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Characterization of Fuzzy Bridges and Fuzzy Cutnodes

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Abstract: ***In this paper, we study fuzzy bridges and fuzzy cutnodes. A characterization is obtained for fuzzy graphs G such that G* is a cycle. A sufficient condition for a node to be a fuzzy cutnode is obtained which becomes also necessary in the case of fuzzy trees. Some significant differences from the crisp theory are pointed out. Note that, bridges and cutnodes of the crisp graph G* are fuzzy bridges and fuzzy cutnodes of the fuzzy graph G respectively.***

Keywords: Fuzzy Bridge, Fuzzy Cutnode, Fuzzy Tree, Crisp Graph.

1. Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld [10] in 1975. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of neural networks, expert systems, information theory, cluster analysis, medical diagnosis, control theory, etc. Rosenfeld [10] has obtained the fuzzy graph-theoretic concepts like bridges, paths, cycles, trees and connectedness and established some of their properties. Bhattacharya [3] has established some connectivity concepts regarding fuzzy cutnodes and fuzzy bridges. First we give the definitions of basic concepts of fuzzy graphs.

Definition 1.1
A fuzzy graph G = (V, σ, μ) is a non-empty set V together with a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0, 1]$ such that for all u, v in V, $\mu(u, v) \le \sigma(\mu) \wedge \sigma(v)$. We call σ the fuzzy vertex set of G and μ the fuzzy edge set of G, respectively. Note that μ is a fuzzy relation on σ.

Definition 1.2
The fuzzy graph H : (τ, ν) is called a fuzzy subgraph of G: (σ, μ) induced by P if $P \subseteq V$, $\tau(u) = \sigma(u)\ \forall\ u \in P$ and $\nu(u, v) = \mu(u, v)\ \forall\ u, v \in P$.

Definition 1.3
A path P in a fuzzy graph G : (σ, μ) is a sequence of distinct nodes $u_0, u_1, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $1 \le i \le n$. Here $n \ge 1$ is called the length of the path P. A single node u may also be considered as a path. In this case the path is of length 0. The consecutive pairs (u_{i-1}, u_i) are called arcs of the path. We call P a cycle if $u_0 = u_n$ and $n \ge 3$.

Definition 1.4
An arc (u, v) is a fuzzy bridge of G : (σ, μ) if the deletion of (u, v) reduces the strength of connectedness between some pair of nodes. Equivalently (u, v) is a fuzzy bridge if and only if there are nodes x, y such that (u, v) is a arc of every strongest
x-y path.

Definition 1.5
A strongest path joining any two nodes u and v is that path which has strength $\mu^{\infty}(u,v)$ and $\mu^{\infty}(u,v)$ is called the strength of connectedness between u and v.

Definition 1.6
A node is a fuzzy cutnode of G : (σ, μ) if removal of it reduces the strength of connectedness between some other pair of nodes. Equivalently w is a fuzzy cutnode if and only if there exist u, v distinct from w such that w is on every strongest u-v path.

Example 1.1

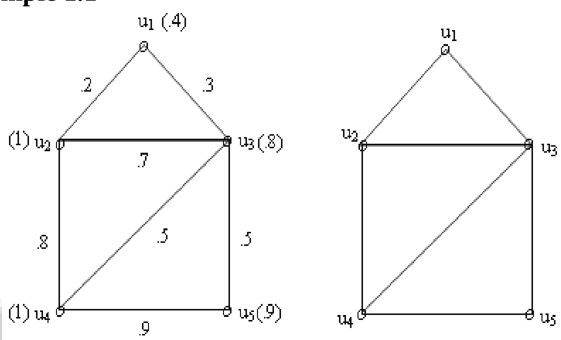

Figure 1.1

A Fuzzy Graph Crisp Graph
G : (σ, μ) G* : (σ*, μ*)
In the above figure (u_1, u_3), (u_3, u_2), (u_2, u_4) and (u_4, u_5) are the fuzzy bridges and u_2, u_3, u_4 are the fuzzy cutnodes of G : (σ, μ).

