gζ*-closed maps and gζ*-open maps in Topological Spaces

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Abstract: ***In this paper, we introduce gζ*-closed map from a topological space X to a topological space Y as the image of every closed set is gζ*-closed and also we obtain some properties of gζ*-closed maps.***

Keywords: gζ*-closed sets, gζ*-open sets, gζ*-closed maps and gζ*-open maps

1. Introduction

In this paper, a new class of maps called generalized ζ*-closed (briefly, gζ*-closed) maps have been introduced and also we obtain some properties of gζ*-closed maps.

Definition 1.1

A subset A of a space (X,τ) is called

i. α-open set [1] if $A \subseteq int(cl(int(A)))$

ii. a generalized closed set [2](briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).

iii. a generalized #α-closed set [3](briefly g#α-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X,τ).

iv. a #generalized α-closed set [4](briefly #gα-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a g#α-open in (X,τ).

v. a generalized ζ*-closed set [5](briefly gζ*-closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a #gα-open in (X,τ).

The complement of above mentioned closed sets are their respective open sets.

Definition 1.2

A map $f:(X,\tau) \to (Y,\sigma)$ is called

i. g-continuous [6] if $f^{-1}(V)$ is g-closed of (X,τ) for every closed set V of (Y,σ).

ii. g#α-continuous [3] if $f^{-1}(V)$ is g#α-closed in (X,τ) for every closed set V of (Y,σ).

iii. #gα-continuous [4] if $f^{-1}(V)$ is #gα-closed in (X,τ) for every closed set V of (Y,σ).

iv. gζ*-continuous[5] if $f^{-1}(V)$ is gζ*-closed in (X,τ) for every closed set V of (Y,σ).

v. #gα-irresolute[4] if $f^{-1}(V)$ is #gα-closed in (X,τ) for every #gα-closed set V of (Y,σ).

(i) gζ*-irresolute [5] if $f^{-1}(V)$ is gζ*-closed in (X,τ) for every gζ*-closed set V of (Y,σ).

2. gζ*-closed maps and gζ*-open maps

In this section, we introduce the concepts of gζ*-closed maps and gζ*-open maps in topological spaces.

Definition 2.1

Let X and Y be two topological spaces. A map $f:(X,\tau) \to (Y,\sigma)$ is called generalized ζ*-closed (briefly, gζ*-closed) map if the image of every closed set in (X,τ) is gζ*-closed in (Y,σ).

Theorem 2.2

Every closed map is gζ*-closed map, but not conversely.

Proof:

Let $f:(X,\tau) \to (Y,\sigma)$ is a closed map and V be a closed set in (X,τ),then $f(V)$ is closed in (Y,σ) and hence gζ*-closed in (Y,σ). Thus f is gζ*-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 2.3

Consider X=Y= {a, b, c} with topologies τ = {X,φ, {a}, {a,b}} and σ = {Y, φ,{a}}. Let $f:(X,\tau) \to (Y,\sigma)$ be the identity map. Then this function is gζ^*-closed but not closed, as the image of closed set {c} in (X,τ) is {c} which is not open set in(Y,σ).

Theorem 2.4

A map$f:(X,\tau) \to (Y,\sigma)$ is gζ^*-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$ there is a gζ^*-open set V of Y such that $S \subseteq V$ and $f^{-1}(S) \subseteq U$.

Proof:

Suppose f is gζ^*-closed. Let S be a subset of Y and U be an open set of X such that$f^{-1}(S) \subseteq U$, $V = Y - f(X - U)$ is gζ^*-open set containing S such that$f^{-1}(V) \subseteq U$.

Converse:

Suppose that F is a closed set of X. Then
$f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is open. By hypothesis, there is a gζ^*-open set V of Y such that $X - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore $F \subseteq X - f^{-1}(V)$, hence $Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$ which implies$f(F) = Y - V$, since $Y - V$ is gζ^*-closed, $f(F)$ is gζ^*-closed and hence f is gζ^*-closed map.

Theorem 2.5

If a map $f:(X,\tau) \to (Y,\sigma)$ is continuous and gζ^*-closed and A is gζ^*-closed set of X, then $f(A)$ is gζ^*-closed in Y.

Proof:

Let $f(A) \subseteq U$ where U is open set in Y, since f is continuous,$f^{-1}(U)$ is an open set containing A. Hence $\alpha cl(A) \subseteq f^{-1}(U)$ as A is gζ^*-closed, since f is gζ^*-closed, $f(\alpha cl(A)) \subseteq U$ is gζ^*-closed, U is an open set which implies $\alpha cl(f(\alpha cl(A))) \subseteq U$ and hence$\alpha cl(f(A)) \subseteq U$, so $f(A)$ is gζ^*-closed set in Y.

