Inventory Model Quadratic Back Order Costs, Lead Time Continuous for the (nQ, R, T) Model

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Abstract: ***The paper considers the continuous lead time to be a gamma distribution backorder costs to be quadratic and demand follows a normal distribution. We obtain the equations for the expected backorder costs and average it over the states of lead time L. The quadratic costs $C_\beta(t) = b_1t+b_2t+b_3t^2$ for t the duration of a backorder. The backorder costs are computed for each component of the quadratic cost separately. Use it made of the Bessel function of imaginary argument in averaging the backorder costs over the states of the lead time. The on hand inventory is also derived the inventory cost is obtained by summing the expected backorder costs and the on hand inventory costs, over the states of the lead times.***

Keywords: Lead Time, Quadratic Backorder Cost, Gamma Distribution, Normal Distribution, Bassel Functions, Inventory on hand, Expected Backorder Costs.

1. Introduction

The paper considers the (nQ, R, T) inventory model with quadratic backorder costs and a continuous lead time that follows a gamma distribution. The demand follows a normal distribution. The expected backorder costs and on hand inventory costs are derived separately for each component of the quadratic backorder costs. Bassel's function of imaginary argument is made use to evaluate extensively in the integrals.

2. Literature Review

Patria (2012) used annual costs to derive expression for the EOQ using price dependent demand in quadratic form. Bertimas (1990)in his paper probabilistic service level guarantee in make-to-stock, considered both linear and quadratic inventory costs and backorder costs. Nasir (2012) utilized an EOQ model with no linear holding cost. Hadley and Whitin (1972) extensively developed the (nQ, R, T) model for constant lead time and linear backorder costs. Zipkin (2006), treat both fixed and random lead times and examines both stationery and limiting distributions under different assumptions.

Probability density function of L, lead time

$$H(L) = \frac{\propto^k L^{k-1} e^{\propto L}}{\Gamma(k)} \quad \propto, L, k > 0 \ (1)$$

$$X \sim N(DL, \sigma^2 L), g(x) = esp - \frac{1}{2}\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)^2 - \infty < x < \infty \ (2)$$

$$C_B(t) = \text{cost of a backorder} = b_1 + b_2 t + b_3 t^2$$

In the inventory position of the system is R+Y immediately after the review at time t, then the expected backorder costs at times t + L, where L is the lead time.

$$= \frac{1}{Q}\int_0^Q D \int_0^L D \int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdy$$

Arrangement over the states of L we have

$$\frac{1}{Q}\int_0^\infty \int_0^Q D \int_0^L D\, H(L) \int_0^t \frac{C_B(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdydL \ (3)$$

For the order placed at time t + T, the lead time is L with probability H (L)

Hence expected backorder cost averaging over the states of L

$$= \frac{1}{Q}\int_0^\infty D \int_0^Q D \int_0^{L+T} \int_0^t H(L) \frac{C_B(t-z)}{\sqrt{\sigma^2 t}}\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdydL \ (4)$$

Substituting for C_B(t-z) from (2) we have

$$\frac{1}{Q}\int_0^\infty \int_0^Q D \int_0^{L_1} D \int_0^t H(L)(b_1 + b_2(L-z) + b_3(L_1 - z)^2)\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdydL \ (5)$$

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and

$$\frac{1}{Q}\int_0^\infty \int_0^Q D \int_0^{L+T} D \int_0^t H(L)(b_1 + b_2(L-z) + b_3(L_2 - z)^2)\, g\left(\frac{R+Y-Dz}{\sqrt{\sigma^2 t}}\right) dzdtdydL \ (6)$$

Hence the expected backorder cost per cycle

$$= \frac{1}{Q}\int_0^\infty \int_0^\infty \int_0^Q D \int_L^{L+T} D \int_0^t \frac{H(L\ \ )H(L)}{\sqrt{\sigma^2 t}}(b_1 + b_2(t-z) + b_3(t-z)^2)\, dzdtdydL \ (7)$$

Integrating with respect to dz, dt, dy in that order we obtain the expected backorder cost for a given L and L the b_1 factor gives

$$\left[\frac{b_1\sigma^2(T+L)D}{2DQ}\left(1+\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)^2\right)F\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)-\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)\right.$$
$$\left.g\left(\frac{R-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)\right)-\frac{b_1\sigma^2(T+L)D}{2DQ}\left[\left(1+\left(\frac{R+Q-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)^2\right)\right.$$
$$\left.F\left(\frac{R+Q-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)-\left(\frac{R+Q-D(T+L)}{\sqrt{\sigma^2(T+L)}}\right)g\left(\frac{R+Q-D(T+L)}{\sqrt{\sigma^2 L}}\right)\right]$$
$$-\left[\frac{b_1\sigma^2 LD}{2DQ}\left(\left(1+\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$$
$$\left.-\frac{b_1\sigma^2 LD}{2DQ}\left(\left(1+\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2\right)F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right] \ (8)$$

Which can written as

$$G_1(R,T+L) - G_1(R,L) - G_1(R+Q,T+L) + G_1(R+Q,L)) \ (9)$$

Where

$$G_1(R,L) = \frac{\sigma^2 L}{2}\left(1+\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

