Approximation by Stancu Type Generalization of Beta Operators

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Abstract: In this paper, we are dealing with Stancu Beta operators \( V_{n}^{\alpha, \beta} \) defined by (1.5). We establish direct and local approximation properties of these operators.

Keywords: Beta operators, Modulus of continuity, Rate of convergence, Direct and local approximation

1. Introduction

Gupta and Ahmad [9] introduced the Durrmeyer variant of the discrete beta operators to approximate Lebesgue integrable functions on the interval \([0, \infty)\). The beta operators from \( C[0, \infty) \) into \( C[0, \infty) \), the class of all bounded and continuous functions on \([0, \infty)\), are defined as

\[
(1.1) \quad (V_{n}f)(x) = \frac{1}{n} \sum_{v=0}^{n} b_{n,v}(x) f \left( \frac{x}{n+1} \right),
\]

where

\[
(1.2) \quad b_{n,v}(x) = \frac{1}{B(v+1,n)} \frac{x^v}{(1+x)^{n+v+1}}, \quad x \in [0, \infty)
\]

and \( B(v+1,n) \) denotes the Beta function given by \( \frac{\Gamma(v+1) \cdot \Gamma(n)}{\Gamma(n+v+1)} \).

In [4] D. D. Stancu introduced the following generalization of Bernstein polynomials

\[
(1.3) \quad S_{n}^{\alpha}(f,x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) P_{n,\alpha}^{k}(x), \quad 0 \leq x \leq 1,
\]

where

\[
P_{n,\alpha}^{k}(x) = \left( \frac{n}{m} \right) \prod_{s=0}^{k-1} \left( n \times \alpha s \right) \prod_{s=k}^{n-1} \left( 1 - \alpha s \right) \prod_{s=0}^{x} (1 + \alpha s).
\]

We get the classical Bernstein polynomials by putting \( \alpha = 0 \) in (1.3). Starting with two parameters \( \alpha, \beta \) satisfying the conditions \( 0 \leq \alpha \leq \beta \) in 1983, the other generalization of Stancu operators was given in [3] and studied the linear positive operators \( S_{n}^{\alpha, \beta} : C[0,1] \to C[0,1] \) defined for any \( f \in C[0,1] \) as follows:

\[
(1.4) \quad S_{n}^{\alpha, \beta}(f,x) = \sum_{k=0}^{n} P_{n,\alpha, \beta}^{k}(x) f \left( \frac{k + \alpha}{n + \beta} \right), \quad 0 \leq x \leq 1,
\]

where

\[
P_{n,\alpha, \beta}^{k}(x) = \left( \frac{n}{k} \right) x^{k} (1 - x)^{n-k} \quad \text{are the fundamental Bernstein polynomials (cf. [8])}.
\]

For \( \alpha = \beta = 0 \), the polynomials in (1.4) are Bernstein polynomials. Atakut [2] gave Stancu type generalisation of Baskakov operators as follows:

\[
L_{n}^{\alpha, \beta}(f,x) = \sum_{k=0}^{x} \frac{(-x)^{k}}{k!} \varphi_{n,k}^{+}(x) f \left( \frac{k + \alpha}{n + \beta} \right),
\]

where \( \varphi_{n,k}(x) = (1 + x)^{-n} \) and established some approximation properties of these operators. She also shown the convergence of the derivative \( \frac{dr}{dx^r} L_{n}^{\alpha, \beta}(f,x) \) to \( f^{r}(x), r = 1,2, \ldots \) as \( n \to \infty \) provided \( f^{r}(x) \) exists.

Recently, Ibrahim [1] introduced Stancu-Chlodowsky polynomials and investigated convergence and approximation properties of these operators. Motivated by such type operators we introduce the operators as follow:

\[
(1.5) \quad V_{n}^{\alpha, \beta}(f,x) = \frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) f \left( \frac{k + \alpha}{n + 1 + \beta} \right),
\]
where \( b_{n,k}(x) \) is given in (1.2). \( (V_n^{\alpha,\beta}f)(x) \) is called Beta Stancu operators. For \( \alpha = 0 = \beta \), we get 1.1.

In the present paper, we study the rate of convergence and approximation properties of these operators by using modulus of continuity and K-functional of Peetre.

