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Variational Methods in a Thin Shell Problem

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Abstract: ***Variational methods and their applications are considered in the solution of problems involving vibrations of thin shell segments consisting of partly positive and partly negative Gaussian curvature parts. The lower part of the spectrum of these shells when the boundary conditions exclude pure bending is investigated. The results obtained by using the Raleigh-Ritz approximation are compared with those obtained by applying the Shooting Method and it is observed that there is good agreement. Results obtained also indicate that the variational method used in the investigation gave large and less accurate results for the lowest frequency parameter while increasing the number of coordinate elements to two gave more accurate values. However, it is shown that increasing the number of elements has the disadvantage of increasing the computations needed to solve the problem.***

Keywords: Gaussian Curvature, Variational Method, Sturm-Liouville Problem, Raleigh-Ritz Approximation, Shooting Method

Nomenclature

A_1, A_2 Lame's coefficients
α_1, α_2 Orthogonal curvilinear coordinates
B_1, B_2 Linear operators
E Young's modulus
$L_k, k = 12,3$ Momentum terms
h Dimensionless thickness
ϑ Poisson's ratio
ρ Density
g Acceleration due to gravity
μ Small geometric parameter
λ Frequency parameter
m Number of waves in circular direction
R_1, R_2 Radii of convergence
s Arc length
φ Angle in the circular direction
Λ Eigenvalue
Λ_0 Smallest eigenvalue
u, v, w Coordinate axes
T_1, T_2, S Stress projections
$u_j(\alpha_1, \alpha_2)$ Displacement projections
ω Natural frequency

1. Introduction

A shell is a body bounded by two curved surfaces, where the distance between the surfaces is small in comparison with other body dimensions [24]. Shell structures have been constructed since ancient times. The Hagia Sophia in Istanbul and the Pantheon in Rome are well-known examples. Shells are very efficient in carrying loads acting perpendicular to their surface by in-plane membrane stresses [14]. Shell structures enjoy the unique position of having extremely high aesthetic value in various architectural designs. The understanding of the behaviour of shell structures enables designers or stress analysts to verify the accuracy of numerical structural analysis results for such structures [24].

Shells have a wide range of applications and uses in engineering. Cylindrical shells find extensive use in tanks, boiler gas, water conduits and aeroplane structures. Examples of shell structures in civil and architectural engineering are large-span roofs, water tanks, containment shells of nuclear power plants and concrete arch domes [24]. In mechanical engineering shell structures find use in piping systems, turbine disks and pressure vessels technology while in aeronautical and marine engineering shell forms are used in the construction of missiles, aircrafts, rockets, ships and submarines [24]. In the field of biomechanics shells are found in various biological forms such as the skull and eye, plant and animal shapes.

Variational methods, also known as calculus of variations or energy methods are a branch of mathematics that involves finding stationary values of functionals. In other words calculus of variations is a field of mathematical analysis that deals with maximizing or minimizing functionals, which are mappings from a set of functions to the real numbers. A functional is defined as an integral that has a specific value for each function from domain substituted into the functional, or a functional is an integral that implicitly contains differential equations that defines a problem.

Variational solutions of shell problems are very useful when the desired result depends on overall rather than local conditions, for example in buckling and vibration problems or general magnitudes of deflections under transverse loads. In particular, an approximate solution of a differential equation can be obtained by using the Raleigh-Ritz method, which involves substituting an approximating function into the variational function, making sure the approximating function satisfies the boundary conditions.

Several studies have recently been carried out involving thin shell theory. Among them are Timoshenko [21], Ventsel [24] and a host of many others. Due to their usefulness in the real world, shells deserve to be studied diligently and carrying out researches about shells is the best way to it.

Other researchers of note who have studied shells include Masashi [13] and Aginam [2] *et al*, who have applied variational methods to analyse thin elastic shells with finite rotations and isotropic thin rectangular plates respectively. Also, Avramov & Brelavski [4] studied on vibrations of shells rectangular in the horizontal projection with two freely supported edges.

Another researcher who has extensively researched on shells is Petrov [19]. Some of his works include the investigation of stability and low-frequency oscillations of thin shells with entirely or partially negative Gaussian curvature among many others. Refer to [15, 16, 17, 18, 20] for more of his research on shells.

2. Problem Formulation

Equations of motion for the vibration analysis of shells may be derived as a simple extension of the static case by adding the inertia forces to body forces and body moment terms that result from accelerations of the mass of the shell according to the D'Alambert's principle. A detailed treatment of shell theory may be found in [10, 19].

