gζ*-closed maps and gζ*-open maps in Topological Spaces

V. Kokilavani[1], M. Myvizhi[2]

[1]Assistant Professor, Dept. of Mathematics, Kongunadu Arts and Science College, Coimbatore, Tamilnadu, India

[2]Assistant Professor, Dept. of Mathematics, Sri Ranganathar Institute of Engineering and Technology, Coimbatore, Tamilnadu, India

Abstract: ***In this paper, we introduce gζ*-closed map from a topological space X to a topological space Y as the image of every closed set is gζ*-closed and also we obtain some properties of gζ*-closed maps.***

Keywords: gζ*-closed sets, gζ*-open sets, gζ*-closed maps and gζ*-open maps

1. Introduction

In this paper, a new class of maps called generalized ζ*-closed (briefly, gζ*-closed) maps have been introduced and also we obtain some properties of gζ*-closed maps.

Definition 1.1

A subset A of a space (X, τ) is called

i. α-open set [1] if $A \subseteq int(cl(int(A)))$
ii. a generalized closed set [2](briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
iii. a generalized #α-closed set [3](briefly g#α-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).
iv. a #generalized α-closed set [4](briefly #gα-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a g#α-open in (X, τ).
v. a generalized ζ*-closed set [5](briefly gζ*-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a #gα-open in (X, τ).

The complement of above mentioned closed sets are their respective open sets.

Definition 1.2

A map $f:(X, \tau) \to (Y, \sigma)$ is called

i. g-continuous [6] if $f^{-1}(V)$ is g-closed of (X, τ) for every closed set V of (Y, σ).
ii. g#α-continuous [3] if $f^{-1}(V)$ is g#α-closed in (X, τ) for every closed set V of (Y, σ).
iii. #gα-continuous [4] if $f^{-1}(V)$ is #gα-closed in (X, τ) for every closed set V of (Y, σ).
iv. gζ*-continuous[5] if $f^{-1}(V)$ is gζ*-closed in (X, τ) for every closed set V of (Y, σ).
v. #gα-irresolute[4] if $f^{-1}(V)$ is #gα-closed in (X, τ) for every #gα-closed set V of (Y, σ).

(i) gζ*-irresolute [5] if $f^{-1}(V)$ is gζ*-closed in (X, τ) for every gζ*-closed set V of (Y, σ).

2. gζ*-closed maps and gζ*-open maps

In this section, we introduce the concepts of gζ*-closed maps and gζ*-open maps in topological spaces.

Definition 2.1

Let X and Y be two topological spaces. A map $f:(X, \tau) \to (Y, \sigma)$ is called generalized ζ*-closed (briefly, gζ*-closed) map if the image of every closed set in (X, τ) is gζ*-closed in (Y, σ).

Theorem 2.2

Every closed map is gζ*-closed map, but not conversely.

Proof:

Let $f:(X, \tau) \to (Y, \sigma)$ is a closed map and V be a closed set in (X, τ),then $f(V)$ is closed in (Y, σ) and hence gζ*-closed in (Y, σ). Thus f is gζ*-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 2.3

Consider X=Y= {a, b, c} with topologies τ = {X,φ, {a}, {a,b}} and σ = {Y, φ,{a}}. Let $f:(X,\tau) \to (Y,\sigma)$ be the identity map. Then this function is gζ*-closed but not closed, as the image of closed set {c} in (X,τ) is {c} which is not open set in(Y,σ).

Theorem 2.4

A map$f:(X,\tau) \to (Y,\sigma)$ is gζ*-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a gζ*-open set V of Y such that $S \subseteq V$ and $f^{-1}(S) \subseteq U$.

Proof:

Suppose f is gζ*-closed. Let S be a subset of Y and U be an open set of X such that$f^{-1}(S) \subseteq U$, $V = Y - f(X-U)$ is gζ*-open set containing S such that$f^{-1}(V) \subseteq U$.

Converse:

Suppose that F is a closed set of X. Then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is open. By hypothesis, there is a gζ*-open set V of Y such that $X - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore $F \subseteq X - f^{-1}(V)$, hence $Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$ which implies$f(F) = Y - V$, since $Y - V$ is gζ*-closed, $f(F)$ is gζ*-closed and hence f is gζ*-closed map.