2. Preliminaries

In this section we require the following results:

**Lemma 1.** For the functions \( t^m, m = 0, 1, 2 \) we have

\[
V_n^{\alpha,\beta}(1,x) = 1, V_n^{\alpha,\beta}(t,x) = \frac{n + 1}{n + 1 + \beta} x + \frac{\alpha}{n + 1 + \beta}
\]

\[
V_n^{\alpha,\beta}(t^2,x) = \frac{(n + 1)(n + 2)}{(n + 1 + \beta)^2} x^2 + \frac{(n + 1)(1 + 2\alpha)}{(n + 1 + \beta)^2} x + \frac{\alpha^2}{(n + 1 + \beta)^2}.
\]

**Proof.** The operators \( V_n^{\alpha,\beta} \) are well defined on functions 1, \( t \), \( t^2 \) and

\[
\sum_{v=0}^{\infty} b_{n,v}(x) = n.\text{ Then for every } n \in \mathbb{N} \text{ and } x \in [0, \infty), \text{ we obtain}
\]

\[
V_n^{\alpha,\beta}(1,x) = \frac{1}{n} \sum_{v=0}^{\infty} b_{n,v}(x) = 1.
\]

Similarly,

\[
V_n^{\alpha,\beta}(t,x) = \frac{1}{n} \sum_{v=0}^{\infty} b_{n,v}(x) \left( \frac{k + \alpha}{n + 1 + \beta} \right)
\]

\[
= \frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \left( \frac{k}{n + 1 + \beta} + \frac{\alpha}{n + 1 + \beta} \right)
\]

\[
= \frac{1}{n(n + 1 + \beta)} \sum_{k=1}^{\infty} \frac{(n + k)!}{(k - 1)!(n - 1)!} \left( 1 + x \right)^{n + k + 1} + \frac{\alpha}{n + 1 + \beta}
\]

\[
= \frac{1}{(n + 1 + \beta)} \sum_{k=0}^{\infty} \frac{(n + k + 1)!}{k! n!} \left( 1 + x \right)^{n + k + 2} + \frac{\alpha}{n + 1 + \beta}
\]

\[
= \frac{x}{(n + 1 + \beta)} \sum_{k=0}^{\infty} b_{n+1,k}(x) + \frac{\alpha}{n + 1 + \beta}
\]

\[
= \frac{(n + 1)x}{(n + 1 + \beta)} + \frac{\alpha}{n + 1 + \beta}.
\]

Finally,

\[
V_n^{\alpha,\beta}(t^2,x) = \frac{1}{n} \sum_{v=0}^{\infty} b_{n,v}(x) \left( \frac{k + \alpha}{n + 1 + \beta} \right)^2
\]

\[
= \frac{1}{n(n + 1 + \beta)^2} \sum_{k=0}^{\infty} \frac{(n + k)!}{k!(n - 1)!} \left( k^2 + 2k\alpha + \alpha^2 \right) \left( 1 + x \right)^{n + k + 1}
\]

\[
= \frac{1}{(n + 1 + \beta)^2} \left[ \frac{1}{n} \sum_{k=1}^{\infty} \frac{(n + k)!}{k!(n - 1)!} k^2 + 2\alpha \frac{1}{n} \sum_{k=1}^{\infty} \frac{(n + k)!}{k!(n - 1)!} \right]
\]

\[
+ \alpha^2 \frac{1}{n} \sum_{k=0}^{\infty} \frac{(n + k)!}{k!(n - 1)!} \left( 1 + x \right)^{n + k + 1}
\]
\[
\frac{1}{(n+1+\beta)^2} \sum_{k=0}^{\infty} \frac{(n+k+1)!}{k!(n-1)!} \left( \frac{x}{1+x} \right)^{n+k+2} + \frac{x}{(n+1+\beta)^2} \sum_{k=0}^{\infty} b_{n,k}(x) + \frac{\alpha^2}{(n+1+\beta)^2}
\]

\[
= \frac{1}{(n+1+\beta)^2} \sum_{k=0}^{\infty} \frac{(n+k+2)!}{k!(n-1)!} \left( \frac{x}{1+x} \right)^{n+k+3} + \frac{1}{(n+1+\beta)^2} \frac{(n+1)(1+2\alpha)x}{(n+1+\beta)^2} \sum_{k=0}^{\infty} b_{n,k}(x) + \frac{\alpha^2}{(n+1+\beta)^2}
\]

\[
= \frac{(n+1)x^2}{(n+1+\beta)^2} \sum_{k=0}^{\infty} b_{n+2,k}(x) + \frac{(n+1)(1+2\alpha)x}{(n+1+\beta)^2} \sum_{k=0}^{\infty} b_{n,k}(x) + \frac{\alpha^2}{(n+1+\beta)^2}
\]