For the b_2 factor for a given L

$$\frac{b_2}{Q}\left(\frac{D^2L^3}{6}-\frac{\sigma^4 R}{6D^3}-\frac{DL^2R}{2}-\frac{\sigma^2R^2}{4D^2}+\frac{\sigma^2L^2}{4}+\frac{LR^2}{2}-\frac{R^3}{6D}-\frac{\sigma^6}{8D^4}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$
$$+b_2\left(\sqrt{\sigma^2 L}\left(\frac{DL^2}{6}-\frac{LR}{3}+\frac{R^2}{6D}+\frac{\sigma^2 L}{12D}+\frac{\sigma^2 R}{4D^2}+\frac{\sigma^4}{4D^3}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right)$$
$$+\frac{\sigma^6}{8D^4Q}esp\left(\frac{2DR}{\sigma^2}\right)F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) \ (10)$$

Which is written as $\frac{b_2}{Q}G_2(R,L)$

The b_3 factors gives from the integration

$$\frac{b_3 D}{Q}\left(\frac{R^4}{12D^3}+\frac{\sigma^2R^3}{6D^4}+\frac{\sigma^4R^2}{4D^5}+\frac{\sigma^6 R}{4D^6}+\frac{\sigma^8}{8D^7}-\frac{L^2\sigma^2R}{2D^2}+\frac{L^2R^2}{2D}-\frac{RL^3}{3}+\frac{L^3\sigma^2}{3D}-\frac{R^3L}{3D^2}+\frac{L^4D}{12}\right)$$
$$F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\frac{b_{3,}2\sqrt{\sigma^2 L,}D}{Q}g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\left(\frac{\sigma^2RL}{24D^3}+\frac{\sigma^4 R}{24D^4}-\frac{R^2L}{6D^2}-\frac{\sigma^2L^2}{8D^2}+\frac{L^2R}{8D}-\frac{L^3}{24}+\frac{R^3}{24D^3}\right.$$
$$\left.+\frac{\sigma^2R^2}{12D^4}+\frac{\sigma^4R}{8D^5}+\frac{\sigma^6}{8D^6}\right)-\frac{1}{8}\frac{\sigma^8}{D^6}\frac{b_3}{Q}esp\left(\frac{2DR}{\sigma^2}\right)F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right) \ (11)$$

We define $G_3(R,L)$in such a way that $\frac{b_3}{Q}G_3(R,L) =$ equation 11

Hence the expected backorder cost for the b_3 factor for given L

$$= \frac{b_3}{Q}G_3(R,L) - \frac{b_3}{Q}G_3(R+Q,L)$$

Hence substituting into equation 7 the b_1, b_2, b_3 factors, we obtain

$$\frac{b_1}{Q}\int_0^\infty H\,(L)(G_1(R,T+L)-(G_1(R,L)-G_1(R+Q,T+L)+\left(G_1(R+Q,L)\right)dL$$
$$+\frac{b_2}{Q}\int_0^\infty H\,(L)(G_2(R,T+L)-G_2(R,L)-G_2(R+Q,T+L)+\left(G_2(R+Q,L)\right)dL$$
$$+\frac{b_3}{Q}\int_0^\infty H\,(L)\left(G_3(R,T+L)-G_3(R,L)-G_3(R+Q,T+L)+G_3(R+Q,L)\right)dL\ (12)$$

Simplifying we have, the expected backorder cost per cycle to be

$$\frac{b_1}{Q}\int_0^\infty H\,(L)\left(G_1(R,T+L)-G_1(R+Q,T+L)\right)dL$$
$$-\frac{b_1}{Q}\int_0^\infty H\,(L)\left(G_1(R,L)-G_1(R+Q,T+L)\right)dL$$
$$+\frac{b_2}{Q}\int_0^\infty H\,(L)\left(G_2(R,T+L)-G_2(R+Q,T+L)\right)dL$$
$$-\frac{b_2}{Q}\int_0^\infty H\,(L)\left(G_2(R,L)-G_2(R+Q,L)\right)dL$$
$$+\frac{b_3}{Q}\int_0^\infty H\,(L)\left(G_3(R,T+L)-G_3(R+Q,T+L)\right)dL$$
$$-\frac{b_3}{Q}\int_0^\infty H\,(L)(G_3(R,L)-(G_3(R+Q,L)dL\ (13)$$

In order to be able to evaluate th expressions involving L_1 we need to evaluate the follwoing integrals

$G_4(R)=\int_0^\infty H\,(L)G_1(R,L)dL\ (14)$
$G_5(R)(R)=\int_0^\infty H\,(L)(G_2(R,L)dL\ (15)$
$G_6=\int_0^\infty H\,(L)G_3(R,L)dL\ (16)$

From equation (8)

$$G_1(R,L)=\frac{\sigma^2L}{2}\left(\left(1+\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)^2\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)\right.$$

Simplifying

$$G_1(R,L)=\left(\frac{\sigma^2L}{2}+\left(\frac{R^2-2DLR+D^2L^2}{2}\right)\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\frac{(R-DL)\sqrt{\sigma^2L}}{2}g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)\ (17)$$

Multiplying by H (L)

Hence H (L) G_1 (R, L)

$$=\frac{\infty^k e^{-\infty L}}{2\,\Gamma(k)}\left(\sigma^2L^k+R^2L^{k-1}-2RDL^k+D^2L^{k+1}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)-\left(RL^{k-\frac{1}{2}}-DL^{k+\frac{1}{2}}\right)\sigma g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)dL\ (18)$$

