected only through contacts with infectious individuals and that immunity is permanent. The transitions between different states are described by the following parameters:

- Λ is the recruitment rate of susceptible;
- β is the effective contact rate;
- μ is the natural mortality rate;
- d is the disease induced death rate;
- r is the recovery rate.

The dynamics of the model are governed by the following system of differential equations subject to non-negative initial conditions

$$\begin{cases} \frac{dS}{dt} = \Lambda - \mu S - \beta \frac{SI}{N} \\ \frac{dI}{dt} = \beta \frac{SI}{N} - (\mu + d + r)I \\ \frac{dR}{dt} = rI - \mu R \end{cases} \quad (1)$$

The strategy of the control we adopt consists of a vaccination program. So, we introduce into the model (1) a control $u(t)$ representing the vaccination rate at time t . The control $u(t)$ is the fraction of susceptibles individuals being vaccinated per unit time. We assume that all susceptibles vaccinees are transferred directly to the removed class. The mathematical system with control is given by the nonlinear differential equations

$$\begin{cases} \frac{dS}{dt} = \Lambda - \mu S - \beta \frac{SI}{N} - uS \\ \frac{dI}{dt} = \beta \frac{SI}{N} - (\mu + d + r)I \\ \frac{dR}{dt} = rI - \mu R + uS \end{cases} \quad (2)$$

With $S(0) \geq 0$, $I(0) \geq 0$, and $R(0) \geq 0$ are given. And $N(t) = S(t) + I(t) + R(t)$ is the total population number at time t .

3. Optimal Control Problem

In practice, it's both difficult and expensive to implement vaccination for large population coverage in large time, especially while considering financial and logistical constraints. That's why we formulate an optimal control problem with free terminal time to derive the optimal duration of vaccination. We first define the objective functional as follows

$$J(u, t_f) = \int_0^{t_f} \left(I(t) - R(t) + \frac{A}{2} u^2(t) \right) dt + B t_f^2 \quad (3)$$

where $A > 0$, $B > 0$ are the weight constants of the control and time respectively. t_f represents the duration of the vaccination program. Our goal is to minimize the duration of vaccination and systemic costs attempting to reduce the number of infeted and increase the removed individuals. We seek an optimal control u^* and an optimal terminal time t_f^* such that

$$J(u^*, t_f^*) = \min \left\{ J(u, t_f) : u \in U, t_f \in \mathbb{R}^+ \right\} \quad (4)$$

Where U is the set of admissible controls defined by

$$U = \left\{ u(t) : 0 \leq u \leq b, u \text{ is measurable, } t \in [0, t_f] \right\}$$

Pontryagin's Maximum Principal converts (2), (3) and (4) into a problem of minimizing a Hamiltonian, defined by

$$H = I - R + \frac{A}{2} u^2 + \sum_{i=1}^3 \lambda_i f_i \quad (5)$$

where f_i is the right side of the differential equation of the i^{th} state variable.

By applying the Pontryagin's maximum principle [2], we obtain the following theorem

Theorem 1: Given an optimal control u^* an optimal terminal time t_f^* , and solutions S^* , I^* and R^* of the corresponding state system, there exists an adjoint vector $\lambda = [\lambda_1, \lambda_2, \lambda_3]$ satisfying

$$\begin{aligned} \dot{\lambda}_1 &= (\lambda_1 - \lambda_2) \beta \frac{I}{N} + (\lambda_1 - \lambda_3) u + \lambda_1 \mu \\ \dot{\lambda}_2 &= \lambda_1 \beta \frac{S}{N} + \lambda_2 \left((\mu + d + r) - \beta \frac{S}{N} \right) - \lambda_3 r - 1 \\ \dot{\lambda}_3 &= \lambda_3 \mu + 1 \end{aligned}$$

with the transversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = 0$$

Futhermore, the optimal control u^* is given by

$$u^* = \min \left(b, \max \left(0, \frac{(\lambda_1 - \lambda_3) S^*}{A} \right) \right) \quad (6)$$

and the optimal final time is given by

$$t_f^* = \frac{R(t_f^*) - I(t_f^*) - \frac{A}{2} (u(t_f^*))^2}{2B} \quad (7)$$

Proof. The adjoint equations and transversality conditions can be obtained by using Pontryagin's Maximum Principle such that

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial H}{\partial S}, & \lambda_1(t_f) = 0 \\ \dot{\lambda}_2 = -\frac{\partial H}{\partial I}, & \lambda_2(t_f) = 0 \\ \dot{\lambda}_3 = -\frac{\partial H}{\partial R}, & \lambda_3(t_f) = 0 \end{cases}$$

The optimal control u^* can be solve from the optimality condition,

$$\frac{\partial H}{\partial u} = 0$$

that is

$$\frac{\partial H}{\partial u} = Au - \lambda_1 S + \lambda_3 S = 0$$

By the bounds in the control U , it is easy to obtain u^* in the form of (6). The transversality condition for t_f to be the optimal terminal time can be stated as

$$H(t_f^*, x(t_f^*), \lambda(t_f^*), u(t_f^*)) = -\frac{\partial g}{\partial t}(t_f^*, x(t_f^*))$$

where $g(t, x(t)) = B t_f^2$ that is

$$I(t_f^*) - R(t_f^*) + \frac{A}{2} u(t_f^*)^2 + 2B t_f^* = 0$$

Thus, t_f^* may be rewritten as in (7).