Definition 1.7
A fuzzy subgraph (τ, ν) spans the fuzzy graph (σ,μ) if σ = τ and $V(u, v) = \begin{cases} \mu(u,v) & \text{if } (u,v) \in v^* \\ 0 & \text{otherwise} \end{cases}$. In this case we call (τ, ν) a fuzzy spanning subgraph of G : (σ, μ).

Definition 1.8
A fuzzy graph G: (σ, μ) is connected if any two nodes are joined by a path. A fuzzy graph G : (σ, μ) is connected if

and only if $\mu^{\infty}(u, v) > 0 \ \forall\ u, v \in V$. also, in a (crisp) graph each path is a strongest path with strength 1.

Definition 1.9

A connected fuzzy graph G : (σ, μ) is a fuzzy tree if it has a fuzzy spanning subgraph F : (σ, v) which is a tree, where for all arcs (u, v) not in F, $\mu(u, v) < v^{\infty}(u, v)$.

Example 1.2

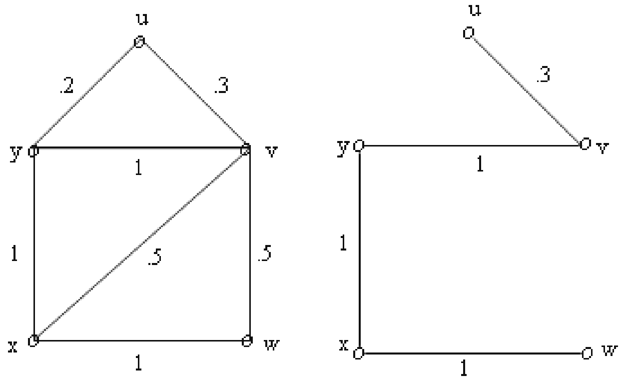

Figure 1.2

Fuzzy tree G:(σ, μ) Spanning Subgraph F:(σ, v)

In the above figure

$\mu(u, y) = .2 < .3 = v^{\infty}(u, y)$,

$\mu(u, w) = .5 < 1 = v^{\infty}(v, w)$ and

$\mu(v, x) = .5 < 1 = v^{\infty}(v, x)$.

Definition 1.10

Let G : (σ, μ) be a fuzzy graph such that G* is a cycle. Then G is called a fuzzy cycle if it has more than one weakest arc.

2. Fuzzy Bridges and Fuzzy Cutnodes

The notation of strength of connectedness plays a significant role in the structure of fuzzy graphs. When a fuzzy bridge (fuzzy cutnode) is removed from a fuzzy graph, the strength of connectedness between some pair of nodes is reduced rather than a disconnection as in the crisp case. If G is a fuzzy graph such that G* is cycle, then all arcs except the weakest are fuzzy bridges.

Theorem 2.1

Let G : (σ, μ) be a fuzzy graph and let (u, v) be a fuzzy bridge of G. Then $\mu^{\infty}(u, v) = \mu(u, v)$.

Proof

Suppose that (u,v) is a fuzzy bridge and that $\mu^{\infty}(u,v)$ exceeds μ(u,v). Then there exists a strongest u-v path with strength greater than μ(u,v) and all arcs of this strongest path have strength greater than μ(u,v). This path together with the arc (u, v) forms a cycle in which (u,v) is the weakest arc, contradicting that (u, v) is a fuzzy bridge.

Characterization of fuzzy graphs

We characterize fuzzy cutnodes in G such that G* is a cycle and then present a sufficient condition for a node to be a fuzzy cutnode.

Theorem 2.2

Let G: (σ, μ) be a fuzzy graph such that G* is a cycle. Then a node is a fuzzy cutnode of G if and only if it is a common node of two fuzzy bridges. Let w be a fuzzy cutnode of G. Then there exist u and v, distinct from w, such that is on every strongest u-v path which is unique since G* is a cycle and it follows that all its arcs are fuzzy bridges. Thus w is a common node of two fuzzy bridges. Conversely, let w be a common node of two fuzzy bridges (u, w) and (w, v). Then both (u, w) and (w, v) are not the weakest arcs of G. Also the path from u to v not containing the arcs (u, w) and (w, v) has strength less than $\mu(u,w) \wedge \mu(w, v)$. Thus the strongest u-v path is the path u, w, v and $\mu^{\infty}(u, v) = \mu(u, w) \wedge \mu(w, v)$. Hence w is a fuzzy cutnode.