Substituting into 14 we have

$$G_4(R)=\frac{\propto^k}{2}\ \frac{1}{\Gamma(k)}\int_0^\infty\left(R^2L^{k-1}+(\sigma^2+2RD)L^k+D^2L^{k+1}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)dL\ (19a)$$
$$-\frac{\propto k\,\sigma}{\Gamma(k)}\int_0^\infty\left(RL^{k-\frac{1}{2}}-DL^{k+\frac{1}{2}}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)dL$$

Integrating and applying $\sigma^2=2\infty\,\sigma^2+D^2$ and noting that

$$(1)\ \int_0^\infty H\,(L)F\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)dL=\frac{\propto^k}{\Gamma(k)}\int_0^\infty L^{k-1}\,esp\,(-\propto L)F\left(\frac{x-DL}{\sqrt{\sigma^2L}}\right)dL$$
$$=\frac{\propto^k}{\sqrt{2\pi}\,2\sigma\ \Gamma(k)}\,esp\,\left(\frac{Dx}{\sigma^2}\right)\sum_{z=1}^k\frac{(k-1)!}{\propto^2(k-z)!}\left[2D\left(\frac{x}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}^{\left(\frac{x\theta}{\sigma^2}\right)}+2x\left(\frac{x}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}^{\left(\frac{x\theta}{\sigma^2}\right)}\right]\ (19b)$$

(2) and $\int_0^\infty H(L) g\left(\frac{x-DL}{\sqrt{\sigma^2 L}}\right)\frac{1}{\sqrt{\sigma^2 L}}dL$

$$= \frac{\propto^k}{\sigma\sqrt{2\pi}}\frac{esp\left(\frac{Dx}{\sigma^2}\right)}{\Gamma(k)}\left[2\left(\frac{x^2}{2\propto\sigma^2+D^2}\right)^{\frac{1}{2}\left(k-\frac{1}{2}\right)} K_{k-z-\frac{1}{2}}^{\left(\frac{x}{\sigma^2}(2\propto\sigma^2+D^2)^{\frac{1}{2}}\right)}\right] \quad (19c)$$

$$G_4(R) = \frac{\propto^k}{2\,\Gamma(k)}\left[\frac{esp\left(\frac{2DR}{\sigma^2}\right)}{2\sigma\sqrt{2\pi}}\left(R^2\sum_{z=1}^{k}\frac{(k-1)!}{\propto^2(k-z)!}\right)\left(2D\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}} K_{k-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right) + \left(\left(\sigma^2-2DR\right)\sum_{z=0}^{k+1}\frac{k!}{\propto^z(k+1-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right)\right) + \left(D^2\sum_{z=0}^{k+2}\frac{k!}{\propto^z(k+2-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k+\frac{5}{2}-z}K_{k+\frac{5}{2}-z}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-z+\frac{3}{2}}K_{k-z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right] - \frac{2\propto^k\sigma}{\sqrt{2\pi}\,\Gamma(k)}\left[\exp\left(\frac{DR}{\sigma^2}\right)\left(R\left(\frac{R}{\sigma}\right)^{k+\frac{1}{2}}K_{k+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} - D\left(\frac{R}{\sigma}\right)^{k+\frac{3}{2}}K_{k+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right] \quad (20)$$

$$G_2(R,L) = \left(\frac{D^2L^3}{6}-\frac{\sigma^4 R}{6D^3}-\frac{DL^2R}{2}-\frac{\sigma^2R^2}{4D^2}+\frac{\sigma^2L^2}{4}+\frac{LR^2}{2}-\frac{R^3}{6D}-\frac{\sigma^6}{8D^4}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)+\frac{DL^{\frac{5}{2}}\sigma}{6} - \frac{\sigma L^{\frac{3}{2}}R}{3}+\frac{\sigma L^{\frac{1}{2}}R^2}{6D}+\frac{\sigma^3L^{\frac{3}{2}}}{2D}+\frac{\sigma^3L^{\frac{1}{2}}R}{4D^2}+\frac{\sigma^5L^{\frac{1}{2}}}{4D^3}\Bigg) g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)+\frac{\sigma^6}{8D^4}\exp\left(\frac{2DR}{\sigma^2}\right)F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right) \quad (21)$$

Simplifying

$$G_2(R,L) = \left[-\left(\frac{\sigma^4R}{6D^3}+\frac{\sigma^2R^2}{4D^2}+\frac{R^3}{6D}+\frac{\sigma^6}{8D^4}\right)+\frac{R^2L}{2}+\left(-\frac{DR}{2}+\frac{\sigma^2}{4}\right)L^2+\frac{D^2L^3}{6}\right]F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right) + \left[\left(\frac{\sigma^5}{4D^3}+\frac{\sigma^3R}{4D^2}+\frac{\sigma R^2}{6D}\right)L^{\frac{1}{2}}+\left(\frac{\sigma^2}{12D}-\frac{\sigma R}{3}\right)L^{\frac{3}{2}}+\frac{D\sigma L^{\frac{5}{2}}}{6}\right]g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)+\frac{\sigma^6}{8D^4}esp\left(\frac{DR}{\sigma^2}\right) F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right) \quad (22)$$

