Finite Difference Method for Transmission Line and Waveguide Problems

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Abstract: ***In the present work we have developed the numerical solutions for various partial differential equations such as electromagnetic problems using finite difference method. The electromagnetic problems under consideration are (a) Transmission lines and (b) waveguide. In case of transmission lines problem the governing equation is Laplace equation. We have computed the characteristic impedance of shielded double micro-strip transmission lines. The computed values of the characteristic impedance are comparable with the theoretical values. In the waveguide problem where the governing equation is Helmholtz equation, we have computed the cut off wave number k_c for the TM modes in the rectangular waveguide. The cut off wave number k_c for the lowest TM modes obtained were compared with theoretical values, and they were found to be satisfactory. All the implementations have done by using Matlab.***

Keywords: Finite difference method, Transmission lines, waveguides, Laplace equation, Helmholtz equation.

1. Introduction

Finite difference method is currently being widely used for solving the partial differential equations. The finite difference method is a choice to numerically solve the elliptic partial differential equations [1]. In the present paper, finite difference method has been used to solve the Laplace and Helmholtz equations. Laplace's equation is a second-order partial differential equation. The application of Laplace equation and Helmholtz equation is to determine of electromagnetic fields in transmission line and waveguide problems. Transmission line is a means of transfer of information from one point to another. The distinguishing feature of most transmission lines is that they have uniform cross sectional dimensions along their length, giving them uniform *impedance*, called the characteristic impedance, to prevent reflections [2-4]. In the transmission lines problem we are computing the characteristic impedance of the shielded double-strip transmission lines. In the waveguide problem the cut off wave number k_c for the lowest rectangular waveguide modes have been computed.

1.2 Transmission Lines

Transmission line is a specialized cable, which is designed to carry alternating current of radio frequency. These are guided conducting structures that are used in power distribution at low frequencies in communications and computer networks at higher frequencies. It is used to connect a source to a load. The source may be transmitter and the load may be a receiver. There are many types of transmission lines, for example micro-strip lines, two-wire parallel lines, coaxial lines, planer lines, optical fibre [5]. Transmission lines have wide applications such as distributing cable television signals, computer network connections and connecting radio transmitters and receivers with their antennas. Transmission line elements are integral parts of antenna. Transmission line is not only used to transmit energy from one place to another it is also used as a circuit element like inductor, resonant circuit, capacitor, filter, transformer and insulator at high frequencies. It is a distributed parameter network and is described by parameters distributed throughout its length [6]. The length of a transmission line is of utmost importance in transmission line analysis.

2. Characteristic Impedance of Transmission Line

The characteristic impedance of a transmission line is the ratio of the amplitudes of voltage and current of a wave travelling along the line. It is usually denoted by z_0. It is independent of length and is determined by geometry of transmission line. The finite difference techniques are suited for computing the characteristic impedance. The characteristic impedance Z_0 and phase velocity u of the line are defined as

$$Z_0 = \sqrt{\frac{L}{C}} \quad \text{and} \quad u = \frac{1}{\sqrt{LC}} \tag{1}$$

Where L and C are the inductance and capacitance per unit length respectively. For the non magnetic dielectric medium the characteristic impedance Z_{00} and phase velocity u_0 with the dielectric removed are given by

$$Z_{00} = \sqrt{\frac{L}{C_0}} \quad \text{and} \quad u_0 = \frac{1}{\sqrt{LC_0}} \tag{2}$$

Where C_0 is the capacitance per unit length without dielectric. Combining equation (1) and (2) yields.

$$Z_0 = \frac{1}{u_0\sqrt{CC_0}} \tag{3}$$

$$u = u_0\sqrt{\frac{C_0}{C}} = \frac{u_0}{\sqrt{\varepsilon_{eff}}} \tag{4}$$

Where ε_{eff} is the effective dielectric constant. Thus to find Z_0 for an inhomogeneous medium, requires calculating the capacitance per unit length of the structure, with and without the dielectric substrate.

$$C = \frac{4Q}{V_d} \tag{5}$$

Where V_d is the potential difference between inner and outer conductors. Now, the problem is reduced to finding the charge per unit length Q. We use the finite difference technique to compute the characteristic impedance phase velocity and attenuation of several transmission lines like as polygon lines, shielded strip lines, micro strip lines, coaxial lines and rectangular lines. Here we are considering the micro strip line. The geometry in the figure is given below