Remark 1. By simple computation we have

\[
V_n^{\alpha,\beta}(t-x,x) = \frac{\alpha - \beta x}{n+1+\beta}
\]

\[
V_n^{\alpha,\beta}\left((t-x)^2,x\right) = \left(\frac{n+1+\beta^2}{(n+1+\beta)^2}\right)x^2 + \frac{n+1-2\alpha\beta}{(n+1+\beta)^2}x + \frac{\alpha^2}{(n+1+\beta)^2}
\]

Lemma 2. For \( n \in \mathbb{N} \), we have

\[
V_n^{\alpha,\beta}\left((t-x)^2,x\right) \leq \frac{(n+1+\beta^2)x(1+x) + \alpha^2}{(n+1+\beta)^2}
\]

Proof. For \( 0 \leq \alpha \leq \beta \) and from the Remark 1, we have

\[
V_n^{\alpha,\beta}\left((t-x)^2,x\right) \leq \left(\frac{n+1+\beta^2}{(n+1+\beta)^2}\right)x^2 + \frac{n+1-2\alpha\beta}{(n+1+\beta)^2}x + \frac{\alpha^2}{(n+1+\beta)^2}
\]

\[
\leq \frac{(n+1+\beta^2)}{(n+1+\beta)^2}x^2 + \frac{(n+1-2\alpha\beta)}{(n+1+\beta)^2}x + \frac{\alpha^2}{(n+1+\beta)^2}
\]

\[
\leq \frac{(n+1+\beta^2)x(1+x) + \alpha^2}{(n+1+\beta)^2}
\]

3. Main Results

In this section we establish direct and local approximation theorems in connection with the operators \( V_n^{\alpha,\beta} \). Let \( C(0,\infty) \) be the space of all real valued, bounded and uniformly continuous function on \( [0,\infty) \) endowed with the norm \( ||f|| = \sup \{|f(x)| : x \in [0,\infty)\} \).

Theorem 1. For any \( f \in C(0,\infty) \), one has for \( n \) sufficiently large. Then, for every \( x \in [0,\infty) \) we have

\[
\left| V_n^{\alpha,\beta}(f,x) - f(x) \right| \leq 2\omega(f,\delta),
\]

where \( \delta = \sqrt{\frac{(n+1+\beta^2)x(1+x) + \alpha^2}{(n+1+\beta)^2}} \) and \( \omega(f,\cdot) \) is the usual modulus of continuity of \( f \).

Proof. Using the relation \( \sum_{\nu=0}^{\infty} b_{n,\nu}(x) = n \), we have

\[
V_n^{\alpha,\beta}(f,x) - f(x) = \frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \left[ f\left(\frac{k+\alpha}{n+1+\beta}\right) - f(x) \right]
\]

and so
\[ V_n^{\alpha,\beta}(f, x) - f(x) \leq \frac{1}{n} \sum_{k=0}^{n} b_{n,k}(x) \left[ f\left( \frac{k + \alpha}{n + 1 + \beta} \right) - f(x) \right] \]

taking \( y = \frac{k + \alpha}{n + 1 + \beta} \) and \( |y - x| \leq \lambda \delta \), we have

\[ |f(y) - f(x)| \leq \omega(f, \lambda \delta) \leq (1 + \lambda) \omega(f, \delta) \]

Thus, we have

\[ |f\left( \frac{k + \alpha}{n + 1 + \beta} \right) - f(x)| \leq \left( 1 + \frac{1}{\delta} \sum_{k=0}^{n} b_{n,k}(x) \left( \frac{k + \alpha}{n + 1 + \beta} - x \right) \right) \omega(f, \delta) \]

applying Cauchy-Schwarz inequality, we have

\[ |V_n^{\alpha,\beta}(f, x) - f(x)| \leq \omega(f, \delta) \left( 1 + \frac{1}{\delta} \left( |V_n^{\alpha,\beta}\left( (t-x)^2, x \right) \right|^{1/2} \right. \]

In view of Lemma 2, by choosing \( \delta = \sqrt{\frac{(n+1+\beta^2)(1+x)+\alpha^2}{(n+1+\beta)^2}} \),

\[ |V_n^{\alpha,\beta}(f, x) - f(x)| \leq 2\omega \left( f, \sqrt{\frac{(n+1+\beta^2)(1+x)+\alpha^2}{(n+1+\beta)^2}} \right) \]

Hence, the required result.

Let \( B_{\alpha}\{0, \infty\} = \{ f : \text{for every } x \in [0, \infty), |f(x)| \leq M_f (1+x^2), M_f \text{ being a constant depending of } f \} \). By \( C_{x}\{0, \infty\} \), we denote the subspace of all continuous functions belonging to \( B_{\alpha}\{0, \infty\} \). Also, \( C_{x}^{*}\{0, \infty\} \) is subspace of all functions \( f \in C_{x}\{0, \infty\} \) for which \( \lim_{x \to \infty} \frac{f(x)}{1+x^2} \) is finite. The norm on \( C_{x}^{*}\{0, \infty\} \) is \( \| f \|_{c^2} = \sup_{x \in [0, \infty)} \left| \frac{f(x)}{1+x^2} \right| \).