Hence H(L) G_2 (R, L)

$$\frac{\propto^k}{\Gamma(k)}\left[-\left(\frac{\sigma^4R}{2D^3}+\frac{\sigma^2R^2}{4D^2}+\frac{R^3}{6D}+\frac{\sigma^6}{8D^4}\right)L^{k-1}+\frac{R^2L^k}{2}+\left(\frac{\sigma^2}{4}-\frac{DR}{2}\right)^{k+1}+\frac{D^2L^{k+2}}{6}\right]exp(-\propto L) F\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right)+\left[\left(\frac{\sigma^5}{4D^3}+\frac{\sigma^3R}{4D^2}+\frac{\sigma R^2}{6D}\right)L^{k-\frac{1}{2}}+\left(\frac{\sigma^3}{12D}-\frac{\sigma R}{3}\right)L^{k+\frac{1}{2}}+\frac{D\sigma L^{k+\frac{3}{2}}}{6}\right]g\left(\frac{R-DL}{\sqrt{\sigma^2L}}\right) + \frac{\sigma^6L^{k-1}\propto^k}{8D^4\,\Gamma(k)}F\left(\frac{R+DL}{\sqrt{\sigma^2L}}\right) \quad (23)$$

Substituting (23) into (15) and integrating we have

$$G_5(R) = \frac{\propto^k}{2\sigma}\frac{exp\left(\frac{DR}{\sigma^2}\right)}{(k)\sqrt{2\pi}}\left[-\left(\frac{\sigma^4R}{2D^3}+\frac{\sigma^2R^2}{4D^2}+\frac{R^3}{6D}+\frac{\sigma^6}{8D^4}\right)\left(\sum_{z=1}^{k}\frac{(k-1)!}{\propto^2(k-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} +2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right)+\frac{R^2}{2}\sum_{z=1}^{k+1}\frac{k!}{\propto^2(k+1-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{3}{2}}K_{k-z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+2R\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right)+\left(\frac{\sigma^2}{4}-\frac{DR}{2}\right)\sum_{z=1}^{k+2}\frac{(k+1)!}{\propto^2(k+z-2)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{5}{2}}K_{k-z+\frac{5}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+2R\left(\frac{R}{\theta}\right)^{k-z+\frac{3}{2}}K_{k-z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)+\frac{D^2}{6}\sum_{z=1}^{k+3}\frac{(k+2)!}{\propto^2(k+3-z)!}\,2D\left(\frac{R}{\theta}\right)^{k-z+\frac{7}{2}}K_{k-z+\frac{7}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+2R\left(\frac{R}{\theta}\right)^{k-z-\frac{5}{2}}K_{k-z-\frac{5}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) + \frac{2exp\left(\frac{DR}{\sigma^2}\right)\propto^k}{\sqrt{2\pi}\,\Gamma(k)}\left[\left(\frac{\sigma^5}{4D^3}+\frac{\sigma^3R}{4D^2}+\frac{\sigma R^2}{6D}\right)\left(\frac{R}{\theta}\right)^{k+\frac{1}{2}}K_{k+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+\left(\frac{\sigma^3}{12D}-\frac{\sigma R}{3}\right)\left(\frac{R}{\theta}\right)^{k+\frac{3}{2}}K_{k+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+\frac{D\sigma}{6}\left(\frac{R}{\theta}\right)^{k+\frac{5}{2}}K_{k+\frac{5}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right]+\frac{\sigma^6\propto^k exp\left(\frac{DR}{\sigma^2}\right)}{8D^4\,\Gamma(k)\,2\sqrt{2\pi}\,\sigma^2}\sum_{z=1}^{k}\frac{(k-1)!}{\propto^2(k-z)!}\left[2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}-2R\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right] \quad (24)$$

From G_3 (R, L) equation (II)

$$G_3(R,L) = \left(\frac{R^4}{12D^3}+\frac{\sigma^2R^3}{6D^4}+\frac{\sigma^4R^2}{4D^5}+\frac{\sigma^6R}{4D^6}+\frac{\sigma^8}{8D^7}-\frac{\sigma^2RL^2}{2D^2}+\frac{L^2R^2}{2D}-\frac{RL^3}{3}+\frac{L^3\sigma^2}{3D}-\frac{R^3L}{3D^2}+\frac{L^4D}{12}\right)$$

$$F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$
$$-2\sqrt{\sigma^2 L}D\left(\frac{\sigma^2 RL}{24D^3}+\frac{\sigma^4 R}{24D^4}-\frac{R^2 L}{8D^2}-\frac{\sigma^2 L^2}{8D^2}+\frac{L^2 R}{8D}-\frac{L^3}{24}+\frac{R^3}{24D^3}+\frac{\sigma^2 R^2}{12D^4}+\frac{\sigma^4 R}{8D^5}+\frac{\sigma^6}{8D^6}\right)g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$
$$-\frac{\sigma^8 D}{8D^7}esp\left(\frac{2DR}{\sigma^2}\right)F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\ (25)$$

Simplifying

$$G_3(R,L)=D\left[\left(\frac{R^4}{12D^3}+\frac{\sigma^2 R^3}{6D^4}+\frac{\sigma^4 R^2}{4D^5}+\frac{\sigma^6 R}{4D^6}+\frac{\sigma^8}{8D^7}\right)-\frac{R^3 L}{3D^2}+\left(-\frac{\sigma^2 RL^2}{2D^2}+\frac{L^2 R^2}{2D}\right)L^2+\left(\frac{\sigma^4}{3D}-\frac{R}{3}\right)L^3\right.$$
$$\left.+\frac{DL^4}{12}\right]F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)-2\sqrt{\sigma^2 L},D\left[\left(\frac{R^3}{24D^3}+\frac{\sigma^2 R^2}{12D^4}+\frac{\sigma^2 R}{8D^5}+\frac{\sigma^6}{8D^6}\right)+\left(\frac{\sigma^2 R}{24D^3}+\frac{\sigma^4}{24D^4}-\frac{R^2}{8D^2}\right)\mathrm{L}\right.$$
$$\left.+\left(-\frac{\sigma^2}{8D^2}+\frac{R}{8D}\right)L^2-\frac{L^3}{24}\right]g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\frac{\sigma^8 D}{8D^7}esp\left(\frac{2DR}{\sigma^2}\right)F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\ (26)$$