Theorem 2.5

If a map $f:(X,\tau) \to (Y,\sigma)$ is continuous and gζ*-closed and A is gζ*-closed set of X, then $f(A)$ is gζ*-closed in Y.

Proof:

Let $f(A) \subseteq U$ where U is open set in Y, since f is continuous,$f^{-1}(U)$ is an open set containing A. Hence $\alpha cl(A) \subseteq f^{-1}(U)$ as A is gζ*-closed, since f is gζ*-closed, $f(\alpha cl(A)) \subseteq U$ is gζ*-closed, U is an open set which implies $\alpha cl(f(\alpha cl(A))) \subseteq U$ and hence$\alpha cl(f(A)) \subseteq U$, so $f(A)$ is gζ*-closed set in Y.

Corollary 2.6

If a map $f:(X,\tau) \to (Y,\sigma)$ is continuous and closed and A is gζ*-closed set of X, then $f(A)$ is gζ*-closed in Y.

Corollary 2.7

If a map $f:(X,\tau) \to (Y,\sigma)$ is gζ*-closed and A is closed set of X, then $f_A:A \to Y$ is gζ*-closed.

Corollary 2.8

If a map $f:(X,\tau) \to (Y,\sigma)$ is gζ*-closed and continuous and A is gζ*-closed set of X, then $f_A:A \to Y$ is continuous and gζ*-closed.

Proof:

Let F be a closed set of A then F is gζ*-closed set of X, by the theorem $f(A)$ is gζ*-closed (Theorem 2.8) hence $f_A(F) = f(F)$ is gζ*-closed set of Y. Here f_A is gζ*-closed and also continuous.

Definition 2.9

A space X is said to be normal if for every two disjoint closed subsets A and B of X, there exists two disjoint #gα-open sets U and V such that $A \subset U$ and $B \subset V$.

Theorem 2.10

If $f:(X,\tau) \to (Y,\sigma)$ is a continuous, gζ*-closed map from a normal space X onto a space Y, then Y is normal.

Proof:

Let A,B be two disjoint closed sets of Y, then $f^{-1}(A), f^{-1}(B)$ are disjoint closed sets of X, since X is normal, there are two disjoint open sets U,V in X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq U$, since f is gζ*-closed by theorem 2.2 , there are gζ*-open sets G,H in Y such that $A \subseteq G, B \subseteq H$ and $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$,since U,V are disjoint open sets α-int(G) and α-int(H) are disjoint open sets since G is open, A is closed and $A \subseteq G$, $A \subseteq \alpha$-int(G) and H is open set is closed and $B \subseteq H$ then $B \subseteq \alpha$-int(H). Hence Y is normal.

Theorem 2.11

If $f:(X,\tau) \to (Y,\sigma)$ is closed map and $g:(Y,\sigma) \to (Z,\eta)$ is gζ*-closed map, then the composition $g \circ f:(X,\tau) \to (Z,\eta)$ is gζ*-closed map.

Proof:

Let F be any closed set in(X,τ), since f is closed map, $f(F)$ is closed set in (Y,σ). Since g is gζ*-closed map, $g(f(F))$ is gζ*-closed set in(Z,η). That is $g \circ f(F) = g(f(F))$ is gζ*-closed and hence $g \circ f$ is gζ*-closed map.

Remark 2.12

If a map $f:(X,\tau) \to (Y,\sigma)$ is gζ*-closed map and $g:(Y,\sigma) \to (Z,\eta)$ is closed map, then the composition need not be gζ*-closed map as seen from the following example.

Example 2.13

Consider X = Y = Z = {a, b, c}, τ = {X, φ, {a}, {b}, {a,b}}, σ = {Y, φ, {a},{b, c}} and η = {Z, φ,{b},{c},{b,c}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map and $g:(Y,\sigma)\to(Z,\eta)$ is defined by $g(a)=g(b)=a$ and $g(c)=b$. Then f is gζ*-closed map and g is a closed map. But their composition $g\circ f:(X,\tau)\to(Z,\eta)$ is not gζ*-closed map, since for the closed set {c} in (X,τ), but $g\circ f(\{c\})=g(f(\{c\}))=g(\{c\})=\{b\}$ which is not gζ*-closed in (Z,η).

