Existence Results for Quasilinear Delay Integrodifferential Equations with Nonlocal Conditions Via Measures of Noncompactness

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Abstract: We study the existence of mild solutions for quasilinear delay integrodifferential equations with nonlocal Cauchy problem in Banach spaces. The results are established by using Hausdorff’s measure of non-compactness.

Keywords: Mild solution, nonlocal conditions; Hausdorff’s measure of noncompactness

1. Introduction

The notion of a measure of non-compactness turns out to be a very important and useful tool in many branches of mathematical analysis. The notion of a measure of weak compactness was introduced by De Blasi [9] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations. El-Sayed [11] proves the existence theorem of monotonic analysis and the theory of differential and integral equations. The study of abstract nonlocal initial value problems was initiated by Byszewski [6, 7]. Balachandran and Paul Samuel [3] give the existence and uniqueness of mild and classical solutions for quasi linear delay integrodifferential equations in Banach space has been studied by several authors [3, 8, 10, 14]. Pazy [14] considered the following problem of existence of solutions of quasi linear evolution equations in Banach space has been studied by several authors [3, 8, 10, 14]. Pazy [14] considered the following problem of existence of solutions of quasi linear evolution equations in Banach space has been studied by several authors.

2. Preliminaries

Let $X$ be a Banach space with norm $|| \cdot ||$. Let $C([0,T];X)$ be the space of $X$-valued continuous functions on $[0,T]$ with the norm $\|u\| = \sup_{t \in [0,T]} \|u(t)\|$ for $u \in C([0,T];X)$, and denoted $L(0,T;X)$ by the space of $X$-valued Bochner integrable functions on $[0,T]$ with the norm $\|u\|_L = \int_0^T \|u(t)\|dt$. The Hausdorff’s measure of non compactness $\chi Y$ is defined by $\chi Y = \inf \{r > 0, B \text{ can be covered by finite number of balls with radii } r\}$ for bounded set $B$ in a Banach space $Y$.

Lemma 2.1 [4]. Let $Y$ be a real Banach space and $B, E \subseteq Y$ be bounded, with the following properties:

1) $B$ is pre compact if and only if $\chi B = 0$.
2) $\chi Y \subseteq \chi \overline{\chi B} \subseteq \chi \overline{\chi Y}$, where $\overline{B}$ means the closure and convex hull of $B$ respectively.
3) $\chi Y \leq \chi (E)$, where $B \subseteq E$.
4) $\chi (B + E) \leq \chi B + \chi E$, where $B + E = \{x + y : x \in B, y \in E\}$.
5) $\chi (B \cup E) \leq \max \{\chi B, \chi E\}$.
6) $\chi (\lambda B) \leq ||\lambda|| \chi B$ for any $\lambda \in \mathbb{R}$.
7) If the map $T: D(T) \subseteq Y \rightarrow Z$ is Lipschitz continuous with constant $k$ the $\chi T(B) \leq k \chi T(Y)$ for any bounded subset $B \subseteq D(T)$, where $Z$ is Banach space.
8) $\chi Y \leq \inf \{d_Y(B, E) : E \subseteq Y\}$ is finite valued, where $d_Y(B, E)$ means the nonsymmetric (or symmetric) Hausdorff distance between $B$ and $E$ in $Y$.
9) If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of $Y$ and $\lim_{n \rightarrow +\infty} \chi Y(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in $Y$.

\[ u(t) + A(t, u(t)) = f(t, u(t)), \]
\[ u(0) = u_0, \]
Lemma 2.2 (Darbo-Sadovskii [4]). If $W \subseteq C([0,T];X)$ is bounded, closed and convex, the continuous map $F : W \to W$ is a $\chi$-contraction, then the map $F$ has at least one fixed point in $W$. In this we denote by $\chi$ the Hausdorff’s measure of non-compactness of $X$ and denote $\overline{\chi}$ by the Hausdorff’s measure of non compactness of $C([a,T];X)$. To discuss the existence, we need the following Lemmas in this paper.

