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Multigrid Approach for Solving Elliptic Type Partial Differential Equations

Ashish Kumar Garg[1], Itendra Kumar[2], Sham Bansal[3], Ishu Goyal[4]

[1]Delhi University, Department of Mathematics, JDM College, Rajinder Nagar, New Delhi – 110060, India

[2]Delhi University, Department of Mathematics, Hansraj College, Mahatma Hansraj Marg Malkaganj, Delhi – 110007, India

[3]Delhi University, Department Of Mathematics, Bharati College, C4, Janakpuri, New Delhi – 110058, India

[4]Panjab University Chandigarh, Department of Physics, Chandigarh – 160014, India

Abstract: ***The present work is to develop numerical solutions for elliptic type partial differential equations using finite difference method. To establish this work we first present and classify the partial differential equations. Next we present and describe the multigrid methods. Applications of multigrid methods for the numerical solution of one-dimensional and two dimensional Poisson's equation have been discussed. All the implementations have done by using Matlab.***

Keywords: Partial differential equations, Finite difference method, Central difference, Multigrid methods, Poisson's equation

1. Introduction

Physical processes can be commonly related to the change in the properties of the substance undergoing the process. Those processes that depend on more than two variables are called partial differential equations. In the second order partial differential equations there are three forms of the partial differential equations, elliptic, hyperbolic and parabolic type. There are various applications of elliptic type partial differential equations in physical phenomena: Quantum mechanics, electrodynamics and thermodynamics. To numerically solve these elliptic partial differential equations, there are many methods and schemes. The finite difference method is a choice to numerically solve the elliptic partial differential equations. The finite difference method uses a topologically square network of lines to construct the discretization of the partial differential equations. In the finite difference method, to solve the large system of equations, the iterative methods are used. The multigrid method is also effective in the finite difference discretization to solve the elliptic type linear partial differential equations. Multigrid methods are the well-developed, flexible and optimal computational complexity tools for the numerical solution of partial differential equations. In contrast to other methods, multigrid methods are general and they can treat arbitrary regions and boundary conditions. Multigrid methods do not depend on the separability of the equations or other special properties of the equation. The main advantage is: it can solve problems to a given accuracy in a number of operations that is proportional to the number of unknowns, so the multigrid methods reduce the solver setup time. The main idea of multigrid is to accelerate the convergence of a basic iterative method by solving a coarse problem. The multigrid method is similar to interpolation between coarser and finer grids [1]. In present work, second order partial differential equations of mathematical physics and the applications of the partial differential equations in physical phenomena have been presented. The finite difference approximations to derivatives have also been described. Multigrid methods can be applied in combination with any of the common discretization techniques. For example, the finite element method may be recast as a multigrid method [2]. Multigrid algorithm has been implemented to find a better approximation to the solution of the Poisson's equation.

2. Finite Difference Method

Among various numerical techniques for solving partial differential equations and initial and boundary problems, the finite difference methods are widely used. These methods are derived from the truncated Taylor's series where a given partial differential equations and boundary and initial conditions are replaced by set of algebraic equations that are then solved by varies well known numerical techniques [3]. These methods have significant advantages over other methods because of their simplicity of analysis and computer codes in solving problems with complex geometrical structures. Different schemes for second order partial derivatives have been discussed and applied to discretize the boundary value problems of the second order partial differential equations.

2.1 Multigrid Methods

Multigrid methods in numerical analysis are being used for solving differential equations using a hierarchy of discretization. Many basic relaxation methods exhibit different rates of convergence for short and long wavelength components, suggesting these different scales to be treated differently, as in a Fourier analysis approach to multigrid [4]. Multigrid methods are among the fastest solution techniques known today. The elliptic and hyperbolic partial differential equations are, by and large, at the heart of most mathematical models used in engineering and physics, giving rise to extensive computations. Often the problems that one would like to solve exceed the capacity of even the most powerful computers, or the time required is too large to allow inclusion of advanced mathematical models in the design process of technical apparatus, from microchips to aircraft, making design optimization more difficult. Multigrid methods are a prime source of important advances in algorithmic efficiency, finding a rapidly increasing number of users. Unlike other known methods, multigrid offers the possibility of solving problems with N unknowns with O (N) work and storage, not just for special cases, but for large classes of problems [5].

2.2 Discretization Scheme

Jacobi iterative solver has been used to specify a tolerance on the residual. The interval has been discretized into n equal subintervals. System of n+ 1 equation has been defined by involving the boundary conditions at the first and last nodes, and the discretized differential equation at the n-1 interior nodes. It has been observed that the solver on a coarse grid quickly approximates the overall behavior of the solution. In present paper, a multigrid approach has been followed to find the solution for the Poisson equation. We first make a very simple experiment, in which we use a single pair of coarse and fine grids. Two geometric grids has been considered in such a way that the fine grid contains all the nodes of the coarse grid, as it has been assumed that the fine grid contains all the nodes of the coarse grid and that the coarse grid can be constructed by selecting just the nodes with odd index from the fine grid.