Consider the two dimensional case of shell theory as presented in [5, 10, 19, 23]:

$$\sum_{j=1}^{3}(\mu^4 N_{ij} + L_{ij})u_j + F_i = 0, i = 1,2,3 \qquad \textbf{(1)}$$

where $u_j(\alpha_1,\alpha_2)$ are the projections of the shear, $N_{ij}(\alpha_1,\alpha_2)$ and $L_{ij}(\alpha_1,\alpha_2)$ are linear differential operators, α_1 and α_2 are orthogonal coordinates on the middle surface, F_i are stress projections and μ is a small geometric parameter given by;

$$\mu^4 = \frac{h^4}{12R^2(1-\vartheta^2)},$$

where ϑ is Poisson's ratio and h represents the shell thickness and is assumed to be very much smaller when compared with the characteristic dimension R. Refer to [5, 10, 23] for a more detailed treatment of the operators , $N_{ij}(\alpha_1,\alpha_2)$ and $L_{ij}(\alpha_1,\alpha_2)$.

The load projections F_i, are proportional to the eigenvalue Λ, that is, $F_i = -\Lambda u_i$, where $\Lambda = \rho R^2\omega^2 E^{-1}$ where ω is the unknown frequency parameter. Also Lame's coefficients and radii of convergence are expressed as

$$A_1 = 1\,, A_2 = B(s),$$
$$R_1 = \frac{ds}{d\theta}, R_2 = \frac{ds}{\sin\theta}.$$

The coefficients of system (1) depend only on s and separation of variables yields $u_i(s,\varphi) = u_i(s)e^{im\varphi}$ and consequently leads to a system of ordinary equations of the form $\mu\frac{dY}{ds} = \mathrm{A}(s,\mu,m,\Lambda)Y$ where

$Y^T = (u_1, u_2, u_3, u'_3, u''_3, T_1, T_2, \mathrm{S})$. See [15, 17, 19, 20].

If $\mu \neq 0$, equation (1) is a system of eighth order and has four boundary effect integrals at each edge of the shell; $B_k(Y(S_k)) = 0, k = 1,2$. We also note from Fig 1.1 that R_1 changes sign at the point s^* and

$R_1(s) > 0$ at $s_1 \leq s < s^*, R_1(s) < 0$ at $s^* \leq s < s_2$, $R_2(s) > 0$ and $R_1^{-1}(s^*)$=0.

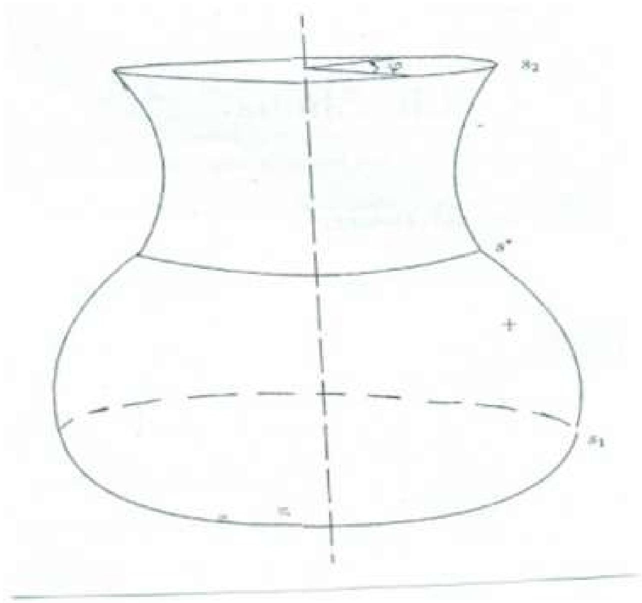

Figure 1: Shell of partly negative and partly positive Gaussian curvature

The stress-strain relations and the equilibrium equations are as follows:

$$\left.\begin{aligned}
\frac{dT_1}{ds} + \frac{B'}{B}(T_1 - T_2) + \frac{m}{B}S &= \frac{h^2}{12}L_1 - Eh\lambda u\\
\frac{dS}{ds} + 2\frac{B'}{B}S - \frac{m}{B}T_2 &= \frac{h^2}{12}L_2 - Eh\lambda v\\
\frac{B'}{B}v + \frac{m}{B}v - \frac{w}{R_2} &= T_2 - \frac{\vartheta T_1}{Eh}\\
\frac{d}{ds}\left(\frac{v}{B}\right) - \frac{m}{B}u &= \frac{2(1+\vartheta)S_1}{Eh}\\
\frac{T_1}{R_1} + \frac{T_2}{R_2} &= \frac{h^2}{12}L_3 - Eh\lambda w\\
\frac{du}{ds} - \frac{w}{R_1} &= T_1 - \frac{\vartheta T_2}{Eh}
\end{aligned}\right\} \qquad \textbf{(2)}$$