Lemma 2.3 [4]. If $W \subseteq C([0,T];X)$ is bounded, Then $\chi(W(t)) \leq \chi(W)$ for all $t \in [0,T]$, where $W(t) = \{u(t); \ u \in W\} \subseteq X$. Furthermore if $W$ is equi-continuous on $[a,T]$, then $\chi(W(t))$ is continuous on $[a,T]$ and $\chi_{a}(W) = \sup\{\chi(W(t)), t \in [a,T]\}$.

Lemma 2.4 [13]. If $\{u_{n}\}_{n=1}^{\infty} \subseteq L^{1}(a,T;X)$ is uniformly integrable, then the function $\chi(\{u_{n}\}_{n=1}^{\infty})$ is measurable and

$$\chi(\{\int_{0}^{t}u_{n}(s)ds\}) \leq 2 \int_{0}^{t} \chi(\{u_{n}\}_{n=1}^{\infty})ds$$

for all $t \in [0,T]$, where $\int_{0}^{t}u_{n}(s)ds = (\int_{0}^{t}u(s)ds; u \in W)$.

The $C_{0}$ - semigroup $U_{t}, (t,s)$ is said to be equi continuous for $T > 0$ for all bounded set $B$ in $X$. The following lemma is obvious.

Lemma 2.6 If the evolution family $\{U_{t}(t,s)\}_{t \in [0,T]}$ is equi-continuous and $\eta \in L(0,T;R^{+})$, then the set $\{\int_{0}^{t}U_{s}(t,s)u(s)ds\}, ||u(s)|| \leq \eta(s)$ for i.e $s \in [0,T]$ is equi-continuous for $t \in [0,T]$.

From [8], we know that for any fixed $\in C([0,T];X)$, there exist a unique continuous function $U_{u_{0}}: [0,T] \times [0,T] \to B(X)$ defined on $[0,T] \times [0,T]$ such that

$$U_{u_{0}}(t,s) = \int_{s}^{t}A_{u_{0}}u_{0}(w,s)dw$$

where $B(X)$ denote the Banach space of bounded linear operators from $X$ to $X$ with the norm $||F|| = sup(||F(u)||; ||u|| = 1)$, and $I$ stands for the identity operator on $X$. $A_{u}(t) = A(t,u(t))$. We have $U_{u_{0}}(t,s) = I_{t, t_{0}}u_{0}(r,s) = U_{u_{0}}(r,s)$, where $(t,s) \in [0,T] \times [0,T]$ and $A_{u_{0}}(t)u_{0}(t,s)$ for almost all $t \in [0,T]$. For a mild solution of (1) - (2) we mean a function $u_{0} \in C([0,T];X)$ and $u_{0}$ $\in X$ satisfying the integral equation

$$u(t) = \int_{0}^{t}u_{0}(0)0h(u) + \int_{0}^{t}u_{0}(t,s)\{g(s,u(\alpha(s))) + \int_{0}^{t}g(s,\tau, u(\beta(\tau)))d\tau\}ds$$

3. The Existence of Results for Compact

In this section, we give some existence results when his compact and $f$ satisfies the conditions with respect to Hausdorff’s measure of non-compactness and its applications in differential equations in Banach spaces. We give some existence results of the nonlocal problem (1)- (2). We assume the following assumptions:

(H1) The evolution family $\{U_{u_{0}}(t,s)\}_{t \in [0,T]}$ generated by $A(t,u)$ is equi-continuous, and $||U_{u_{0}}(t,s)|| \leq M_{p}$ for almost $t,s \in [0,T]$

(H2) $f : [0,T] \times X \to X$ satisfies the Caratheodory type conditions and there exist $m_{1} \in L[0,T]$ and $b_{1} \geq 0$ such that

$$||f(t,u)|| \leq m_{1}(t)||u||, t.a.e.\ \in [0,T], u \in R^{+}$$

(H3) (a) $h : C([0,T];X) \to X$ is continuous and compact.

(b) There exist $N > 0$ such that $||h(u)|| \leq N_{0}$ for all $u \in C([0,T];X)$.