Theorem 2.14

Let (X,τ), (Z,η) be two topological spaces, and (Y,σ) be topological spaces where "Every gζ*-closed subset is closed". Then the composition $g\circ f:(X,\tau)\to(Z,\eta)$ of the gζ*-closed maps $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ is gζ*-closed.

Proof:

Let A be a closed set of (X,τ). Since f is gζ*-closed, $f(A)$ is gζ*-closed in (Y,σ). Then by hypothesis, $f(A)$ is closed. Since g is gζ*-closed, $g(f(A))$ is gζ*-closed in (Z,η) and $g(f(A))=g\circ f(A)$. Therefore $g\circ f$ is gζ*-closed.

Theorem 2.15

Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two mappings such that their composition $g\circ f:(X,\tau)\to(Z,\eta)$ be gζ*-closed mapping. Then the following statements are true.

(i) If f is continuous and surjective, then g is gζ*closed.
(ii) If g is gζ*-irresolute and injective, then f is gζ* closed.

Proof:

(i) Let A be a closed set of (Y,σ). Since f is continuous, $f^{-1}(A)$ is closed in (X,τ) and since $g\circ f$ is gζ*-closed, $(g\circ f)(f^{-1}(A))$ is gζ*-closed in (Z,η). That is $g(A)$ is gζ*-closed in (Z,η), since f is surjective. Therefore g is gζ*-closed.
(ii) Let B be a closed set of (X,τ). Since $g\circ f$ is gζ*-closed, $g\circ f(B)$ is gζ*-closed in (Z,η). Since g is gζ*-irresolute, $g^{-1}(g\circ f(B))$ is gζ*-closed set in (Y,σ). That is $f(B)$ is gζ*-closed in (Y,σ), since f is injective. Therefore f is gζ*-closed.

Theorem 2.16

If $f:(X,\tau)\to(Y,\sigma)$ is an open, continuous, gζ*-closed surjection and $cl(F)=F$ for every gζ*-closed set in (Y,σ), where X is regular, then Y is regular.

Proof:

Let U be an open set in Y and $p\in U$. Since f is surjection, there exists a point $x\in X$ such that $f(x)=p$. Since X is regular and f is continuous, there is an open set V in X such that $x\in V\subset cl(V)\subset f^{-1}(U)$. Here, $p\in f(V)\subset f(cl(V))\subset U\to(i)$. Since f is gζ*-closed, $f(cl(V))$ is gζ*-closed set contained in the open set U. By hypothesis, $cl(f(cl(V)))=f(cl(V))$ and $cl(f(V))=cl(f(cl(V)))\to(ii)$. From (i) and (ii), we have $p\in f(V)\subset cl(f(V))\subset U$ and $f(V)$ is open, since f is open. Hence Y is regular.

Definition 2.17

Let X and Y be two topological spaces. A map $f:(X,\tau)\to(Y,\sigma)$ is called generalized ζ^*- open (briefly, $g\zeta^*$-open) map if the image of every open set in (X,τ) is $g\zeta^*$-open in (Y,σ).

Theorem 2.18

Every open map is $g\zeta^*$-open map, but not conversely.

Proof:

Let $f:(X,\tau)\to(Y,\sigma)$ is an open map and V be an open set in (X,τ), then $f(V)$ is open in (Y,σ) and hence $g\zeta^*$-open in (Y,σ). Thus f is $g\zeta^*$-open.

The converse of the above theorem need not be true as seen from the following example.

Example 2.19

Consider X=Y= {a, b, c} with topologies τ = {X,φ, {a}, {a,b}} and σ = {Y, φ,{a,b}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then this function is $g\zeta^*$-open but not open, as the image of open set {a} in (X,τ) is {a} which is not open set in (Y,σ).

Theorem 2.20

For any bijection map $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent.

(i) $f^{-1}:(Y,\sigma)\to(X,\tau)$ is $g\zeta^*$-continuous.
(ii) f is $g\zeta^*$-open map and
(iii) f is $g\zeta^*$-closed map.

Proof:

(i) ⇒ (ii) Let U be an open set of (X,τ). By assumption, $(f^{-1})^{-1}$(U) = f (U) is $g\zeta^*$-open in (Y,σ) and so f is $g\zeta^*$-open.