For any positive number \( a \), by \( \omega_a(f, \delta) = \sup_{t \in [x,x+\delta]} \left| f(t) - f(x) \right| \),

we denote the usual modulus of continuity of \( f \) on the closed interval \([0, a]\). We know that for a function \( f \in C_{x}\{0, \infty\} \), modulus of continuity \( \omega_a(f, \delta) \) tends to zero as \( \delta \to 0 \).

**Theorem 2.** Let \( f \in C_{x}\{0, \infty\} \) and \( \omega_{a+1} \) be its modulus of continuity of finite interval \([0, a+1] \subset [0, \infty) \) where \( a > 0 \).

Then for every \( n \)

\[ \| V_n^{\alpha,\beta}(f) - f \|_{c_{[0, a]}} \leq \kappa \left( \frac{n+1+\beta^2}{n+1+\beta} \right) \left( \frac{a (1+a) + \alpha^2}{(n+1+\beta)^2} \right) \]
\[ +2 \omega_{n+1} \left( f, \sqrt{\frac{(n+1+2\beta^2)a(1+a)+\alpha^2}{(n+1+\beta)^2}} \right), \]

where \( K = 6M_f \left( 1 + a^2 \right). \)

**Proof.** For \( x \in [0,a] \) and \( t > a + 1 \), we have
\[
|f(t) - f(x)| \leq M_f \left( 2 + x^2 + t^2 \right) \leq M_f \left( 2 + 3x^2 + 2(t-x)^2 \right) \leq 3M_f \left( 1 + x^2 + (t-x)^2 \right) \leq 6M_f \left( 1 + x^2 \right)(t-x)^2.
\]
(3.1) \leq 6M_f \left( 1 + a^2 \right)(t-x)^2.

For \( x \in [0,a] \) and \( t \leq a + 1 \), we have
\[
|f(t) - f(x)| \leq \omega_{n+1} \left( f, \left\lfloor \frac{t-x}{\delta} \right\rfloor \right) \omega_{n+1} \left( f, \delta \right)
\]
with \( \delta > 0 \).

From (3.1) and (3.2), we can write
\[
|f(t) - f(x)| \leq 6M_f \left( 1 + a^2 \right)(t-x)^2 + \left( 1 + \frac{x}{\delta} \right) \omega_{n+1} \left( f, \delta \right)
\]
(3.3)

Hence, by Schwartz's inequality and Lemma 2, for every \( x \in [0,a] \)
\[
|f(t) - f(x)| \leq 6M_f \left( 1 + a^2 \right)(t-x)^2 + \left( 1 + \frac{x}{\delta} \right) \omega_{n+1} \left( f, \delta \right)
\]
(3.2)

For \( x \in [0,a] \) and \( t \leq a + 0 \)
\[
V^{u,v}(f,x) - f(x) \leq V^{u,v}(t-x)^2 + \omega_{n+1} \left( f, \delta \right) \left( 1 + \frac{1}{\delta} \right) \omega_{n+1} \left( f, \delta \right)
\]
(3.3)

Hence, by Schwartz’s inequality and Lemma 2, for every \( x \in [0,a] \)
\[
|f(t) - f(x)| \leq 6M_f \left( 1 + a^2 \right)V^{u,v}(t-x)^2 + \omega_{n+1} \left( f, \delta \right) \left( 1 + \frac{1}{\delta} \right) \omega_{n+1} \left( f, \delta \right)
\]
(3.2)

For \( x \in [0,a] \) and \( t \leq a + 0 \)
\[
|f(t) - f(x)| \leq 6M_f \left( 1 + a^2 \right)(t-x)^2 + \left( 1 + \frac{x}{\delta} \right) \omega_{n+1} \left( f, \delta \right)
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\]
(3.2)

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\[
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\]
(3.2)

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\[
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\]
(3.2)

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\[
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\]
(3.2)

For \( x \in [0,a] \) and \( t \leq a + 0 \)
\[
|f(t) - f(x)| \leq 6M_f \left( 1 + a^2 \right)(t-x)^2 + \left( 1 + \frac{x}{\delta} \right) \omega_{n+1} \left( f, \delta \right)
\]
(3.3)