(H4) (a) $g : [0,T] \times [0,T] \times X \to X$ satisfies the Caratheodory-type conditions, (i.e.) $g(.,.,u)$ is measurable for all $u \in X$ and $g(t,s,.)$ is continuous for a.e $t,s \in [a,T]$, there exist $m_{2} \in L[0,T]$ and $b_{2} \geq 0$ such that

$$\int_{0}^{t}g(s,t,u)ds \leq \int_{0}^{s}m_{2}(t,s)b_{2}||u||$$

for a.e $\in [0,T], u \in R^{+}$

(b) There exist $\eta \in L(0,T;R^{+}), \xi \in L(0,\infty)$, such that $\eta(t) = \eta(t)(\xi)D(\eta)$ for $a.e t,s \in [0,T]$, and for any bounded subset $D \subseteq C([0,T],X)$. Herefore let $\eta(t) \leq K_{1}$ and $\xi(t) \leq K_{2}$.

(H5) $\alpha, \beta : [0,T] \to [0,T]$ is a non decreasing and there exist positive constants $\alpha_{1}$ and $\beta_{2}$ such that $\alpha(t) \geq \alpha_{1}$ and $\beta(t) \geq \beta_{2}$ respectively for $t \in [0,T]$.

Theorem: 3.1 Under the assumptions (H1) - (H2) are satisfied, then the nonlocal initial value problem. Let $u_{0} \in Y$ and the family $A(t,b)$ of linear operators for $t \in [0,T]$ and (1) - (2) has at least one mild solution.

Proof.

Let $\Omega(t)$ be a solution of the scalar equation

$$\Omega(t) = M_{h}N_{0} + M_{h}m_{1}(t)b_{1}/\alpha_{1} + m_{2}(t,s)b_{2}/\beta_{2} \int_{0}^{t}\Omega(s)ds$$

for $t \in [0,T]$.

Consider the map $F : C([0,T];X) \to C([0,T];X)$ defined by

$$(Fu)(t) = U_{0}(t,0)0h(u) + \int_{0}^{t}U_{t}(t,s)\{f(s,u(\alpha(s))) + \int_{0}^{t}g(s,\tau, u(\beta(\tau)))d\tau\}ds$$

for all $u \in C([0,T];X)$. We can show that $F$ is continuous by the usual techniques.

Let us take

$$W_{0} = \{u \in C([0,T];X) ||u(t)|| \leq \Omega(t), t \in [0,T]\}$$

Then $W_{0} \subseteq C([0,T];X)$ is bounded and convex. We define $W_{1} = \text{conv}(W_{0})$, where $\text{conv}$ means the closure of the convex hull in $C([0,T];X)$. As $U_{0}(t,s)$ is equi-continuous, $g$ is compact and $\Omega_{0} \subseteq C([0,T];X)$ is bounded, due to Lemma 2.7 and the assumption (H4)(b), $W_{1} \subseteq C([0,T];X) is bounded closed convex non empty and equi-continuous on $[0,T]$. 

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For any $u \in F(W_0)$, we know
\[
||u(t)|| \leq M_0 N_0 + M_0 \int_0^t \left| f(s, u(\alpha(s))) \right| ds + M_0 \int_0^t \left| g(s, \tau, u(\beta(\tau))) \right| d\tau ds
\]
\[
\leq M_0 N_0 + M_0 \int_0^t m_1(s)b_1||u(\alpha(s))||ds
+ M_0 \int_0^t m_2(s, \tau)b_2||u(\beta(\tau))||d\tau ds
\]
\[
\leq M_0 N_0 + M_0 m_1(t)b_1\int_0^t ||u(\alpha(s))||a'(s)/\delta_1 ds
+ M_2(s, \tau)b_2\int_0^s ||u(\beta(\tau))|||\beta'(\tau)/\delta_2 d\tau ds
\]
\[
\leq M_0 N_0 + M_0 m_1(t)b_1/\delta_1 + M_2(s, \tau)b_2/\delta_2 \int_0^t \Omega(s)ds,
=\Omega(t), \text{ for } t, s, \tau \in [0, T].
\]

It follows that $W_1 \subset W_0$. We define $W_{n+1} = \overline{co W_n}$, for $n = 1, 2, 3, \cdots$. From above we know that $W_n$ is a decreasing sequence of bounded, closed, convex, equicontinuous on $[0, T]$ and non empty subsets of $C([0, T]; X)$.