Proposition 2.1 [4] : If G : (σ, μ) is a complete fuzzy graph then $\mu^{\infty}(u, v) = \mu(u, v)$

Proposition 2.2 [4]: A complete fuzzy graph has no fuzzy cutnodes.

Remark 2.1 From proposition 2.1 we have in a complete fuzzy graph that each arc (u, v) is a strongest u-v path. But the converse does not hold as we see in the below figure. Also it follows from proposition 2.2 that if in G = (σ, μ), $\mu^{\infty}(u,v) = \mu(u,v)$ for all u, v then G has no fuzzy cutnodes.

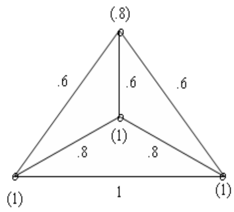

Figure 1.3

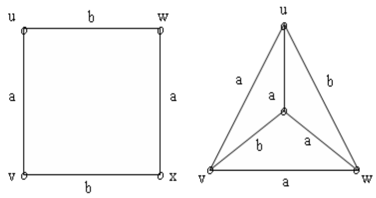

Figure 1.4

Note that a fuzzy graph with fuzzy bridge need not have fuzzy nodes (fig 1.4) and a complete fuzzy graph has no fuzzy cut nodes (proposition 2.2) but we have.

Theorem 2.3

A complete fuzzy graph has atmost one fuzzy bridge.

Proof

Let G: (σ, μ) be a complete fuzzy graph with | V | = 3. Then G can have atmost one fuzzy bridge. Now, let | V | ≥ 4 and let u_1, u_2, u_3 and u_4 be any four nodes of G. without loss of generality, let u_1 be such that $\sigma(u_1)$ in least among $\sigma(u_i)$'s i = 1, 2, 3, 4. Then (u_1, u_2) (u_1, u_3) and (u_1, u_4) are not fuzzy bridges, they bring the weakest arcs of some cycle in the fuzzy subgraph induced by u_1, u_2, u_3, u_4 Now the arcs (u_2, u_3), (u_2, u_4) and (u_3, u_4) are adjacent to each other and it follows that atmost one of them can be a fuzzy bridge. We also have

Theorem 2.4

Let G: (σ, μ) be a complete fuzzy graph with | V | = n. Then G has a fuzzy bridge if and only if there exists an increasing sequence $\{t_1, t_2, \ldots, t_{n-1}, t_n\}$ such that $t_{n-2} < t_{n-1} \leq t_n$ where $t_i = \sigma(u_i)\ \forall$ i = 1, 2, 3, ….n. Also, the arc (u_{n-1}, u_n) is the fuzzy bridge of G.

Proof

Assume that G: (σ, μ) is a complete fuzzy graph and that G has a fuzzy bridge (u, v). Now $\mu(u, v) = \sigma(u) \wedge \sigma(v)$. Without loss of generality, let $\sigma(u) \le \sigma(v)$, so that $\mu(u, v) = \sigma(u)$. Also, note that (u, v) is not a weakest arc of any cycle in G. Now we have to prove that $\sigma(u) > \sigma(w)\ \forall\ w \neq v$. On the contrary assume that there is atleast one node $w \neq v$ such that $\sigma(u) \le \sigma(w)$. Now consider the cycle C : u, v, w, u. Then $\mu(u, v) = \mu(u, w) = \sigma(u)$ and

$$\mu(v, w) = \begin{cases} \sigma(v), \text{if } \sigma(u) = \sigma(v) \text{ or if } \sigma(u) < \sigma(v) \le \sigma(w) \\ \sigma(w) \text{ if } \sigma(u) < \sigma(w) < \sigma(v). \end{cases}$$

In either case the arc(u, v) becomes a weakest arc of the cycle which contradicts out assumption that (u,v) is a fuzzy bridge.