Corollary 2.6
If a map $f:(X,\tau) \to (Y,\sigma)$ is continuous and closed and A is gζ^*-closed set of X, then $f(A)$ is gζ^*-closed in Y.

Corollary 2.7
If a map $f:(X,\tau) \to (Y,\sigma)$ is gζ^*-closed and A is closed set of X, then $f_A:A \to Y$ is gζ^*-closed.

Corollary 2.8

If a map $f:(X,\tau) \to (Y,\sigma)$ is gζ^*-closed and continuous and A is gζ^*-closed set of X, then $f_A:A \to Y$ is continuous and gζ^*-closed.

Proof:

Let F be a closed set of A then F is gζ^*-closed set of X, by the theorem $f(A)$ is gζ^*-closed (Theorem 2.8) hence $f_A(F) = f(F)$ is gζ^*-closed set of Y. Here f_A is gζ^*-closed and also continuous.

Definition 2.9

A space X is said to be normal if for every two disjoint closed subsets A and B of X, there exists two disjoint $^{\#}$gα-open sets U and V such that $A \subset U$ and $B \subset V$.

Theorem 2.10

If $f:(X,\tau) \to (Y,\sigma)$ is a continuous, gζ^*-closed map from a normal space X onto a space Y, then Y is normal.

Proof:

Let A,B be two disjoint closed sets of Y, then $f^{-1}(A), f^{-1}(B)$ are disjoint closed sets of X, since X is normal, there are two disjoint open sets U,V in X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq U$, since f is gζ^*-closed by theorem 2.2 , there are gζ^*-open sets G,H in Y such that $A \subseteq G, B \subseteq H$ and $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$,since U,V are disjoint open sets α-int(G) and α-int(H) are disjoint open sets since G is open, A is closed and $A \subseteq G, A \subseteq \alpha$-int(G) and H is open set is closed and $B \subseteq H$ then $B \subseteq \alpha$-int(H). Hence Y is normal.

Theorem 2.11

If $f:(X,\tau) \to (Y,\sigma)$ is closed map and $g:(Y,\sigma) \to (Z,\eta)$ is gζ^*-closed map, then the composition $g \circ f:(X,\tau) \to (Z,\eta)$ is gζ^*-closed map.

Proof:

Let F be any closed set in(X,τ), since f is closed map, $f(F)$ is closed set in (Y,σ), Since g is gζ^*-closed map, $g(f(F))$ is gζ^*-closed set in(Z,η). That is $g \circ f(F) = g(f(F))$ is gζ^*-closed and hence $g \circ f$ is gζ*-closed map.

Remark 2.12

If a map $f:(X,\tau) \to (Y,\sigma)$ is gζ^*-closed map and $g:(Y,\sigma) \to (Z,\eta)$ is closed map, then the composition need not be gζ^*-closed map as seen from the following example.

Example 2.13

Consider X = Y = Z = {a, b, c}, τ = {X, φ, {a}, {b}, {a,b}}, σ = {Y, φ, {a},{b, c}} and η = {Z, φ,{b},{c},{b,c}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map and $g:(Y,\sigma)\to(Z,\eta)$ is defined by $g(a)=g(b)=a$ and $g(c)=b$. Then f is gζ*-closed map and g is a closed map. But their composition $g\circ f:(X,\tau)\to(Z,\eta)$ is not gζ*-closed map, since for the closed set {c} in (X,τ), but $g\circ f(\{c\})=g(f(\{c\}))=g(\{c\})=\{b\}$ which is not gζ*-closed in (Z,η).

Theorem 2.14

Let (X,τ), (Z,η) be two topological spaces, and (Y,σ) be topological spaces where "Every gζ*-closed subset is closed". Then the composition $g\circ f:(X,\tau)\to(Z,\eta)$ of the gζ*-closed maps $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ is gζ*-closed.

Proof:

Let A be a closed set of (X,τ). Since f is gζ*-closed, $f(A)$ is gζ*-closed in (Y,σ). Then by hypothesis, $f(A)$ is closed. Since g is gζ*-closed, $g(f(A))$ is gζ*-closed in (Z,η) and $g(f(A))=g\circ f(A)$. Therefore $g\circ f$ is gζ*-closed.

Theorem 2.15

Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two mappings such that their composition $g\circ f:(X,\tau)\to(Z,\eta)$ be gζ*-closed mapping. Then the following statements are true.

(i) If f is continuous and surjective, then g is gζ*closed.
(ii) If g is gζ*-irresolute and injective, then f is gζ* closed.