Multiplying by H (L)

$$H(L)G_3(R,L)=\frac{D\propto^k esp\,(-\infty L)}{\Gamma(k)}\left[\left(\frac{R^4}{12D^3}+\frac{\sigma^2 R^3}{6D^4}+\frac{\sigma^4 R^2}{4D^5}+\frac{\sigma^6 R}{4D^6}+\frac{\sigma^8}{8D^7}\right)L^{k-1}-\frac{R^3}{3D^2}\ L^k\right.$$
$$\left.\left(\frac{R^2}{2D}-\frac{\sigma^2 R}{2D^2}\right)L^{k+1}+\left(\frac{\sigma^2}{3D}+\frac{R}{3}+\right)L^{k-2}+\frac{DL^{k+3}}{12}\right]F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\frac{2D\infty^k esp\,(-\infty L)}{\sqrt{(k)}}$$
$$\left[\left(\frac{R^3}{24D^3}+\frac{\sigma^2 R^2}{12D^4}+\frac{\sigma^2 R}{8D^5}+\frac{\sigma^6}{8D^6}\right)L^{k-\frac{1}{2}}\left(\frac{\sigma^2 R}{24D^3}+\frac{\sigma^4}{24D^4}-\frac{R^2}{8D^2}\right)L^{k+\frac{1}{2}}+\left(\frac{R}{8D}+\frac{\sigma^2}{8D^2}\right)L^{k+\frac{3}{2}}-\frac{L^{k+\frac{5}{2}}}{24}\right]$$
$$g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\frac{\sigma^8}{8D^7}esp\left(\frac{2DR}{\sigma^2}\right)\frac{\propto^k L^{k-1}D}{\Gamma(k)}\ F\left(\frac{R+DL}{\sqrt{\sigma^2 L}}\right)\ (27)$$

Substituting into equation (16) and integrating and applying (19B), (19C) using the fact that

$$\int_Q^\infty\left(\frac{2x\theta}{\sigma^2}\right)^{y-j}esp\left(-x\left(\frac{\theta-D}{\sigma^2}\right)\right)dx$$

Integrating by parts

$$=\left(\frac{2\theta}{\sigma^2}\right)^{y-j}\sum_{z=0}^{y-j}\frac{(y-j)!}{(y-j-1)!}\frac{x^{y-j-1}}{\left(\frac{\theta-D}{\sigma^2}\right)^{j+1}}esp\left[\left(-x\left(\frac{\theta-D}{\sigma^2}\right)\right)\right]_Q^\infty\ (28)$$
$$=\left(\frac{2\theta}{\sigma^2}\right)^{y-j}\sum_{z=0}^{y-j}\frac{(y-j)!}{(y-j-1)!}\frac{Q^{y-j-1}}{\left(\frac{\theta-D}{\sigma^2}\right)^{i+1}}esp\left(-Q\left(\frac{\theta-D}{\sigma^2}\right)\right)\ (29)$$

Then

$$G_6(R)=D\ \propto^k\ esp\left(\frac{DR}{\sigma^2}\left(\frac{R^4}{12D^2}+\frac{\sigma^2 R^3}{6D^4}+\frac{\sigma^4 R^2}{4D^5}+\frac{\sigma^6 R}{4D^6}+\frac{\sigma^8}{8D^7}\right)\sum_{z=1}^{k}\frac{(k-j)!}{\propto^z(k-z)!}\right.$$
$$\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)-\frac{R^3}{3D^2}\ \sum_{z=1}^{k+1}\frac{k!}{\propto^z(k+1-z)!}$$
$$\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{3}{2}}+2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)+\left(\frac{R^2}{2D}-\frac{\sigma^2 R}{2D^2}\right)\ \sum_{2=1}^{k+z}\frac{(k+1)!}{\propto^z(k+z-2)!}$$
$$\left.\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{5}{2}}K_{k-z+\frac{5}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+2R\left(\frac{R}{\theta}\right)^{k-z+\frac{3}{2}}K_{k-z-\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right)+\left(\frac{\sigma^2}{3D}-\frac{R}{3}\right)\ \sum_{z=1}^{k+3}\frac{(k+2)!}{\propto^z(k+3-2)!}$$
$$\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{7}{2}}K_{k-z+\frac{7}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+2R\left(\frac{R}{\theta}\right)K_{k-z-\frac{5}{2}(R\sigma^L)}K_{k-z-\frac{5}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)+\frac{D}{12}\ \sum_{z=1}^{k+4}\frac{(k+3)!}{\propto^z(k+4-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{9}{2}}K_{k-z+\frac{9}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+2R\right.$$
$$\left.\left.\left(\frac{R}{\theta}\right)^{k-z+\frac{7}{2}}K_{k-z+\frac{7}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right]-\frac{2\infty^k esp\left(\frac{DR}{\sigma^2}\right)}{\Gamma(k)\sqrt{2\pi}}D\left[\left(2\left(\frac{R}{\theta}\right)^{k+\frac{1}{2}}K_{k+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\left(\frac{R^3}{24D^3}+\frac{\sigma^2 R^2}{12D^4}+\frac{\sigma^2 R}{8D^5}+\frac{\sigma^6}{8D^6}\right)\right.$$
$$+\left(\frac{\sigma^2 R}{24D^3}+\frac{\sigma^2}{24D^4}-\frac{R^2}{8D^2}\right)\left(2\left(\frac{R}{\theta}\right)^{k+\frac{3}{2}}K_{k+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\Bigg)+2\left(\frac{R}{8D}-\frac{\sigma^2}{8D^2}\right)\left(\frac{R}{\theta}\right)^{k+\frac{5}{2}}K_{k+\frac{5}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}-\frac{2}{24}\left(\frac{R}{\theta}\right)^{k+\frac{7}{2}}$$
$$K_{k+\frac{7}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}-\frac{\propto^k}{\Gamma(k)}\frac{\sigma^8 D}{8D^7}\frac{1}{2\sigma\sqrt{2\pi}}\sum_{z=1}^{k}\frac{(k-1)!}{\propto^z(k-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-z+\frac{1}{2}}K_{k-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}+2R\left(\frac{R}{\theta}\right)^{k-z-\frac{1}{2}}K_{k-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\ (30)$$

