A Second Order Runge Kutta Method to Solve Fuzzy Differential Equations with Fuzzy Initial Condition

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Abstract: ***This paper presents solution for first order fuzzy differential equation by Runge –Kutta method of order two with new parameters that increase the order of accuracy of the solution. This method is discussed in detail and followed by a complete error analysis. The accuracy and efficiency of the proposed method is illustrated by solving a fuzzy initial value problem with trapezoidal fuzzy number.***

Keywords: Fuzzy differential equations, multi-step Runge-Kutta method, higher order derivative approximations.

1. Introduction

Fuzzy Differential Equation (FDE) models have wide range of applications in many branches of engineering and in the field of medicine. The concept of fuzzy derivative was first introduced by S.L.Change and L.A.Zadeh in [6].D.Dubois and Prade in [7] discussed differentiation with fuzzy features.M.L.puri and D.A.Ralesec in [8] and R.Goetschel and W.Voxman in [9] contributed towards the differential of fuzzy functions. The fuzzy differential equation and initial value problems were extensively studied by O.Kaleva in [10],[11] and by S.Seikkala in [12].Recently many research papers are focused on numerical solution of fuzzy initial value problems (FIVPS).Numerical Solution of fuzzy differential equations has been introduced by M.Ma, M. Friedman, A. Kandel in [13] through Euler method and by S.Abbasbandy and T.Allahviranloo in [14] by Taylor method.Runge – Kutta methods have also been studied by authors in [1],[2].

2. Preliminaries

Definition 2.1. A fuzzy number u is a fuzzy subset of R (ie) $u: R \to [0,1]$ satisfying the following conditions:

1. u is normal (ie) $\exists\, x_0 \in R$ with $u(x_0) = 1$.
2. u is convex fuzzy set (ie) $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1], x, y \in R$.
3. u is upper semi continuous on R.
4. $\overline{\{x \in R, u(x) > 0\}}$ is compact.

Let E be the class of all fuzzy subsets of R.Then E is called the space of fuzzy numbers [10].
Clearly, $R \subset E$ and $R \subset E$ is understood as $R = \{\aleph_x : \aleph\ is\ usual\ real\ number\}$.
An arbitrary fuzzy number is represented by an ordered pair of functions
$(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$ that satisfies the following requirements.

1. $\underline{u}$(r)is a bounded left continous non-decreasing function over [0,1],with respect to any 'r'.
2. $\overline{u}$(r) is a bounded right continuous non-increasing function over [0,1] with respect to any 'r'.
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

Then the r-level set is $[u]_r = \{x \setminus u(x) \geq r\}, 0 \leq r \leq 1$ is a closed and bounded interval,
denoted by $[u]_r = [\underline{u}(r), \overline{u}(r)]$.
And clearly, $[u]_0 = \{x \setminus u(x) > 0\}$ is compact.

Definition 2.2. A trapezoidal fuzzy number u is defined by four real numbers of the trapezoidal is the interval $[k, n]$ and its vertices at $x = l$,
$x = m$.Trapezoidal fuzzy number will be written as $u = (k, l, m, n)$. The membership function for the trapezoidal fuzzy number $u = (k, l, m, n)$ is defined as the following:

$$u(x) = \begin{cases} \frac{x-k}{l-k}, k \leq x \leq l \\ 1, l \leq x \leq m \\ \frac{x-n}{m-n}, m \leq x \leq n \end{cases}$$

We will have: (1) u>0 if k>0 (2) u>0 if l>0
(3) u>0 if m>0 and (4) u>0 if n>0.

Definition 2.3 Let $F: (a,b) \to E^1$ and $x_0 \in (a,b)$.It is said that F is strongly generalized differentiable on x_0,if there exists an element $F'(x_0) \in E^1$,such that
(i) for all $h > 0$ sufficiently small, $\exists F(x_0 + h) - F(x_0)$, $F(x_0) - F(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\searrow 0} \frac{F(x_0+h)-F(x_0)}{h} = \lim_{h\searrow 0} \frac{F(x_0)-F(x_0-h)}{h} = F'(x_0),$$

(or)
(ii) for all $h > 0$ sufficiently small,$\exists F(x_0) - F(x_0 + h)$, $F(x_0 - h) - F(x_0)$ and the limits

$$\lim_{h\searrow 0} \frac{F(x_0)-F(x_0+h)}{(-h)} = \lim_{h\searrow 0} \frac{F(x_0-h)-F(x_0)}{(-h)} = F'(x_0),$$