2.3 Restriction Operator

The restriction operator is a mapping from fine grid to coarse grid $R: \Omega_h \rightarrow \Omega_H$, let u and $\bar{u}$ defined on Ω_h (fine grid) and Ω_H (coarse grid) [5, 6, 7]. Then $Ru = \bar{u}$.

For restriction operator in one dimensional

$$\bar{u}_i = \frac{1}{4}u_{2i-1} + \frac{1}{4}u_{2i} + \frac{1}{4}u_{2i+1}$$

For restriction operator in two dimensional

$$\bar{u}(i,j) = \frac{1}{16}\left(u(2i+1,2j+1) + u(2i-1,2j+1) + u(2i+1,2j-1) + u(2i-1,2j-1)\right)$$

$$+\frac{1}{8}\left(u(2i+1,2j) + u(2i,2j+1) + u(2i,2j-1) + u(2i-1,2j)\right) + \frac{1}{4}u(2i,2j).$$

2.4 Prolongation operator

Prolongation operator is a mapping from coarse grid to fine grid I: $\Omega_h \rightarrow \Omega_H$, let u and $\bar{u}$ defined on Ω_h (fine grid) and Ω_H (coarse grid) [5, 6, 7]. Then $I\bar{u} = u$.

For restriction operator in one dimensional

$$u_{2i} = \bar{u}_i \ \ and \ \ u_{2i+1} = \frac{1}{2}\bar{u}_{i+1} + \frac{1}{2}\bar{u}_i$$

For restriction operator in two dimensional

$$u(2i,2j) = \frac{1}{2}\bar{u}(i,j) \text{ , } u(2i+1,2j) = \frac{1}{4}\left(\bar{u}(i,j) + \bar{u}(i+1,j)\right) \text{ , }$$

$$u(2i,2j+1) = \frac{1}{4}\left(\bar{u}(i,j) + \bar{u}(i,j+1)\right) \ and$$

$$u(2i+1,2j+1) = \frac{1}{8}\left(\bar{u}(i,j) + \bar{u}(i+1,j) + \bar{u}(i,j+1) + \bar{u}(i+1,j+1)\right)$$

Two-grid Algorithm

Two-grid method for solving $A_h u_h = f_h$ [5]

- Pre-smoothing steps on the fine grid: $u_h^l = S(u_h^{l-1}, f_h); l = 1,2,3,4,\ldots\ldots, v_1$
- Applying iterative methods, Jacobi or Gauss seidel method.
- Computational of residual: $r_h = f_h - A_h u_h^{v_1}$
- Restriction of residual: $r_H = Rr_h$
- Solution of the coarse grid problem: $A_H e_H = r_H$
- Coarse grid correction: $u_h^{v_1+1} = u_h^{v_1} + Ie_H$
- Post-smoothing steps on the fine grid: $u_h^l = S(u_h^{l-1}, f_h); l = v_1 + 2,\ldots\ldots, v_1 + v_2 + 1$

A_h is the fine grid matrix.
A_H is the coarse grid matrix.
R is matrix from the restriction operator.
I is matrix from the prolongation operator.

H is the step length of the coarse grid and h is the step length of the fine grid. A two-grid scheme starts at the fine level with pre-smoothing, performs a coarse grid correction for solving a coarse grid auxiliary problem, and ends with post-smoothing. A pictorial representation of this process where 'fine' is a high level and 'coarse' is a low level looks like a *V* work flow. This is called *V* cycle. To solve the problem to a given tolerance, one has to apply the two grid *V*-cycle iteratively.

3. Multigrid Algorithm

In the Two-grid scheme the size of the coarse grid is twice larger than the fine one, thus the coarse problem may be very large. However, the coarse problem has the same form as the residual problem on the fine level. The sequence of grids with mesh size $h_1 > h_2 > h_3 > h_4 > \ldots.. > h_L > 0$ so that $h_{k-1} = 2h_k$. Here $k = 1, 2, \ldots.., L$, is called the level number. The number of interior grid point will be n_k. On each level k we denote the problem $A_k u_k = f_k$. Here A_k is a $n_k \times n_k$ matrix, and u_k, f_k *are* vectors of size n_k. The transfer among levels has been performed by two linear mappings, the restriction *R* (*i.e.* I_k^{k-1}) and I (*i.e.* I_{k-1}^k) prolongation operators. We denote $u_k = S_k(u_k, f_k)$ as a smoothing iteration.