Also to be able to evaluate the expressions involving L we need the following integrals

$$\text{Let } G_7(R,T) = \int_0^\infty \mathrm{H}\,(\mathrm{L})\mathrm{G}_1(\mathrm{R},\mathrm{T}+\mathrm{L})\mathrm{dL}$$
$$\text{Let } G_8(R,T) = \int_0^\infty \mathrm{H}\,(\mathrm{L})\mathrm{G}_2(\mathrm{R},\mathrm{T}+\mathrm{L})\mathrm{dL}$$

Let $G_9(R,T) = \int_0^\infty \mathrm{H}\,(\mathrm{L})\mathrm{G}_3(\mathrm{R},\mathrm{T}+\mathrm{L})\mathrm{dL}$ (31)

Following the integral procedures so far

$$G_7(R,T) = \frac{esp\left(\propto T+\frac{DR}{\sigma^2}\right)\propto^k}{2\sigma\ \Gamma(k)\sqrt{2\pi}}\Big[(\sigma^2 T + R^2 - 2DRT + D^2T^2)\sum_{z=0}^{k-1}(-T)^j\binom{K-1}{j}\sum_{j=0}^{k-j}\frac{(k-1-j)!}{\propto^z(k-j-z)!}$$
$$\left(2D\left(\frac{R}{\theta}\right)^{k-j+\frac{1}{2}}K_{k-j-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-j-z-\frac{1}{2}}K_{k-j-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)(\sigma^2 - 2DR + 2D^2T)\sum_{j=0}^{k}(-T)^j\binom{k}{j}$$
$$\sum_{z=1}^{k+1-j}\frac{(k-j)!}{\propto^z(k+1-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{3}{2}}K_{k-j-z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) + D^2$$
$$\sum_{j=0}^{k+1}(-T)^j\binom{k-1}{j}\sum_{z=1}^{k+2-j}\frac{(k+1-j)!}{\propto^z(k+z-j-2)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{5}{2}}K_{k-j-z+\frac{5}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-j-z+\frac{3}{2}}K_{k-j-z-\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right]$$
$$\frac{2\sigma\propto^k esp\left(\propto T+\frac{DR}{\sigma^2}\right)}{\Gamma(k)\sqrt{2\pi}}\left[(R-DT)\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}\left(\frac{R}{\theta}\right)^{k+\frac{1}{2}-j}K_{k+\frac{1}{2}-j}^{\left(\frac{R\theta}{\sigma^2}\right)} - D\sum_{j=0}^{k}(-T)^j\binom{k}{j}\left(\frac{R}{\theta}\right)^{k+\frac{3}{2}-j}K_{k+\frac{3}{2}-j}^{\left(\frac{R\theta}{\sigma^2}\right)}\right] \quad (32)$$