(or)
(iii) for all $h > 0$ sufficiently small,$\exists F(x_0 + h) - F(x_0)$, $F(x_0 - h) - F(x_0)$ and the limits

$\lim_{h\searrow 0}\frac{F(x_0+h)-F(x_0)}{h}=\lim_{h\searrow 0}\frac{F(x_0-h)-F(x_0)}{(-h)}=F'(x_0)$,
(or)
(iv) for all $h>0$ sufficiently small, $\exists F(x_0)-F(x_0+h)$, $F(x_0)-F(x_0-h)$ and the limits
$\lim_{h\searrow 0}\frac{F(x_0)-F(x_0+h)}{(-h)}=\lim_{h\searrow 0}\frac{F(x_0)-F(x_0-h)}{h}=F'(x_0)$.
(h and $(-h)$ at denominators mean $\frac{1}{h}$ and $-\frac{1}{h}$, respectively).

Remark 2.1 A function that is strongly differentiable as in cases(i) and (ii) of definition 2.3, will be referred as (i)-differentiable or as (ii) –differentiable, respectively.

Lemma 2.1 If $u(t)=(x(t),y(t),z(t),w(t))$ is a trapezoidal fuzzy number valued function, then (a) if u is (i)-differentiable (Hukuhara differentiable), then $u'=(x',y',z',w')$. (b) if u is (ii) –differentiable, then $u'=(w',z',y',x')$.

3. The Runge –Kutta Method of Order Two

This chapter is devoted for finding numerical solutions of **FIVP**
$y'=f(t,y(t)), t\in[a,b], y(a)=y_0$ (3.1) by the Runge-Kutta methods with higher order derivative approximations. The basis of all Runge-Kutta methods is to express the difference between the values of y at t_{n+1} and t_n as

$$y_{n+1}-y_n=\sum_{i=0}^{m} w_i k_i \quad (3.2)$$

Where w_is are constants for all i and

$$k_i=hf(t_n+a_ih, y_n+\sum_{j=1}^{i-1}c_{ij}k_j) \quad (3.3)$$

with $h=t_{n+1}-t_n$ and $a_1=c_{11}=0$.

Second order Runge-Kutta method proposed in [16]:
Consider

$$y_{n+1}=y_n+w_1k_1+w_2k_2 \quad (3.4)$$

where

$$k_1=hf(t_n,y_n) \quad (3.5)$$

$$k_2=hf\{t_n+c_{21}h, y(t_n)+a_{21}k_1+ha_{22}(f_y(t_n,y(t_n))k_1+hf_t(t_n,y(t_n)))\} \quad (3.6)$$

where f_y is evaluated at (t_n,y_n).

Utilizing the Taylor's series expansion techniques, the above is uniquely satisfied as follows:
The parameters $w_1,w_2,c_{21},a_{21},a_{22}$ are chosen to make y_{n+1} closer to $y(t_{n+1})$. There are five parameters to be determined. The Taylor's series expansion of Equation (3.4) about t_n gives:

$$w_1+w_2=1, c_{21}=a_{21}, w_2c_{21}=\frac{1}{2}, w_2c_{21}^2=\frac{1}{3}, w_2a_{22}=\frac{1}{6}.$$

Solving this system of equations,

$$w_1=\frac{1}{4}, w_2=\frac{3}{4}, c_{21}=a_{21}=\frac{2}{3}, a_{22}=\frac{2}{9}.$$

∴ From Equations (3.4) to(3.6), Second order Runge-Kutta is obtained as
Runge-Kutta method of order two is given by:

$$y(t_{n+1})=y(t_n)+\frac{1}{4}[k_1+3k_2] \quad (3.7)$$

where

$$k_1=hf(t_n,y(t_n)) \quad (3.8)$$

$$k_2=hf\left\{t_n+\frac{2}{3}h, y(t_n)+\frac{2}{3}k_1+h\frac{2}{9}(f_y(t_n,y(t_n))k_1+hf_t(t_n,y(t_n)))\right\} \quad (3.9)$$

Here, $hf'=f_y(t_n,y(t_n))k_1+hf_t(t_n,y(t_n))$.