(ii) ⇒ (iii) Let F be a closed set of (X,τ). Then F^c is open set in(X,τ). By assumption, $f(F^c)$ is gζ^*-open in (Y,σ) and therefore $f(F)$ is gζ^*-closed in(Y,σ). Hence f is gζ^*-closed.
(iii) ⇒ (i) Let F be a closed set of (X,τ). By assumption, $f(F)$ is gζ^*-closed in(Y,σ). But $f(F) = (f^{-1})^{-1}$ (F) and therefore f^{-1} is continuous.

Theorem 2.21

A map $f:(X,\tau) \to (Y,\sigma)$ is gζ^*-open if and only if for any subset S of (Y,σ) and any closed set of (X,τ) containing f^{-1}(S), there exists a gζ^*-closed set K of (Y,σ) containing S such that $f^{-1}(K) \subset F$.

Proof:

Suppose f is gζ^*-open map. Let $S \subset Y$ and F be a closed set of (X,τ) such that$f^{-1}(S) \subset F$. Now $X - F$ is an open set in(X,τ). Since f is gζ^*-open map, $f(X-F)$ is gζ^*-open set in(Y,σ). Then $K = Y - f(X-F)$ is a gζ^*-closed set in(Y,σ). Note that$f^{-1}(S) \subset F$ implies $S \subset K$ and $f^{-1}(K) = X - f^{-1}(X-F) \subset X - (X-F) = F$.

That is$f^{-1}(K) \subset F$.

For the converse, let U be an open set of(X,τ). Then $f^{-1}((f(U))^c) \subset U^c$ and U^c is a closed set in(X,τ). By hypothesis, there exists a gζ^*-closed set K of (Y,σ) such that $(f(U))^c \subset K$ and $f^{-1}(K) \subset U^c$ and so $U \subset (f^{-1}(K))^c$. Hence $K^c \subset f(U) \subset f((f^{-1}(K)))^c \subset K^c$ which implies$f(U) = K^c$. Since K^c is a gζ^*-open, $f(U)$ is gζ^*-open in (Y,σ) and therefore f is gζ^*-open map.

References

[1] Njastad, On some classes of nearly open sets, Pacific J Math., 15(1965), 961-970
[2] N.Levine, generalized Closed sets in Topology, Rend. Circ. Math. Palermo, 19, (1970), 89-96
[3] K.Nono, R.Devi, M.Devipriya, K.Muthukumaraswamy and H.Maki, On g#α- closed sets and the Digital plane, Bull. Fukuoka Univ. Ed. Part III, 53, (2004), 15-24
[4] R.Devi, H.Maki and V.Kokilavani, The group structure of #gα-closed sets in topological spaces, International Journal of General Topology, 2(2009), 21-30
[5] V.Kokilavani, M.Myvizhi and M.Vivek prabu, Generalized ζ^*-closed sets in topological spaces, International Journal of Mathematical Archive, 4950, 2013, 274 – 279.
[6] K.Balachandran, P.Sundaram and H.Maki, On Generalized Continuous Maps in Topological Spaces, Mem. Fac. Kochi Uni. Ser. A Math., 12, (1991), 5-13.
[7] R.Devi, V.Kokilavani and P.Basker, On Strongly- $\alpha\delta$-Super-Irresolute Functions in Topological Spaces, International Journal of Computer Applications, 40, (2012), 38-42
[8] A.Pushpalatha and K.Anitha, g*s-Closed sets in Topological Spaces, Int.J.Contemp.Math.Sciences, 6, (2011), 917-929
[9] rgα-Closed and rgα-Open Maps in Topological Spaces, Int. Journal of Math. Analysis, 4, (2010), 453-468

Author Profile

M Myvizhi received B.Sc degree in Mathematics from Nirmala College for Women, Coimbatore, India. She received M.Phil degree in Mathematics from Bharathiar University, Coimbatore, India. She is pursuing Ph.D in Mathematics (Topology) from Kongunadu Arts and Science College, Coimbatore, India. Currently she is working as Assistant Professor in the Department of Mathemtics, Sri Ranganathar Institute of Engineering and Technology, Coimbatore, India.