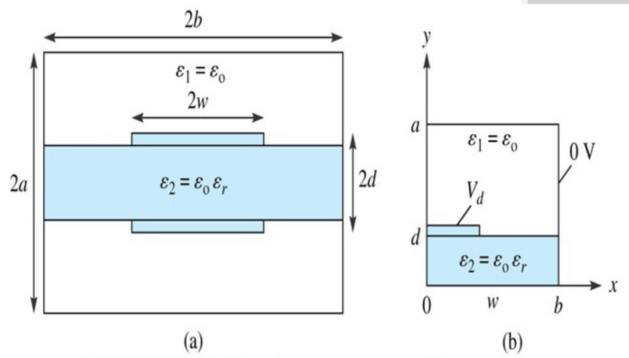

Figure 1: (a) Shielded double-strip line with partial dielectric support; (b) problem in (a) simplified by making full use of symmetry [7]

Algorithm [7]:

- First calculate potential V without dielectric space.
- Charge Q_0 without dielectric medium.
- Calculate $C_0 = \frac{4Q_0}{V_d}$
- The potential V with the dielectric space.
- Charge Q with the dielectric medium.
- Calculate $C = \frac{4Q}{V_d}$
- Finally calculate $Z_0 = \frac{1}{u_0\sqrt{CC_0}}$

Where u_0 is the speed of light.

3. Helmholtz Equation

The Helmholtz equation is an elliptic partial differential equation given by

$$\nabla^2 u + k^2 u = 0 \tag{6}$$

Where ∇^2 a Laplace operator, k is is the wave number and u is a scalar function. The Helmholtz equation often arises in the study of physical problems involving partial differential equations in both space and time. In this section, we are computing the cut off wave number k_c for TM (Transverse Magnetic) mode in the rectangular waveguide by using Helmholtz equation.

3.1 Wave Guides

Wave-guide is the most efficient way to transfer electro-magnetic energy. It is used to transmit the electrical waves at microwave frequency. A wave-guide is a structure that guides waves, such as electromagnetic waves. Wave guides are essentially coaxial lines without centre conductors. They are constructed from conductive material and may be rectangular, circular or elliptical in shape [6].

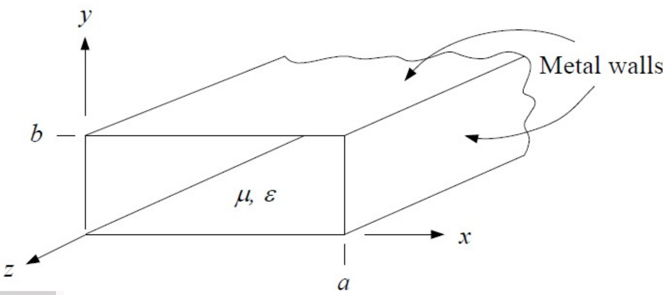

Figure 2: Rectangular waveguide

The solution of waveguide problems is well suited for finite difference schemes because the solution region is closed. This amounts to solving the Helmholtz or wave equation

$$\nabla^2 v + k^2 v = 0 \tag{7}$$

Where for $v = E_z$ transverse magnetic (TM) modes or $v = E_z$ transverse electric (TE) modes, while wave number k is:

$$k^2 = \omega^2 \mu\varepsilon - \beta^2 \tag{8}$$

The permittivity ε of the dielectric medium can be real for a lossless medium or complex for a lossy medium [5]. To apply the finite difference method, we discretize the cross section of the waveguide by a suitable square mesh. Using central difference approximation to the partial derivatives of equation (7), gives

$$\frac{v(i,j+1)+v(i,j-1)-2v(i,j)}{h^2} + \frac{v(i+1,j)+v(i-1,j)-2v(i,j)}{h^2} + k^2 v(i,j) = 0 \tag{9}$$

Where $\Delta x = \Delta y = h$ is the mesh size. Equation (9) applies to all the free or interior nodes. At the boundary points, we apply Dirichlet condition ($v=0$) for the TM modes and Neumann condition for the TE modes:

$$\frac{\partial v}{\partial n} = 0 \tag{10}$$

This implies that at boundary *v=0* for TM modes. By applying equation (9) and Dirichlet conditions to all mesh points in the wave-guide cross section, m simultaneous equations involving the m unknowns $(v_1, v_2, v_3, ..., v_m)$ have been obtained. These equations may be conveniently cast in to matrix equation