Hence, by Schwartz’s inequality and Lemma 2, for every \( x \in [0,a] \)
\[
|f(t) - f(x)| \leq 6M_f \left( 1 + a^2 \right)(t-x)^2 + \left( 1 + \frac{x}{\delta} \right) \omega_{n+1} \left( f, \delta \right)
\]
(3.2)
Theorem 3. Let $f \in C[0,\infty)$. Then, for every $x \in [0,\infty)$, we have
\[
V_n^{\alpha,\beta} (f, x) - f(x) \leq 2K(f, \delta) + \omega \left( f, \frac{\alpha - \beta x}{n+1+\beta} \right),
\]
where $K(f, \delta)$ is Peetre's $K$ functional defined above and
\[
\delta = \frac{(n+1+2\beta^2)x^2(n+1-4\alpha\beta)x + \alpha^2 + \alpha}{(n+1+\beta)}.
\]

Proof. We introduce the auxiliary operators defined by
\[
V_n^{\alpha,\beta} (f, x) = V_n^{\alpha,\beta} (f, x) - f \left( \frac{(n+1)x + \alpha}{n+1+\beta} \right) + f(x),
\]
\[x \in [0,\infty).\] These operators are linear and preserves the linear functions i.e.
\[V_n^{\alpha,\beta} (t-x, x) = 0\]

Let $g \in C[0,\infty)$. From Taylor's expansion of $g$
\[
g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, t \in [0,\infty)
\]
we have
\[
V_n^{\alpha,\beta} (g, x) - g(x) = V_n^{\alpha,\beta} \left( \int_x^t (t-u)g''(u)du, x \right).
\]
\[
\begin{align*}
|V_n^{\alpha,\beta} (g, x) - g(x)| &\leq |V_n^{\alpha,\beta} \left( \int_x^t (t-u)g''(u)du, x \right) + \int_x^t \left( \frac{(n+1)x + \alpha}{n+1+\beta} - u \right)g''(u)du| \\
&\leq V_n^{\alpha,\beta} \left( \int_x^t (t-u)g''(u)du, x \right) + \int_x^t \left( \frac{(n+1)x + \alpha}{n+1+\beta} - u \right)g''(u)du \\
&\leq V_n^{\alpha,\beta} \left( \int_x^t (t-x)g''(u)du, x \right) + \int_x^t \frac{\alpha - \beta x}{n+1+\beta}g''(u)du \\
&\leq \left[ V_n^{\alpha,\beta} \left( ((t-x)^2, x) + \frac{\alpha - \beta x}{n+1+\beta} \right)^2 \right] \|g''\| \\
&\leq \left[ \frac{(n+1+2\beta^2)x(1+x) + \alpha^2 + (\alpha - \beta x)^2}{(n+1+\beta)^2} \right] \|g''\| \\
&\leq \left[ \frac{(n+1+4\beta^2)x(1+x) + 2\alpha^2}{(n+1+\beta)^2} \right] \|g''\|
\end{align*}
\]
\[
|V_n^{\alpha,\beta} (f, x) - f(x)| \leq \left| V_n^{\alpha,\beta} (f-g, x) - (f-g)(x) \right| + \left| V_n^{\alpha,\beta} (g, x) - g(x) \right|
\]
\[
\begin{align*}
&\leq 2\|f-g\| + \left| V_n^{\alpha,\beta} (g, x) - g(x) \right| + \left| f \left( \frac{(n+1)x + \alpha}{n+1+\beta} \right) - f(x) \right| \\
&\leq 2\|f-g\| + \left[ \frac{(n+1+4\beta^2)x(1+x) + 2\alpha^2}{(n+1+\beta)^2} \right] \|g''\| \\
&+ \left| f \left( \frac{(n+1)x + \alpha}{n+1+\beta} \right) - f(x) \right|
\end{align*}
\]
\[2 \|f - g\| + \left(\frac{(n + 1 + 4\beta^2)x(1 + x) + 2\alpha^2}{(n + 1 + \beta)^2}\right)\|g^*\| + \omega\left(f, \frac{|\alpha - \beta x|}{n + 1 + \beta}\right),\]

where \(\omega(f, \cdot)\) is the usual modulus of continuity of \(f\).

Taking infimum over all \(g \in C^0[0, \infty)\), we have

\[|\mathcal{V}_{n^a\beta} f(x) - f(x)| \leq 2K \left(\frac{(n + 1 + 4\beta^2)x(1 + x) + 2\alpha^2}{(n + 1 + \beta)^2}\right) + \omega\left(f, \frac{|\alpha - \beta x|}{n + 1 + \beta}\right).\]

This completes the proof of the theorem.

References


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