Following the same integral procedures

$$G_8(R,T) = \frac{\propto^k}{\Gamma(k)}\frac{esp\left(\propto T+\frac{DR}{\sigma^2}\right)}{2\sigma\sqrt{2\pi}}\left[-\left(\frac{\sigma^4R}{2D^3}+\frac{\sigma^2R^2}{4D^2}+\frac{R^3}{6D}+\frac{\sigma^6}{8D^4}-\frac{RT}{2}\right)\right]\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}$$
$$\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\propto^z(k-j-2)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) + \frac{R^2}{2}\sum_{j=0}^{k}(-T)^j\binom{k}{j}$$
$$\sum_{z=1}^{k+1-j}\frac{(k-j)!}{\propto^z(k-1-j-2)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{3}{2}}K_{k-j-z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) + \left(\frac{\sigma^2}{4}-\frac{DR}{2}\right)$$
$$\sum_{i=0}^{2}\binom{2}{i}T^i\sum_{j=0}^{k+1-j}(-T)^j\binom{k+1-2}{j}\sum_{z=1}^{k+2-i-j}\frac{(k+1-i-j)!}{\propto^z(k+2-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-i-2+\frac{5}{2}-j}K_{k-i-2+\frac{5}{2}-j}^{\left(\frac{R\theta}{\sigma^2}\right)}\right.$$
$$\left.+2R\left(\frac{R}{\theta}\right)^{k-i-2+\frac{3}{2}-j}K_{k-i-2+\frac{3}{2}-j}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) + \frac{D^2}{6}\sum_{2=0}^{3}\binom{3}{i}T^i\sum_{j=0}^{k-i+2}(-T)^j\binom{k-i-2}{j}$$
$$\sum_{z=1}^{k-i+3-j}\frac{(k+z-i-j)!}{\propto^z(k-i+3-z+j)!}\left(2D\left(\frac{R}{\theta}\right)^{k-i-2+\frac{7}{2}-j}K_{k-i-2+\frac{7}{2}-j}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-i-z+\frac{5}{2}-j}K_{k-i-z+\frac{5}{2}-j}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\right]$$
$$+\frac{2\propto^k esp\left(T+\frac{DR}{\sigma^2}\right)}{\sqrt{2\pi}}\left[\left(\frac{\sigma^5}{4D^3}+\frac{\sigma^3R}{4D^2}+\frac{\sigma R^2}{6D}+\frac{\sigma^3T}{12D}-\frac{\sigma RT}{3}\right)\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}\left(\frac{R}{\theta}\right)^{k-J+\frac{1}{2}}K_{k-J+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right.$$
$$\left.+\left(\frac{\sigma^3}{12D}-\frac{\sigma R}{3}\right)\sum_{j=0}^{k}(-T)^j\binom{k}{j}\left(\frac{R}{\theta}\right)^{k-J+\frac{3}{2}}K_{k-J+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + \frac{2D\sigma}{6}\sum_{2=0}^{2}\binom{2}{i}T^i\left(\frac{R}{\theta}\right)^{k-i+\frac{3}{2}}K_{k-i+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right]$$
$$-\frac{\propto^k}{\Gamma(k)}\frac{esp\left(\propto T+\frac{DR}{\sigma^2}\right)}{\sigma\sqrt{2\pi}}\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\propto^z(k-j-z)!}\left(D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right.$$
$$\left.+R\left(\frac{R}{\theta}\right)^{k-j-2-\frac{1}{2}}K_{k-j-2-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) \quad (33)$$

$\mathrm{G}_9(\mathrm{R},\mathrm{T})$from (31) gives

$$G_8(R,T) = \frac{\propto^k}{2\sigma}\frac{esp\left(\propto T+\frac{DR}{\sigma^2}\right)}{\Gamma(k)\sqrt{2\pi}}\left(\frac{R^4}{12D^3}+\frac{\sigma^2R^3}{6D^4}+\frac{\sigma^4R^2}{4D^5}+\frac{\sigma^6R}{4D^6}+\frac{\sigma^8}{8D^7}-\frac{R^3T}{3D^2}\right)\sum_{j=0}^{k-1}(-T)^j\binom{k-1}{j}$$
$$\sum_{z=1}^{k-j}\frac{(k-1-j)!}{\propto^z(k-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-j-z-\frac{1}{2}}K_{k-j-z-\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) - \frac{R^3}{3D^2}\sum_{j=0}^{k}(-T)^j\binom{k}{j}$$
$$\sum_{z=1}^{k-j}\frac{(k-j)!}{\propto^z(k-1-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{3}{2}}K_{k-j-z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}}K_{k-j-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) + \left(\frac{R^2}{2D^2}-\frac{\sigma^2R}{2D^2}\right)$$
$$\sum_{i=0}^{2}\binom{2}{i}T^i\sum_{j=0}^{k-i+1}(-T)^j\binom{k-i-j}{j}\sum_{i=0}^{k-i+2}\frac{(k-i+1-j)!}{\infty^z(k-1+2-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-i-j-z+\frac{3}{2}}K_{k-i-j+z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right.$$
$$\left.+2R\left(\frac{R}{\theta}\right)^{k-i-j-z+\frac{1}{2}}K_{k-i-j-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) + \left(\frac{\sigma^2}{3D}-\frac{R}{3}\right)\sum_{i=0}^{3}\binom{3}{i}T^i\sum_{j=0}^{k-i+2}(-T)^j\binom{k-i-2}{j}$$
$$\sum_{z=0}^{k-i+3}\frac{(k-i+z-j)!}{\propto^z(k-i+3-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-i-j-z+\frac{3}{2}}K_{k-j-z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-i-j-z+\frac{3}{2}}K_{k-i-j-z+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) + \frac{D}{12}$$