Proof:

(i) Let A be a closed set of (Y,σ). Since f is continuous, $f^{-1}(A)$ is closed in (X,τ) and since $g\circ f$ is gζ*-closed, $(g\circ f)(f^{-1}(A))$ is gζ*-closed in (Z,η). That is $g(A)$ is gζ*-closed in (Z,η), since f is surjective. Therefore g is gζ*-closed.
(ii) Let B be a closed set of (X,τ). Since $g\circ f$ is gζ*-closed, $g\circ f(B)$ is gζ*-closed in (Z,η). Since g is gζ*-irresolute, $g^{-1}(g\circ f(B))$ is gζ*-closed set in (Y,σ). That is $f(B)$ is gζ*-closed in (Y,σ), since f is injective. Therefore f is gζ*-closed.

Theorem 2.16

If $f:(X,\tau)\to(Y,\sigma)$ is an open, continuous, gζ*-closed surjection and $cl(F)=F$ for every gζ*-closed set in (Y,σ), where X is regular, then Y is regular.

Proof:

Let U be an open set in Y and $p\in U$. Since f is surjection, there exists a point $x\in X$ such that $f(x)=p$. Since X is regular and f is continuous, there is an open set V in X such that $x\in V\subset cl(V)\subset f^{-1}(U)$. Here, $p\in f(V)\subset f(cl(V))\subset U\to(i)$. Since f is gζ*-closed, $f(cl(V))$ is gζ*-closed set contained in the open set U. By hypothesis, $cl(f(cl(V)))=f(cl(V))$ and $cl(f(V))=cl(f(cl(V)))\to(ii)$. From (i) and (ii), we have $p\in f(V)\subset cl(f(V))\subset U$ and $f(V)$ is open, since f is open. Hence Y is regular.

Definition 2.17

Let X and Y be two topological spaces. A map $f:(X,\tau)\to(Y,\sigma)$ is called generalized ζ^*- open (briefly, gζ^*-open) map if the image of every open set in (X,τ) is gζ^*-open in (Y,σ).

Theorem 2.18

Every open map is gζ^*-open map, but not conversely.

Proof:

Let $f:(X,\tau)\to(Y,\sigma)$ is an open map and V be an open set in (X,τ), then $f(V)$ is open in (Y,σ) and hence gζ^*-open in (Y,σ). Thus f is gζ^*-open.

The converse of the above theorem need not be true as seen from the following example.

Example 2.19

Consider X=Y= {a, b, c} with topologies τ = {X,φ, {a}, {a,b}} and σ = {Y, φ,{a,b}}. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then this function is gζ^*-open but not open, as the image of open set {a} in (X,τ) is {a} which is not open set in (Y,σ).

Theorem 2.20

For any bijection map $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent.

(i) $f^{-1}:(Y,\sigma)\to(X,\tau)$ is gζ^*-continuous.
(ii) f is gζ^*-open map and
(iii) f is gζ^*-closed map.

Proof:

(i) ⇒ (ii) Let U be an open set of (X,τ). By assumption, $(f^{-1})^{-1}$(U) = f (U) is gζ^*-open in (Y,σ) and so f is gζ^*-open.

(ii) ⇒ (iii) Let F be a closed set of (X, τ). Then F^c is open set in (X, τ). By assumption, $f(F^c)$ is gζ*-open in (Y, σ) and therefore $f(F)$ is gζ*-closed in (Y, σ). Hence f is gζ*-closed.
(iii) ⇒ (i) Let F be a closed set of (X, τ). By assumption, $f(F)$ is gζ*-closed in (Y, σ). But $f(F) = (f^{-1})^{-1}$ (F) and therefore f^{-1} is continuous.

Theorem 2.21

A map $f:(X, \tau) \to (Y, \sigma)$ is gζ*-open if and only if for any subset S of (Y, σ) and any closed set of (X, τ) containing f^{-1}(S), there exists a gζ*-closed set K of (Y, σ) containing S such that $f^{-1}(K) \subset F$.

Proof:

Suppose f is gζ*-open map. Let $S \subset Y$ and F be a closed set of (X, τ) such that $f^{-1}(S) \subset F$. Now $X - F$ is an open set in (X, τ). Since f is gζ*-open map, $f(X - F)$ is gζ*-open set in (Y, σ). Then $K = Y - f(X - F)$ is a gζ*-closed set in (Y, σ). Note that $f^{-1}(S) \subset F$ implies $S \subset K$ and $f^{-1}(K) = X - f^{-1}(X - F) \subset X - (X - F) = F$.

That is $f^{-1}(K) \subset F$.

For the converse, let U be an open set of (X, τ). Then $f^{-1}((f(U))^c) \subset U^c$ and U^c is a closed set in (X, τ). By hypothesis, there exists a gζ*-closed set K of (Y, σ) such that $(f(U))^c \subset K$ and $f^{-1}(K) \subset U^c$ and so $U \subset (f^{-1}(K))^c$. Hence $K^c \subset f(U) \subset f((f^{-1}(K)))^c \subset K^c$ which implies $f(U) = K^c$. Since K^c is a gζ*-open, $f(U)$ is gζ*-open in (Y, σ) and therefore f is gζ*-open map.

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