Conversely, let $t_1 \le t_2 \le \dots < t_{n-2} < t_{n-1} \le t_n$ and $t_i = \sigma(u_i)\ \forall$ i. Now we have to prove that arc (u_{n-1}, u_n) is the fuzzy bridge of G. Now, $\mu(u_{n-1}, u_n) = \sigma(u_{n-1}) \wedge \sigma(u_n) = \sigma(\mu_{n-1})$ and clearly by hypothesis, all other arcs of G will have strength strictly less than $\sigma(\mu_{n-1})$. Thus the arc (μ_{n-1}, μ_n) is not a weakest arc of any cycle in G and hence it is a fuzzy bridge.

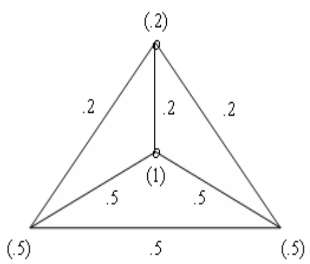

Figure 1.5

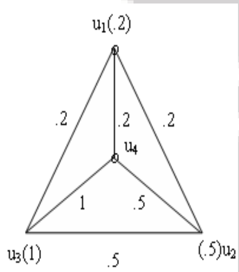

Figure 1.6

Fig 1.5 and Fig 1.6 are complete fuzzy graph where Fig 1.5 has no fuzzy bridges. The increasing sequence $\{t_i\}$ in fig 1.6 is {.2, .5, 1, 1} and $\{u_3, u_4)$ is the fuzzy bridge. Based on the concept of maximum spanning tree of a fuzzy graph, we present a characterization of fuzzy bridge and fuzzy cutnode. Also, in a (crisp) graph G*, each spanning tree is a maximum spanning tree. The following are characterizations of fuzzy bridge and fuzzy cut node.

Theorem 2.5
An arc (u, v) is a fuzzy bridge of G : (σ, μ) if and only if (u, v) is in every maximum spanning tree of G.

Proof

Let (u, v) be a fuzzy bridge of G. Then arc (u, v) is the unique strongest u-v path and hence it is in every maximum spanning tree of G. Conversely, let (u,v) be in every maximum spanning tree T of G and assume that (u, v) is not a fuzzy bridge. Then (u, v) is a weakest arc of some cycle in G and $\mu^{\infty}(u, v) > \mu(u,v)$ which implies that (u, v) is in no maximum spanning tree of G.

Remark 2.2

From the above theorem, it follows that arcs not in T are not fuzzy bridges of G and we have the following corollary. **Corollary:** If G : (σ, μ) is a connected fuzzy graph with $|V| = n$ then G has atmost n-1 fuzzy bridges.

Theorem 2.6

A node w is a fuzzy cutnode of G: (σ, μ) if and only if w is an internal node of every maximum spanning tree of G.

Proof

Let w be a fuzzy cutnode of G. Then there exist u, v distinct from w such that w is an every strongest u-v path. Now each maximum spanning tree of G contains unique strongest u-v path and hence w is an internal node of each maximum spanning tree of G.

Conversely, let w be an internal node of every maximum spanning tree. Let T be a maximum spanning tree and let (u,w) and (w,v) be arcs in T. Note that the path u, w, v is a strongest u-v path in T. If possible assume that w is not a fuzzy cutnode. Then between every pair of nodes u, v, there exist atleast one strongest u-v path not containing w. Consider one such u-v path P which clearly contains arcs not in T. Now, without loss of generality, let $\mu^{\infty}(u, v) = \mu(u,w)$ in T. Then arcs in P have strength $\ge \mu(u,w)$. Removal of (u,w) and adding P in T will result in another maximum spanning tree of G for which w is an end node, which contradicts our assumption.

Remark 2.3
From the above theorem, it follow that the end nodes of maximum spanning tree T of G are not fuzzy cutnodes of G. This results in the following corollary

Corollary

Every fuzzy graph has atleast two nodes which are not fuzzy cut nodes of G.

3. Conclusion

Thus in this paper, thus characterize fuzzy cutnodes (fuzzy bridges) and we derived the sufficient condition for a node to be a fuzzy cutnode.

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