For shells of negative Gaussian curvature we have

$$m = h^{-\frac{1}{3}}$$

and

$$\lambda = h^{\frac{2}{3}}$$

and the waves cover the whole middle surface. For shells of positive Gaussian curvature $\lambda = 1$ and the waves are located near a weak parallel. It is thus important to pay more attention to the concave part of the shell.

Definition 1: *A Sturm-Liouville* is a real second-order differential equation of the form;

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y = 0 \quad \textbf{(3)}$$

where y is a function of the variable x on some interval $[a,b]$ and y and x satisfy the following boundary conditions:

$$\left.\begin{aligned} k_1y(a) + k_2y'(a) = 0\\ l_1y(b) + l_2y'(b) = 0 \end{aligned}\right\} \qquad \textbf{(4)}$$

where k_1, k_2, l_1 and l_2 are none-zero constants. The value of λ if it exists, is called its *eigenvalue* of the problem, and the corresponding solutions of such a λ are called *eigenfunctions* of the Sturm-Liouville problem.

Definition 2: A *functional* f is a mapping from a vector space X into its underlying scalar field R, i.e. $f: X \longrightarrow \mathbb{R}$.

Definition 3: A *Euler-Lagrange equation* is a differential equation whose solutions are the functions for which a given functional is stationary. If J is defined by an integral of the form $J = \int f(t, y, \dot{y})dt$, where $\dot{y} = \frac{dy}{dx}$, then J has a stationary value if the following Euler-Lagrange differential equation is satisfied:

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0. \quad (5)$$

The form (5) is also referred to as the *variational form* of the differential equation.\

Consider J= $\int_{x_1}^{x_2} F(y, y', x)dx$. It can easily be shown that functional that possesses the Euler-Lagrange equation of the form that is needed for the system (3) and (4) is given by $K = \int_0^L (p(y')^2 - sy^2 + \lambda r(x)y^2)dx$. If $y(x)$ of the Sturm-Liouville equation (3), then it must make stationary the integral $\int_0^L (p(y')^2 + sy^2)\, dx$ subject to the constraint $N = \int_0^L r(x)y^2\, dx = constant$. The value of the constant N is irrelevant to the variational formulation and is taken to be unity in most cases.

3. Methodology

The Raleigh-Ritz Method

The Raleigh-Ritz method makes use of an approximating function which is substituted into the variational form of the problem. It must be borne in mind that the approximating function should satisfy the boundary conditions of the problem under consideration. The major utility of the Raleigh-Ritz method lies in approximating lies in approximating a solution rather than evaluating it exactly and this feature is very useful for eigenvalue problems.

In the Raleigh-Ritz approximation, a finite set of N linearly independent functions $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ and a linear combination $y = \sum_{n=1}^{N} c_n\, \varphi_n(x)$ of the functions is then formed when each function $\varphi_n(x)$ satisfies the boundary conditions of of the Sturm-Liouville eigenvalue problem and the stationary values of K are determined by setting and solving $\frac{\partial K}{\partial c_i} = 0$ where $i = 1,2, \ldots, N$.

If $y(x)$ is a solution of the Sturm-Liouville equation, then K vanishes and;

$$\lambda = \frac{\int_0^L (p(y')^2 + sy^2)}{\int_0^L ry^2\, dx} dx$$

$$= \frac{J\{y\}}{N\{y\}} \quad (6)$$

It must be noted here that since the eigenvalues λ_i of the Sturm-Liouville equation are the stationary values of $\frac{J\{y\}}{N\{y\}}$, then any evaluation of this ratio yields a value that lies between the lowest and highest eigenvalues of the corresponding Sturm-Liouville equation, that is,

$$\lambda_{min} \le \frac{J}{N} \le \lambda_{max},$$

where, depending on the equation under consideration, either, λ_{min} is finite or λ_{max} is finite. As an example, for an equation with a finite lowest eigenvalue λ_0, any evaluation of $\frac{J}{N}$ provides an upper bound on λ_0.