Multigrid method for solving $A_k u_k = f_k$ [5]

- Pre-smoothing steps on the fine grid: $u_k^l = S(u_k^{l-1}, f_k); l = 1,2,3,4,\ldots\ldots, v_1$
- Computational of residual: $r_k = f_k - A_k u_k^{v_1}$
- Restriction of residual: $r_{k-1} = I_k^{k-1} r_k$
- $u_{k-1} = 0$
- Apply γ times multigrid scheme to solve $A_{k-1} u_{k-1} = r_{k-1}$
- Coarse grid correction: $u_k^{n+1} = u_k^n + I_{k-1}^k u_{k-1}$
- Post-smoothing steps on the fine grid: $u_k^l = S(u_k^{l-1}, f_k); l = v_1 + 2,\ldots\ldots, v_1 + v_2 + 1$

The parameter γ represents the number of times the multi grid procedure is applied to coarse level problem. Since this procedure converges very fast, γ = 1 or γ = 2 are the typical values used. For γ = 1 the multigrid scheme is called *V*-cycle, whereas γ = 2 is named *W*-cycle. It turns out that with a reasonable γ, the coarse problem is solved almost exactly.

Therefore in this case the convergence factor of a multigrid cycle equals that of the corresponding two grid method.

4. Results

For one dimensional Poisson equation

$$\frac{\partial^2 u}{\partial x^2} = -f(x),\ f(x) = \pi^2 \sin(\pi x)$$

And the boundary conditions are $u(0) = 0, u(1) = 0.$

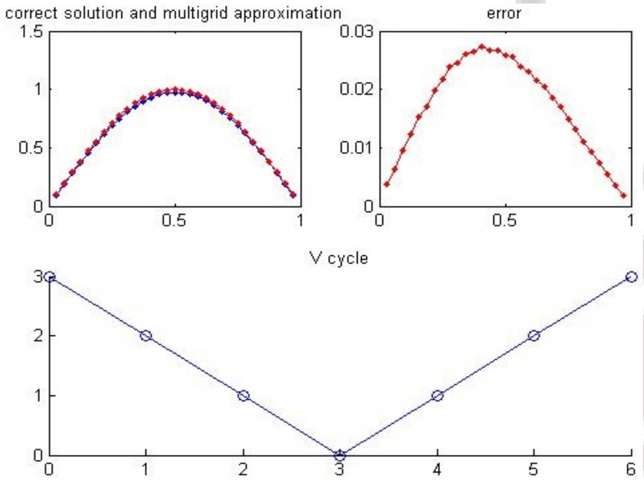

Figure 1: Exact and approximate Solution of one dimensional Poisson equation

The numerical solution of one dimensional Poisson equation by multigrid method is shown in the Fig. 1. In the figure, blue curve shows the approximate solution computed by the multigrid method and red curve shows the exact solution. From the above plot we can say, the computed numerical solution is converging to the exact solution. Therefore multigrid methods reduce the error. So multigrid methods are much more efficient for computing the solution of elliptic partial differential equations.

For Two dimensional Poisson equation

$$\nabla^2 u = -f(x,y),\ f(x,y) = \exp(-\cos((4x)^2)) - 3/2$$

The boundary conditions are $u(0,y) = 0, u(x,0) = 0$, $u(1,y) = 0, u(x,1) = 0$; $0 \le x \le 1, 0 \le y \le 1$.

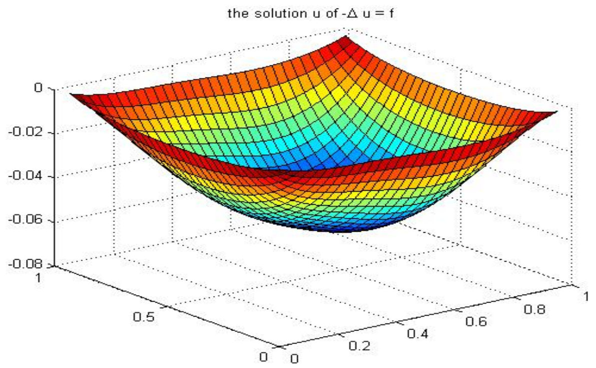

Figure 2: Solution of two dimensional Poisson equation by multigrid method

The numerical solution of two dimensional Poisson's equation by multigrid methods shown in the Fig. 2. The multigrid method is more efficient to numerically solve the partial differential equations. The convergence rate of the multigrid methods is faster than the other simple iterative methods and gives the better approximation to the numerical solution.

5. Conclusion

The finite difference method is used to solve the elliptic partial differential equations. The numerical solution of one dimensional and two dimensional Poisson's equations are computed by multigrid methods in the finite difference discretization techniques. Multigrid methods are very well suited methods to solve the elliptic partial differential equations. Multigrid methods are among the fastest solution techniques and give the better approximation to the numerical solutions. It can solve problems to a given accuracy in a number of operations that is proportional to the number of unknowns.

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Author Profile

Ashish Garg received the M.Sc. degree in Mathematics from Indian Institute of Technology Madras in 2011 and M. Tech. degree from Indian Institute of Technology Kharagpur in 2013.