4. Second Order Runge-Kutta Method For FIVPS

Case(i)
Assume that $y'(t;r)$ given in Equation (3.1) is (i)-differentiable.
Let the exact solution of the FIVP given in Equation (3.1) $[Y(t)]_r=[\underline{Y}(t;r),\overline{Y}(t;r)]$ be approximated by some $[y(t)]_r=[\underline{y}(t;r),\overline{y}(t;r)]$
From the Equations (3.8) to (3.10), it is defined that

$$\underline{y}(t_{n+1};r)=\underline{y}(t_n;r)+\sum_{i=1}^{2} w_i\underline{k_i}, \quad (4.1)$$

$$\overline{y}(t_{n+1};r)=\overline{y}(t_n;r)+\sum_{i=1}^{2} w_i\overline{k_i}, \quad (4.2)$$

where w_is are constants and

$$[k_i(t,y(t;r))]_r=\left[\underline{k_i}(t,y(t;r)),\overline{k_i}(t,y(t;r))\right], i=1,2,. \quad (4.3)$$

$$\underline{k_1}(t,y(t;r))=\min\{h.f(t,u)\backslash u\in[\underline{y}(t;r),\overline{y}(t;r)]\} \quad (4.4)$$

$$\overline{k_1}(t,y(t;r))=\max\{h.f(t,u)\backslash u\in[\underline{y}(t;r),\overline{y}(t;r)]\} \quad (4.5)$$

$$\underline{k_2}(t,y(t;r))=\min\{h.f\left(t+\frac{2}{3}h,u\right)\backslash u\in[\underline{z_1}(t,y(t;r)),\overline{z_1}(t,y(t;r))]\} \quad (4.6)$$

$$\overline{k_2}(t,y(t;r))=\max\{h.f\left(t+\frac{2}{3}h,u\right)\backslash u\in[\underline{z_1}(t,y(t;r)),\overline{z_1}(t,y(t;r))]\} \quad (4.7)$$

$$\underline{z_1}(t,y(t;r))=\underline{y}(t;r)+\frac{2}{3}\underline{k_1}(t,y(t;r))+\frac{2}{9}(\underline{a}+\underline{b})$$

$$\overline{z_1}(t,y(t;r))=\overline{y}(t;r)+\frac{2}{3}\overline{k_1}(t,y(t;r))+\frac{2}{9}(\overline{a}+\overline{b})$$

$$\underline{a}=\min\{h.f_y(t,u).v\backslash u\in[\underline{y}(t;r),\overline{y}(t;r)]\ \&\ v\in[\underline{k_1}(t,y(t;r)),\overline{k_1}(t,y(t;r))]\}$$

$$\overline{a}=\max\{h.f_y(t,u).v\backslash u\in[\underline{y}(t;r),\overline{y}(t;r)]\ \&\ v\in[\underline{k_1}(t,y(t;r)),\overline{k_1}(t,y(t;r))]\}$$

$$\underline{b}=\min\{h.h.f_t(t,u)\backslash u\in[\underline{y}(t;r),\overline{y}(t;r)]\}$$

$$\overline{b}=\max\{h.h.f_t(t,u)\backslash u\in[\underline{y}(t;r),\overline{y}(t;r)]\}$$

Define,

$$F[t,y(t;r)]=\underline{k_1}[t,y(t;r)]+3\underline{k_2}[t,y(t;r)], \quad (4.8)$$

$$G[t,y(t;r)]=\overline{k_1}[t,y(t;r)]+3\overline{k_2}[t,y(t;r)] \quad (4.9)$$

The exact and approximate solutions at $t_n, 0\le n\le N$ are denoted by $[Y(t_n)]_r=[\underline{Y}(t_n;r),\overline{Y}(t_n;r)]$ and $[y(t_n)]_r=[\underline{y}(t_n;r),\overline{y}(t_n;r)]$, respectively. The solution is calculated by grid points at

$$a=t_0\le t_1\le t_2\le\cdots\ldots\ldots\ldots\ldots\ldots\le t_N=b \text{ and}$$

$$h=\frac{(b-a)}{N}=t_{n+1}-t_n.$$

By Equations (4.1) to (4.9), let

$$\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) + \frac{1}{4}F[t_n, Y(t_n;r)] \quad (4.10)$$

$$\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) + \frac{1}{4}G[t_n, Y(t_n;r)] \quad (4.11)$$

and

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{1}{4}F[t_n, y(t_n;r)] \quad 4.12)$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{1}{4}G[t_n, y(t_n;r)] \quad (4.13)$$

The following lemmas will be applied to show the convergences of theses approximates i.e., $\lim_{h\to 0} \underline{y}(t,r) = \underline{Y}(t,r)$ and $\lim_{h\to 0} \overline{y}(t,r) = \overline{Y}(t,r)$.