The asymptotic solution of the system of equations (2) yields the following Sturm-Liouville differential equation

$$\frac{d}{ds}\left[rg\frac{dy}{ds}\right] + \frac{f}{r}y = 0 \quad (7)$$

where $g(s) = \frac{2R_1^2 B^4}{R_2(R_1 - R_2)^2}$, $f(s) = \frac{\Lambda R_2 m^2}{2} - \frac{m^6 h^2}{12(1-\vartheta^2)g}$. Refer to [19].

It is helpful to mention here that the spectrum of equation (3) when taken together with the boundary conditions (4) is discrete and the eigenvalues are positive if the following conditions are satisfied:

(a) $p(x) \ge 0$, p_0 is a positive constant,
(b) $q(x) \ge 0$,
(c) $r(x)$ lies between certain positive numbers t_0 and t_1,
(d) k_1, k_2, l_1, l_2 are non-zero and at least of the constants k_1 and l_1 are non-zero.

The smallest eigenvalue λ is equal to the minimum of the functional;

$$\int_a^b [\, p(x)(y')^2 + q(x)y^2]dx + \frac{k_1}{k_2}p(a)y^2 + \frac{l_1}{l_2}p(b)y^2 \quad (8)$$

$$\text{under the condition } \int_a^b r(x)[y(x)]^2\, dx = 1 \quad (9)$$

If $k_2 = l_2 = 0$, the terms containing k_2 and l_2 vanish and $y(x)$ satisfies condition (9) together with $y(a) = y(b) = 0$. Now considering equation (8), it is observed that condition (a) above is not satisfied at $s = s^*$ in accordance with $r(s^*) = 0$. However there is a discrete spectrum of (7) together with (3) even when $p(a) = 0$ provided the improper integral

$$I = \int_a^b \frac{x-a}{p(x)} dx \quad (10)$$

converges and $r(s) \approx k(s - s^*)^{\frac{1}{2}}$.
Therefore $I = \int_a^b \frac{s-s^*}{g(s)r(s)} ds \approx \frac{2}{3k}(s - s^*)^{\frac{3}{2}} < 0$.
Substituting for $f(s)$, the Sturm-Liouville equation (7) is transformed to;

$$\frac{d}{ds}\left[rg\frac{dy}{ds}\right]-\frac{m^6h^2}{12(1-\vartheta^2)rg}y+\frac{\Lambda m^2R_2}{2r}y=0 \quad \textbf{(11)}$$

where $p(x)=rg$, $q(x)=\frac{m^6h^2}{12(1-\vartheta^2)rg}$ and $t(x)=\frac{\Lambda m^2R_2}{2r}$.

The functional of the Sturm-Liouville equation (11) is;

$$K=\int_0^L(-p(y')^2-qy^2+\Lambda ty^2)ds \quad \textbf{(12)}$$

Thus $F(s,y',y)=-p(y')^2-qy^2+\Lambda ty^2$

Now

$\frac{\partial F}{\partial y}=\frac{\Lambda R_2m^2}{r}y-\frac{m^6h^2}{12(1-\vartheta^2)rg}$, $\frac{\partial F}{\partial y'}=-2gry'$ and

$\frac{d}{ds}\left(\frac{\partial F}{\partial y'}\right)=-2\frac{d}{ds}(gry')$.

It can easily be shown that the Euler-Lagrange equation $\frac{\partial F}{\partial y}-\frac{d}{ds}\left(\frac{\partial F}{\partial y'}\right)=0$ is satisfied, hence $F(s,y',y)$ is a variational function of the Sturm-Liouville equation (11).

The functional K is given by

$$K=\int_{s^*}^{s_2}\frac{\Lambda R_2m^2}{r}(y')^2-gr(y')^2+\frac{m^6h^2}{12(1-\vartheta^2)rg}y^2\,dx.$$

Hence equation (6) implies that the smallest eigenvalue, Λ_0 is equal to the minimum of $\frac{\int_{s^*}^{s_2}(p(y')^2+qy^2)ds}{\int_{s^*}^{s_2}ty^2ds}$.