4. Numerical simulations

In this section we present the results obtained by solving numerically the optimality system. This system consists of a two-point boundary value problem, with separated boundary conditions at times $t=0$ and $t=t_f$. There were initial conditions for the state variables and terminal conditions for the adjoints. In addition, the final time t_f itself is now a variable and must satisfy the transversality condition (7). The Algorithm we used proceeds as follows

Algorithm 2

- Start with an initial guess of the final time;
- Choose initial guess of the state variables, the adjoint variables and the control;
- Forward solving of the state system;
- Backward solving of the adjoint system ;
- Update the control using the characterization (6);
- Update the terminal time using the characterization (7);
- Continue until the optimality condition is achieved.

For a detailed description of the semi-implicit finite difference method used for solving the optimality system, we

refer the interested reader to [1]. The numerical simulations are carried out using Matlab and using the following parameter values and initial conditions taken for [1].

- Initial conditions: $S(0) = 3 \times 10^6$, $I(0) = 30$, $R(0) = 28$
- Parameter values: $\beta = 0.3095$, $\delta = 1174.17$, $\mu = 3.9139 \times 10^{-5}$, $d = 0.63$, $r = 0.2$

Note that the parameter β is calculated from $\beta = R_0(\delta + d + r)$ with $R_0 = 1.9$. The critical level of vaccination needed to protect the population is defined by $V_C = 1 - (1/R_0)$ (see [5]), so one has $b = V_C = 0.47$. As defined above, we attempt to give the optimal free final time needed to reduce the infected group. Considering the critical level of vaccination, we give a final time sufficient to eradicate definitively disease. By this way, numerical simulations suggest 148 days as final time of the vaccination campaign.

The graphs below, allow us to compare changes in the number of infected, susceptible and removed individuals before and after the introduction of control. Figure1 gives an example of the evolution of the number I with and without control. We notice that in absence of control, the infected group grew to extremely high levels and in presence of the control, this group decrease greatly, where the maximum number of infected individuals is 77 infections.

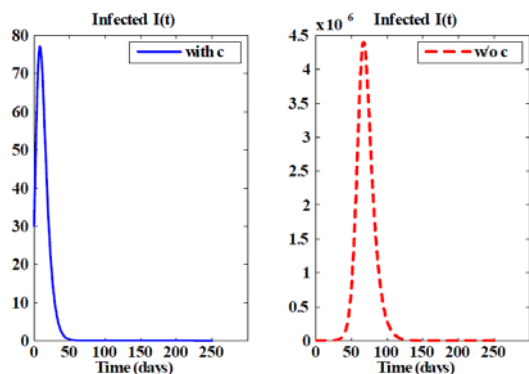


Figure 1: The function I with and without control.

Figure2 also shows the effect of control by indicating that the number S decreases more rapidly during the vaccination campaign.

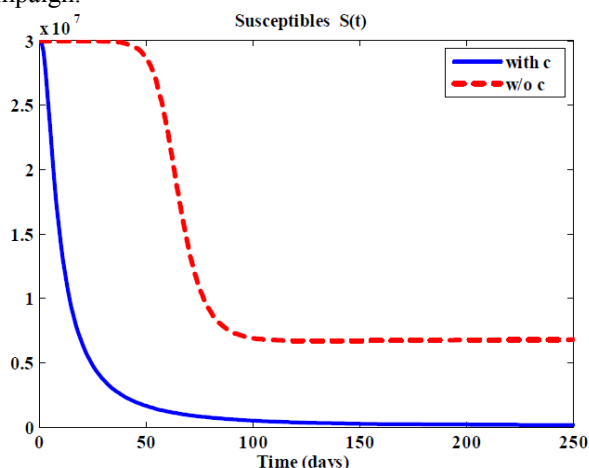


Figure 2: The function S with and without control.

Figure3, show that the number of people removed begins to grow notably from the first day, instead of the 48st day in absence of control. In the end of the vaccination campaign, all population is removed. Which show the effectiveness of the control to eradicate definitively the infection.

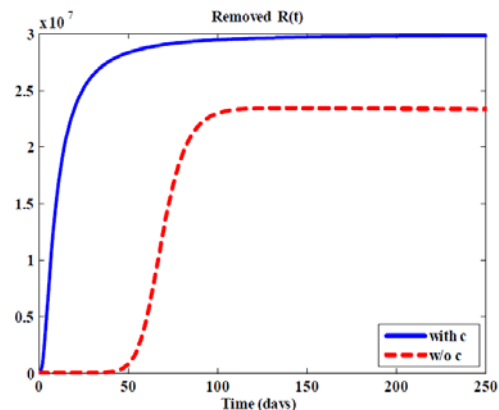


Figure 3: The function R with and without control.

Finally, Figure4 gives a representation of the optimal control u^*

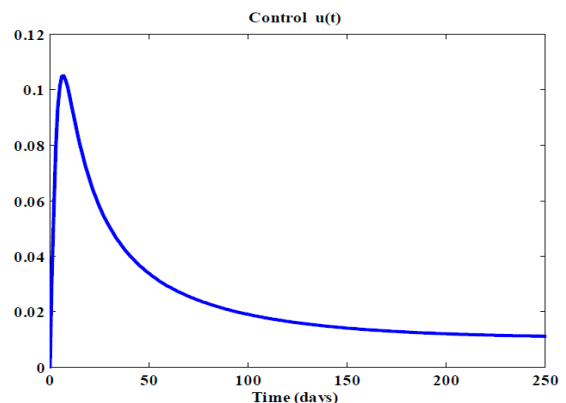


Figure 4: The optimal control u^*

5. Conclusion

The purpose of this work is to derive a new control strategy for an SIR epidemic model, considering a control simulating vaccination program. By this way, both the terminal time, and a control u representing the pourcentage of susceptible individuals being vaccinated per time unit, were objectif to minimise. Numerical simulations demonstrate an interesting result : The infected groupe and the terminal time are reduced, which confirm the effectiveness of the approach.

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