Lemma4.1 Let the sequence of numbers $\{W_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \le A|W_n| + B, 0 \le n \le N-1$, or some given positive constants A and B, then $|W_n| \le A^n|W_0| + B\frac{A^n-1}{A-1}, 0 \le n \le N-1$.

The proof of Lemma (4.1) follows Lemma 1 of Ming Ma et al (1999).

Lemma 4.2 Let the sequence of numbers $\{W_n\}_{n=0}^{N}, \{V_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \le |W_n| + A max\{|W_n|, |V_n|\} + B$, $|V_{n+1}| \le |V_n| + A max\{|W_n|, |V_n|\} + B$, for some given positive constants A and B, and denote $U_n = |W_n| + |V_n|, 0 \le n \le N$. Then $U_n \le \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A}-1}, 0 \le n \le N$, where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$.

The proof of Lemma (4.2) follows Lemma 2 of Ming Ma et al (1999).

Let $F(t,u,v)$ and $G(t,u,v)$ be obtained by substituting $[y(t)]_r = [u,v]$ in the Equations (4.8) & (4.9),

$$F[t,u,v] = \underline{k_1}[t,u,v] + 3\underline{k_2}[t,u,v],$$

$$G[t,u,v] = \overline{k_1}[t,u,v] + 3\overline{k_2}[t,u,v].$$

The domain where F and G are defined is therefore

$$K = \{(t,u,v)\backslash 0 \le t \le T, -\infty < v < \infty, -\infty < u \le v\}.$$

Theorem 4.4 let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^2(K)$ and let the partial derivatives of F and G be bounded over K.Then, for arbitrary fixed $r, 0 \le r \le 1$, the approximate solutions given in Equations (4.12) converge to the exact solutions $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ uniformly in t.

The proof of Theorem (4.1) follows Theorem 1 of Ming Ma et al(1999).

Case (ii)

Assume that $y'(t;r)$ given in equation (3.1) is (ii) -differentiable.

Let the exact solution of the FIVP given in equation (3.1) $[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)]$ be approximated by some $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$.

From the Equations (3.8) to (3.10), it is defined that

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \sum_{i=1}^{2} w_i \overline{k_i}, \quad (4.14)$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \sum_{i=1}^{2} w_i \underline{k_i}, \quad (4.15)$$

where w_is are constants and $\underline{k_i}$ and $\overline{k_i}$,$i = 1,2$ are given as in Equations (4.4) to (4.7).

Using Equations (4.14) and (4.15) together with Equations (4.8) and (4.9), The exact solution of the FIVP (3.1) is given by

$$\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) + \frac{1}{4}G[t_n, Y(t_n;r)] \quad (4.16)$$

$$\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) + \frac{1}{4}F[t_n, Y(t_n;r)] \quad (4.17)$$

And the approximate solution is given by

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{1}{4}G[t_n, y(t_n;r)] \quad (4.18)$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{1}{4}F[t_n, y(t_n;r)] \quad (4.19)$$

As in case (i), it can be shown that the approximate solutions $\underline{y}(t_{n+1};r)$ and $\overline{y}(t_{n+1};r)$ approach the exact solutions $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ respectively.

5. Numerical Example

Example 1. Consider the fuzzy initial value problem,

$$\begin{cases} y'(t) = y(t), t \in [0,1], \\ y(0) = (0.8 + 0.125r, 1.1 - 0.1r), 0 < r \le 1. \end{cases}$$

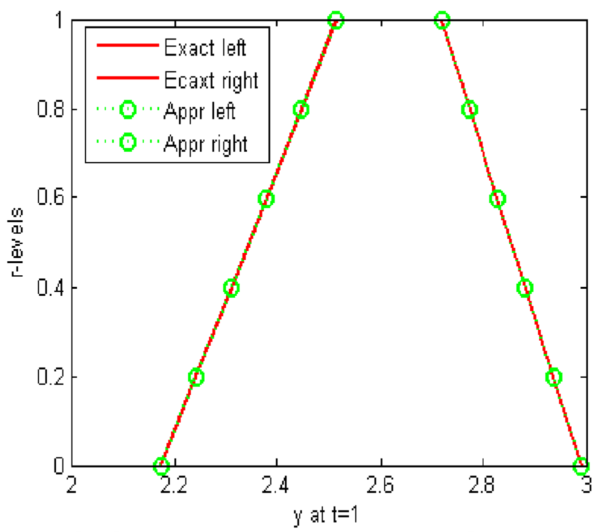
The exact solution is given by$\underline{Y}(t;r) = \underline{y}(t;r)e^t, \overline{Y}(t;r) = \overline{y}(t;r)e^t$

Which at $t = 1$, $Y(1;r) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e], 0 < r \le 1$.