Therefore

$$\Lambda_0=\min_{m,y}\frac{\int_{s^*}^{s_2}\left[gr(y')^2+\frac{m^6h^2}{12(1-\vartheta^2)rg}y^2\right]ds}{m^2\int_{s^*}^{s_2}\frac{R_2y}{2r}ds} \quad \textbf{(13)}$$

Rearranging (3) gives

$$\Lambda_0=\min_{m,y}\frac{I_1}{m^2I_3}+\min_{m,y}\frac{m^4h^2I_2}{12(1-\vartheta^2)I_3} \quad \textbf{(14)}$$

where $I_1=\int_{s^*}^{s_2}[gr(y')^2]ds$, $I_2=\int_{s^*}^{s_2}\frac{y^2}{rg}ds$ and

$I_3=\int_{s^*}^{s_2}\frac{R_2y^2}{2r}ds$.

The minimum is evaluated for all $y(s)$ satisfying the boundary conditions $y(s^*)=y(s_2)=0$. Differentiating (14) with respect to m and equating the result to zero, gives

$m^*=\left(\frac{6(1-\vartheta^2)I_1}{h^2I_2}\right)^{\frac{1}{6}}$. Now substituting for m^* for m in (14)

gives $\Lambda_0^*=\frac{3}{2}\left(\frac{h^2}{6[1-\vartheta^2]}\right)^{\frac{1}{3}}\min_m\frac{I_1^{\frac{2}{3}}I_2^{\frac{1}{3}}}{I_3}$.

For our Sturm-Liouville problem (11), we let the approximating function be;

$$y(x)=\alpha(x-x^*)(x-x_2)+\beta\sin\frac{\pi(x-x^*)}{x-x_2}. \quad \textbf{(15)}$$

We note here that the approximating function satisfies the boundary conditions $y(x^*)=y(x_2)=0$. Now letting $x^*=0$ and $x_2=\frac{\pi}{2}$ gives

$y(x)=\alpha\left(2x-\frac{\pi}{2}\right)+\beta\sin 2x$. We use this equation for y to get an equation for the functional K as given by (12).

4. Results and Discussion

For the functional K we let $h=0.01, m=5$ and $\vartheta=0.5$ to get

$$K=17.0781\alpha^2\Lambda-0.06759\alpha^2-0.8935\alpha^2-52.869\alpha\beta\Lambda+0.636\alpha\beta-0.4\alpha\beta+41.009\beta^2\Lambda-0.388\beta^2-2.211\beta^2$$

We now take partial derivatives of K with respect to α and β in turn, and equate to zero. As a result we obtain two simultaneous equations which we solve to get $\Lambda=0.0276$ or $\Lambda=57.349$. We discard the larger value because we are looking for the smallest frequency parameter. The above procedure is repeated for the cases m=6, 7, 8,9,10 and a comparison is made with results obtained by using the Shooting method, denoted as Λ_s. We follow the same procedure for $m=9,10,11,12,13,14$ when $h=0.003$.

The values obtained by using two coordinate elements are denoted as Λ_{v_2} while results obtained using one coordinate element $y(x)=\alpha(x-x^*)(x-x_2)$ are denoted Λ_{v_1}.

Table 1: Results for $h=0.01$

m	5	6	7	8	9	10
Λ_s	0.0437	0.0341	0.0307	0.0306	0.0328	0.0342
Λ_{v_1}	0.0563	0.0445	0.0419	0.0464	0.0477	0.0513
Λ_{v_2}	0.0276	0.0288	0.0396	0.0421	0.0451	0.0493

Table 2: Results for $h=0.003$

m	9	10	11	12	13	14
Λ_s	0.0153	0.0139	0.0135	0.0136	0.0141	0.0148
Λ_{v_1}	0.0199	0.0188	0.0192	0.0209	0.0240	0.0286
Λ_{v_2}	0.0130	0.0141	0.0162	0.0190	0.0229	0.0279

5. Conclusion

The variational method gives values for the lowest frequency parameter which are comparable to those obtained by applying the Shooting method. Increasing the number of coordinate elements to two gave lower values, thereby increasing the accuracy of the method. However increasing the number of coordinate elements also implied increasing the number of computations required to solve the problem. Nevertheless, the variational or energy method has the advantage of consuming much less computer time to evaluate the frequencies when compared to the Shooting method.

We conclude that the variational method is an effective method for dynamics or stability investigation of rather complicated structures such as shells and beams. Even though the results obtained are less accurate when compared to proper numerical ones, the variational method is desirable because high accuracy would demand huge computer time.

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