Now for $n \geq 1$ and $t \in [0, T], W_n(t)$ and $F(W_n(t))$ are bounded subsets of $X$, hence, for any $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset W_n$ such that (see, e.g.[5], pp125).

\[
\chi(W_{n+1}) = \chi(FW_n(t))
\]
\[
\leq \chi(\int_0^t U_u(t, s)f(s, u_k(u(s)))_{k=1}^{\infty}) ds
+ \int_0^t g(s, \tau, u_k(\beta(\tau)))_{k=1}^{\infty} d\tau ds
\]
\[
\leq 2M_0 \int_0^t \chi(f(s, u_k(u(s))))_{k=1}^{\infty} d\tau ds
+ 4M_0 \int_0^t \chi(g(s, \tau, u_k(\beta(\tau))))_{k=1}^{\infty} d\tau ds
\]
\[
\leq 2M_0 K_1 \int_0^t \chi((u_k(\alpha(s))))_{k=1}^{\infty} ds
+ 4M_0 K_2 \int_0^t \chi((u_k(\beta(\tau))))_{k=1}^{\infty} d\tau ds
\]
\[
\leq 2M_0 K_1/\delta_1 \int_0^t \chi((u_k(s)))_{k=1}^{\infty} ds
+ 4M_0 K_2/\delta_2 \int_0^t \chi((u_k(s)))_{k=1}^{\infty} ds + \varepsilon
\]
\[
= (2M_0 K_1 \frac{1}{\delta_1} + \frac{2K_2}{\delta_2}) \int_0^t \chi((u_k(s)))_{k=1}^{\infty} ds + \varepsilon
\]

Since $\varepsilon > 0$ is arbitrary, it follows that from the above inequality that
\[
\chi(W_{n+1}(t)) \leq (2M_0 K_1 \frac{1}{\delta_1} + \frac{2K_2}{\delta_2}) \int_0^t \chi(W_n(s)) ds + \varepsilon
\]
for all $t \in [0, T]$. Because $W_n$ is decreasing for $n$, we have
\[
\sigma(t) = \lim_{n \to \infty} \chi(W_n(t))
\]
for all $t \in [0, T]$. From (9), we have
\[
\sigma(t) \leq (2M_0 K_1 \frac{1}{\delta_1} + \frac{2K_2}{\delta_2}) \int_0^t \sigma(s) ds
\]
for $t \in [0, T]$, which implies that $\sigma(t) = 0$ for all $t \in [0, T]$. By Lemma 2.3, we know that $\lim_{n \to \infty} \chi(W_n(t)) = 0$.

Using Lemma 2.1 we know that $W = \bigcap_{m=1}^{\infty} W_n$ is compact and nonempty $C([0, T]; X)$ and $F(W) \subset W$. By the famous Schauder’s fixed point theorem, there exist at least one mild solution $u$ of the initial value problem (1) – (2), where $u \in W$ is a fixed point of the continuous map $F$.

**Remark 3.2.** If the functions $f$ and $g$ are compact Lipschitz continuous (see e.g. [6, 7, 15], then (H4) is automatically satisfied. In some of the early related results in references and above results, it is supposed that the map $h$ is uniformly bounded. We indicate here that this condition can be released. In fact, if $h$ is compact, then it must be bounded on bounded set. Here we give an existence result under growth condition of $f$ and $g$, when $h$ is not uniformly bounded. Precisely, we replace the assumptions (H4) by

\[
(H6) \text{ There exist a functions } p \in L(0, T; R^+) \text{ and } q \in L(0, T; R^+). \text{ The constants } b_1, b_2 > 0 \text{ such that}
\]
\[
||f(t, u)|| \leq p(t)b_1||u||,
\]
\[
\int_0^t ||g(t, s, u)|| ds \leq q(t)b_2||u||
\]
for $t \in [0, T]$ and all $u \in C([0, T]; X)$.