The exact and approximate solutions obtained by Runge-Kutta method of order 2and by the proposed second –order Runge-Kutta method‘$h = 0.001$’ are given in Table: 1

Table 1:

r	t	Exact solutions at t=1		Approximated solutions at h=0.001	
		y1	y2	y1	y2
0	1	2.174625463	2.990110011	2.174625101	2.990109513
0.2	1	2.242582508	2.935744375	2.242582135	2.935743886
0.4	1	2.310539554	2.881378738	2.310539169	2.881378258
0.6	1	2.3784966	2.827013102	2.378496204	2.827012631
0.8	1	2.446453646	2.772647465	2.446453238	2.772647003
1	1	2.514410691	2.718281828	2.514410273	2.718281376



6. Conclusions

In this work, we have used the proposed second-order Runge-Kutta method to find a numerical solution of fuzzy differential equations using trapezoidal fuzzy number. Taking into account the convergence order $O(h^2)$ is obtained by the proposed method. Comparison of the solutions of example 5.1 shows that the proposed method gives a better solution.

References

[1] S.Abbasbandy, T.Allahviranloo, Numerical solution of fuzzy differential equations by Runge-Kutta method, *Nonlinear studies* 11(2004) No.1, 117-129.

[2] Bede, B & Gal, SG 2005, ‘Generalizations of the differentiability of fuzzy-number valued functions with applications to fuzzy differential equations’, Fuzzy sets and systems, vol. 151, pp. 581-599.

[3] J.J.Buckley, E.Eslami, and T.Feuring, 2002, Fuzzy Mathematics in economics and engineering, *Physica-verlag*, Heidelberg, Germany.

[4] J.J.Buckley, Thomas Fairing, Fuzzy differential equations, *Fuzzy sets and Systems* 110(2000), 43-54.

[5] J.C.Butcher, The Numerical Analysis of Ordinary Differential equations Runge-Kutta and General Linear Methods, *Wiley*, New York, 1987.

[6] S.L.Chang, L.A.Zadeh, On fuzzy mapping and control, *IEEETrans, Systems Man cybernet.*2 (1972), 30-34.

[7] D.Dubois, H.Prade, towards fuzzy differential calculus: Part3, differentiation, *Fuzzy sets and systems* 8(1982), 25-233.

[8] M.L.puri, D.A.Ralescu, Differentials of fuzzy functions, *J.Math.Anal.Appl.*91 (1983)552-558.

[9] R.Goetschel, W.Voxman.Elementry Fuzzy Calculus, *Fuzzy Sets and Systems* 18(1986).31-43.

[10] O.Kaleva, Fuzzy differential equations, Fuzzy sets and systems 24(1987).301-317.

[11] O.Kaleva, The Cauchy problem for Fuzzy differential equations, *Fuzzy sets and systems* 35(1990), 389-396.

[12] S.Seikkala, On the Fuzzy initial value problem,*Fuzzy sets and systems* 24(1987)319-330.

[13] M.Ma, M.Friedman, A.Kandel, Numerical solutions of fuzzy differential equations, *Fuzzy sets and System* 105(1999), 133-138.

[14] S.Abbasbandy,T.allahviranloo,Numerical Solution of fuzzy differential equations by Taylor method,J.*of Computational methods in Applied Mathematics* 2 (2002),113-124.

[15] J.D.Lambert, Numerical methods for Ordinary differential systems, *John Wiley&Sons*, 1990

[16] David Goeken, Olin Johnson, “Runge–Kutta with higher order derivative Approximations” *Applied Numerical mathematics* 34(2000) 207-218.

[17] V.Nirmala,N.Saveetha,S.Chenthurpandian, “Numerical Solution of Fuzzy Differential Equation by Runge-Kutta Method with Higher Order Derivative Approximations”,*Proceedins of the International Conference on Emerging Trends in Mathematics and Computer Applications*,MEPCO Schlenk Engineering College,Sivakasi,India.Dec.16-18,2010.PP.131-134(ISBN:978-81-8424-649-0).

[18] S.Ch.Palligkinis, G.Papageorgiou, I.TH.Famelis, Runge-Kutta methods for fuzzy differential equations, *Applied mathematics and Computation* 209(2009)97-105.

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