$$\sum_{i=0}^{4}\binom{4}{i} T^i \sum_{j=0}^{k-i+3}(-T)^j \binom{k-i-3}{j} \sum_{z=0}^{k-i+3} \frac{(k-i+2-j)!}{\propto^z (k-1+4-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-i-j-z+\frac{7}{2}} K_{k-i-j-z+\frac{7}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right.$$
$$\left.+2R\left(\frac{R}{\theta}\right)^{k-i-j-z+\frac{5}{2}} K_{k-i-j-z+\frac{5}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right)\Bigg] - \frac{\sqrt{2}\,\sigma \propto^k esp\left(\infty T + \frac{DR}{\sigma^2}\right)}{\sqrt{\pi}\ \Gamma(k)}$$
$$\left(\frac{R^3}{24D^3} + \frac{\sigma^2 R^2}{12D^4} + \frac{\sigma^2 R}{8D^5} + \frac{\sigma^6}{8D^6} - \frac{\sigma^2 RT}{24D^3} - \frac{\sigma^4 T}{24D^4} - \frac{R^2 T}{8D^2}\right) \sum_{j=0}^{k-1}(-T)^j \binom{k-1}{j}$$
$$\left(2D\left(\frac{R}{\theta}\right)^{k-j+\frac{1}{2}} K_{k-j+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + \left(\frac{\sigma^2 R}{24D^3} + \frac{\sigma^4}{24D^4} - \frac{R^2}{8D^2}\right)\right) \sum_{j=0}^{k}(-T)^j \binom{k}{j}\left(2\left(\frac{R}{\theta}\right)^{k-j+\frac{3}{2}} K_{k-j+\frac{3}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right.$$
$$+2\left(\frac{R}{8D} - \frac{\sigma^2}{8D^2}\right) \sum_{i=0}^{2}\binom{2}{i} T^i \sum_{j=0}^{k-i+1}\binom{k-i+1}{j}(-T)^j \left(\frac{R}{\theta}\right)^{k-j+\frac{5}{2}-i} K_{k-j+\frac{5}{2}-i}^{\left(\frac{R\theta}{\sigma^2}\right)} - \frac{2}{24}\sum_{j=0}^{3}\binom{3}{i} T^i$$
$$\sum_{j=0}^{k-i+2}\binom{k-i+2}{j}(-T)^j \left(\frac{R}{\theta}\right)^{k-j+\frac{7}{2}} K_{k-j+\frac{7}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} - \frac{\sigma^2}{8D^7}\frac{\propto^k esp\left(\propto T + \frac{DR}{\sigma^2}\right)}{2\sigma\ \Gamma(k)\sqrt{2\pi}}\left[\sum_{j=0}^{k-1}(-T)^j \binom{k-1}{j}\right.$$
$$\sum_{z=1}^{k-i} \frac{(k-1-j)!}{\infty^z (k-j-z)!}\left(2D\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)} + 2R\left(\frac{R}{\theta}\right)^{k-j-z+\frac{1}{2}} K_{k-j-z+\frac{1}{2}}^{\left(\frac{R\theta}{\sigma^2}\right)}\right) \quad (34)$$

Hence the backorder cost per cycle

$$= \frac{b_1}{Q}\left(G_7(R,T) - G_4(R) - G_7(R+Q,T) + G_4(R+Q)\right)$$
$$+ \frac{b_2}{Q}\left(G_8(R,T) - G_5(R) - G_8(R+Q,T) + G_5(R+Q)\right)$$
$$= \frac{b_3}{Q}\left(G_9(R,T) - G_6(R) - G_9(R+Q,T) + G_6(R+Q)\right) \quad (35)$$

Probability of a stock out

$$= \frac{DT}{Q}\left(1 - F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)\right) + F\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right) - \frac{\sqrt{\sigma^2 L}}{Q}\ g\left(\frac{Q-DT}{\sqrt{\sigma^2 T}}\right)$$

Which is independent of lead time?
The expected on hand inventory for fixed lead times L is

$$D(Q,R,T) = \frac{Q}{2} + \left(R - DL - \frac{DT}{2} + B(Q,R,T)\right)$$

Using equation (9)

$$B(Q,R,T) = G_1(R, L+T) - G_1(R,L) - G_1(R+Q, L+T) + G_1(R+Q, L) \quad (37)$$

Averaging $D(Q,R,T)$ over the states of L we have

$$\int_0^\infty D\,(Q,R,T)H(L)dL \quad (38)$$

Substituting for D (Q,R,T) integral

$$= \int_0^\infty \int_0^\infty \left[\left(\frac{Q}{2} + R + DL - \frac{DT}{2}\right)\right.$$
$$+ \frac{1}{QT}\left(G_1(R, L+T) - G_1(R,L) - G_1(R+Q, L+T) + G_1(R+Q, L)\right)H(L)\,dL \quad (39)$$

Noting that

$$G_8(R,T) = \int_0^\infty G_2\,(R, L+T)\ H(L)\ dL \text{ and } G_5(R) = \int_0^\infty G_2\,(R,L)\ H(L)\ dL$$

Then on hand inventory

$$D(Q,R,T) = \frac{Q}{2} + R - \frac{Dk}{\propto} + \left(G_8(R,T) - G_5(R) - G_8(R+Q,T) + G_5(R+Q)\right)$$

No of cycles $= \dfrac{1}{T}$

Hence the inventory cost for model (nQ,R,T) when lead time is continuous and the cost of a backorder is a quadratic function of the length of time of a backorder is

$$C = \frac{Rc}{T} + \frac{S.P\ out}{T} + hc\left(\frac{Q}{2} + R - \frac{Dk}{\propto} - \frac{DT}{2}\right) + \frac{b_1}{QT}\left(G_7(R,T) - G_4(R) - G_7(R+Q,T) + G_4(R+Q)\right)$$
$$+ \frac{(hc+b_2)}{QT}\left(G_8(R,T) - G_5(R) - G_8(R+Q,T) + G_5(R+Q)\right)$$
$$+ \frac{b_3}{QT}\left(G_9(R,T) - G_6(R) - G_9(R+Q,T) + G_6(R+Q)\right) \quad (40)$$

3. Impact of Study

The study will enable industries, or organizations with thousands of items in their warehouses in different locations to express their backorder costs more accurately in non linear formulations. This would give more realistic inventory costs for holding items in various locations of the world.

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