**Theorem 3.2.** Suppose that the assumptions (H1) – (H6) are satisfied, then the equation (1) – (2) has at least one mild solution if

\[
\lim sup_{r \to \infty} \frac{M_0}{r} \left( \varphi(r) + rT \frac{p b_1 + q b_2}{\delta_1 + \delta_2} \right) < 1. (10)
\]
Where $\varphi(r) = \sup(\|h(u)\|, ||u|| \leq r)$. 

**Proof.** The inequality (10) implies that there exist a constant $r > 0$ such that
\[
M_0 (\varphi(r) + rT \frac{p b_1 + q b_2}{\delta_1 + \delta_2}) < r.
\]
Just as in the proof of Theorem 3.1, let $W_0 = \{u \in C([0, T]; X), ||u(t)|| \leq r\}$ and $W_1 = \overline{co W_0}$. Then for any $u \in W_1$, we have
\[
||u(t)|| \leq ||U_0(t, 0) h(u)|| + \int_0^t U_0(t, s) f(s, u(a(s))) ds + \int_0^t g(s, \tau, u(\beta(\tau))) d\tau ds,
\]
\[
\leq (2M_0 K_1 \frac{1}{\delta_1} + \frac{2K_2}{\delta_2}) \int_0^t \chi((u_k(s)))_{k=1}^{\infty} ds + \varepsilon
\]
4. The Existence of Results for Lipschitz

In the previous section, we obtained the existence results when \( h \) is compact but without the compactness of \( U_0(t,s) \) is satisfied, then the equation \( (1) \) - \( (2) \) has at least one mild solution provided that for \( t \in [0,T] \). It means that \( W_1 \subset \subset W_0 \). So, we can complete the proof similarly to Theorem 3.1.

\[
\|h(u) - h(v)\| \le L_0 \|u - v\|, \; u, v \in C([0,T];X).
\]

**Theorem:** 4.1 Suppose that the assumptions \( H_1 \), \( H_2 \), \( H_4 \), \( H_6 \) are satisfied, then the equation \( (1) \) - \( (2) \) has at least one mild solution provided that

\[
(M_0L_0 + TM_0K_1 \frac{1}{\delta_1} + \frac{2K_2}{\delta_2}) < 1. \quad (11)
\]

**Proof.** Consider the map \( F_1, F_2 : C([0,B]; X) \rightarrow C([0,B];X) \) defined by \( F_1 + F_2 = F \), where

\[
F_1(u)(t) = U_0(t,0)h(u),
\]

\[
F_2(u)(t) = \int_0^t U_0(t,s) f(s, u(\alpha(s))) ds + \int_0^s g(s, \tau, u(\beta(\tau))) d\tau ds,
\]

for \( u \in C([0,B]; X) \). As defined in the proof of Theorem 3.1. We define \( W_0 = \{ u \in C([0,B]; X) : \|u(t)\| \le \Omega(t) \text{ for all } t \in [0,T] \} \) and let \( W = \overline{C_{00}F(W)} \). Then from the proof of Theorem 3.1 we know that \( W \) is an abounded closed convex and equi continuous subset of \( C([0,T];X) \) and \( F \subset W \). We shall prove that \( F \) is a contraction on \( W \). Then Darbo-Sadovskii's fixed point theorem can be used to get a fixed point of \( F_{\infty} \), which is a mild solution of \( (1) \) - \( (2) \). First, for every bounded subset \( B \subset W \), from the \( (H_7) \) and \( (H_6) \), we have

\[
\|x(\varphi(t))\| \le M_0 + TM_0K_1 \left( \frac{1}{\delta_1} + \frac{2K_2}{\delta_2} \right) < 1.
\]

Now, for any subset \( B \subset W \), due to Lemma 2.1, (12) and (13) we have

\[
X_c(B) \le X_e(F_1(B)) + X_e(F_2(B)) \quad \text{(13)}
\]

We know that \( F \) is an contraction on \( W \). By Lemma 2.2, there is a fixed point \( u \) of \( F \) in \( W \), which is a solution of \( (1) \) - \( (2) \). This completes the proof.

**References**


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