$$(A - \lambda I) = 0 \cdot \qquad (11)$$

Where *A* is an *m x m* band matrix of known integer elements, *I* is an identity matrix $v = (v_1, v_2, v_3, ..., v_m)$ is the eigenvector and $\lambda = (kh)^2$ is the eigen-value. There are several ways of determining λ and the corresponding *v*.
The method is iterative method. In this method the matrix elements are usually generated rather than stored. We begin with $v_1 = v_2 = v_3 = ... = v_m = 1$ and a guessed value for k. The field v_{ij}^{r+1} at the *(i, j)th* node in the *(r+1)th* iteration is obtained from its known value in the r^{th} iteration using

$$v_{ij}^{r+1} = v_{ij}^{r} + \frac{\omega R_{ij}}{4 - h^2 k^2} \qquad (12)$$

Where ω is the acceleration factor, $1 < \omega < 2$, and R_{ij} is the residual at the (i, j)th node given by

$$R_{ij} = v(i, j+1) + v(i, j-1) + v(i+1, j) + v(i-1, j) - (4 - h^2k^2)v(i, j) \qquad (13)$$

$$k^2 = \frac{-\int_s v\nabla^2 v ds}{\int_s v^2 d} \qquad (14)$$

The finite difference equation is:

$$k^2 = -\frac{\sum_{i=1}\sum_{j=1} v(i, j)[v(i, j+1) + v(i, j-1) + v(i+1, j) + v(i-1, j) - 4v(i, j)]}{h^2 \sum_{i=1}\sum_{j=1} v(i, j)} \qquad (15)$$

The new value *k* is obtained from the equation (15) is now used in applying equation (12), over the mesh for another more accurate values, which are again substituted in to the equation (15) for update *k*.

4. Results

4.1 Transmission line problem:

To calculate the characteristic impedance Z_0 for the micro-strip transmission line in Fig.1 by using the algorithm described above with

$$a = b = 2.5cm, d = 0.5cm,$$
$$w = 1cm, \varepsilon_1 = \varepsilon_0, \varepsilon_2 = 2.35\varepsilon_0$$

Finite difference method is used to determine the characteristic impedance Z_0.

Table 1: Characteristic Impedance of a micro-strip line

h	*Number of iterations*	Z_0
0.25	700	47.9242
0.05	1000	50.1185
0.025	1000	50.8198
0.01	1000	56.7084

Computed characteristic impedance of shielded double-micro strip transmission lines, are shown in the above table. The values of the characteristic impedance given by Mathew N. O. Sadiku [7] for h=0.25 is 49.05 and for h=0.05 is 61.53, and the number of iterations are 700 and 1000 respectively. We obtained that the computed results are compatible with these results.

4.2 Waveguide problem for Helmholtz equation

In this case, the cut-off wave number k_c has been computed for different values of 'a' and 'b' for the rectangular waveguide as shown in Fig.2.

Table 2: Values of cut-off wave number K_c for the TM modes in the rectangular waveguide

a	b	a/b	Guess value of k	Calculated value of K_c	Value of K_c from analytical solution	$\lambda_c = \frac{2\pi}{k_c}$
2.0	4.0	0.50	60	1.8312	1.7562	3.4312
3.0	4.0	0.75	70	1.3306	1.3089	4.7222
5.0	5.0	1.00	80	0.9555	0.8886	6.5758
5.0	4.0	1.25	85	1.0477	1.0058	5.9971
8.0	6.0	1.33	90	0.7555	0.6545	8.3166
9.0	6.0	1.50	100	0.7338	0.6293	8.5625

In the eigen-value problem (7), the cut off wave number K_c for the TM modes (Dirichlet boundary condition v=0, at boundary) in the rectangular waveguide has been computed. The width of the waveguide is 'a' cm and the height of waveguide is 'b' cm as shown in Fig.2. The values of K_c are calculated for the different-different dimensions of the rectangular waveguide. The results are shown in table 2. The computed results are compatible with theoretical values of lowest rectangular wave guide mode.

5. Conclusion

In the transmission lines problem the governing equation is Laplace equation. We computed the characteristic impedance of shielded double micro-strip transmission lines. The computed values of the characteristic impedance are 47.92 and 56.7 for h = 0.25 and h = 0.05 respectively. These values are compatible to compare with 49.05 and 61.53 for h = 0.25 and h = 0.05 respectively given by Mathew N. O. Sadiku [7].
In the waveguide problem the governing equation is Helmholtz equation. We computed the cut off wave number k_c for the lowest TM modes in the rectangular waveguide. The cut off wave number k_c for the lowest TM modes obtained were compared with theoretical values, and they were found to be satisfactory.

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Author Profile

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